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GAUGE FIELD ASSOCIATED WITH THE THERMAL BOGOLIUBOV TRANSFORMATION

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Thermo field dynamics (TFD), which is a real time quantum field theory with themal degrees of freedom, is well established as a theory of equilibrium systems [1]. TFD is also extended to describe time-dependent phnomena and yields many interesting results [2][3]. So far, however, most arguments are limited to the spatially homogeneous cases. In the paper in Ref. [4], we proposed a way to extend TFD to spatially inhomogeneous systems.

In TFD we have two kinds of fields, the field by which the dynamics of the system is described and one which has the thermal vacuum as its vacuum. These two fields are related with each other by the thermal Bogoliubov transformation which depends on the parameters characterizing the thermal state of the system. For the spatially inhomogeneous system, these parameters and hence the thermal Bogoliubov transformation must depend on the space-time coordinates. The space-time dependence of the thermal Bogoliubov transformation leads the noncommutativity of the derivation with respect to space-time coordinates and the transformation. This motivates us to introduce a gauge field associated with the thermal Bogoliubov transformation to construct the covariant derivative. The purpose of this note is to show how it can be done.

Let us begin by recapitulating the results of Ref. [4] to show how to construct the space-time dependent thermal Bogliubov transformation.

As is known [2], in the spatially homogeneous TFD, the thermal Bogoliubov transformation matrix B contains the three parameters, n_k, α_k, s_k.

For simplicity, we choose the paticular values of $\alpha_k = 1$ and $s_k = (1/2)\ln(1+n_k)$. For this choice, B has a simple form as

$$B[n_k] = \begin{pmatrix} 1+\sigma n_k & -n_k \\ -\sigma & 1 \end{pmatrix}, \tag{1}$$

with σ being 1 (-1) for bosonic (fermionic) operators.

The transformation by B is generated formally by the following operator transformations:

$$\begin{aligned} a_k^\mu &= B^{-1}[n_k]^{\mu\nu}\xi_k^\nu \\ &= e^{-G[n_k]}e^{-G_0}\xi_k^\mu e^{G_0}e^{G[n_k]}, \\ \bar{a}_k^\mu &= \bar{\xi}_k^\nu B[n_k]^{\nu\mu} \\ &= e^{-G[n_k]}e^{-G_0}\bar{\xi}_k^\mu e^{G_0}e^{G[n_k]}, \end{aligned} \tag{2}$$

where

$$G_0 = \int d\vec{k}\bar{\xi}^\mu_k \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{\mu\nu} \xi^\nu_k, \tag{3a}$$

$$G[n_k] = \int d\vec{k} n_k \bar{\xi}^\mu_k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{\mu\nu} \xi^\nu_k. \tag{3b}$$

Here we use the usual thermal doublet notation [3].

To generalize the above trasformation to spatially inhomogeneous situations, we note that G in eq. (3b) can be rewritten [5] as

$$G[n_k] = \frac{1}{(2\pi)^3} \int d\vec{x} \int d\vec{k} n_k J_k(t, \vec{x}; \vec{k}) \tag{4}$$

with

$$J_k(t, \vec{x}; \vec{k}) = \frac{1}{(2\pi)^3} \int d\vec{\eta} e^{i\vec{k}\cdot\vec{\eta}} \bar{\xi}(t, \vec{x} + \frac{1}{2}\vec{\eta})^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{\mu\nu} \xi(t, \vec{x} - \frac{1}{2}\vec{\eta})^\nu. \tag{5}$$

Here the fields $\bar{\xi}(t, \vec{x})^\mu$ and $\xi(t, \vec{x})^\mu$ are defined by

$$\xi(t, \vec{x})^\mu = \frac{1}{(2\pi)^{3/2}} \int d\vec{k} \xi^\mu_k e^{i\vec{k}\cdot\vec{x} - i\omega_k t}, \tag{6a}$$

$$\bar{\xi}(t, \vec{x})^\mu = \frac{1}{(2\pi)^{3/2}} \int d\vec{k} \bar{\xi}^\mu_k e^{-i\vec{k}\cdot\vec{x} + i\omega_k t}, \tag{6b}$$

which obey the equal time commutation relation,

$$[\xi(t,\vec{x})^{\mu},\ \bar{\xi}(t,\vec{y})^{\nu}]_{\sigma} = \delta^{\mu\nu}\delta(\vec{x}-\vec{y}), \tag{7}$$

where $[A, B]_{\sigma}$ means $AB - \sigma BA$.

In the time dependent but spatially homogeneous case [2], we introcuce the time dependent Bogoliubov transformation by considering n_k in $G[n_k]$, hence in $B[n_k]$, to be time dependent. For spatially inhomogeneous situation, it seems quite natural to modify the generator $G[n_k]$ by replacing n_k in (4) with the space-time dependent $n_k(t, \vec{x}) \equiv n(t, \vec{x}; \vec{k})$.

The transformations (2) with thus generalized $G[n(t, \vec{x}; \vec{k})]$ lead the momentum mixing Bogoliubov transformations [4, 6]

$$a_k(t)^{\mu} = \frac{1}{(2\pi)^3}\int d\vec{k}' \int d\vec{x} e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} B^{-1}[n(t,\vec{x};\frac{1}{2}(\vec{k}+\vec{k}'))]^{\mu\nu} e^{-i\omega_{k'}t}\xi^{\nu}_{k'}, \tag{8a}$$

$$\bar{a}_k(t)^{\mu} = \frac{1}{(2\pi)^3}\int d\vec{k}' \int d\vec{x} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \bar{\xi}^{\nu}_{k'} e^{i\omega_{k'}t} B[n(t,\vec{x};\frac{1}{2}(\vec{k}+\vec{k}'))]^{\nu\mu}, \tag{8b}$$

which assure the commutation relation

$$[a_k(t)^{\mu},\ \bar{a}_l(t)^{\nu}]_{\sigma} = \delta^{\mu\nu}\delta(\vec{k}-\vec{l}), \tag{9}$$

We note that $n(t, \vec{x}; \vec{k})$ is expressed as

$$n(t,\vec{x};\vec{k}) = \int d\vec{K} e^{i\vec{K}\cdot\vec{x}} \langle 0 | \bar{a}_{k+\frac{1}{2}K}(t)^{1}\, a_{k-\frac{1}{2}K}(t)^{1} | 0 \rangle.$$

Now let us construct the field operators $\phi(t, \vec{x})$ and $\bar{\phi}(t, \vec{x})$ for the oscillator variables $a_k(t)$ and $\bar{a}_k(t)$ as

$$\phi(t,\vec{x})^\mu = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{k} a_k(t)^\mu e^{i\vec{k}\cdot\vec{x}},$$

$$\bar{\phi}(t,\vec{x})^\mu = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d\vec{k} \bar{a}_k(t)^\mu e^{-i\vec{k}\cdot\vec{x}}.$$

Because of eq. (9), the field operators introduced above satisfy the canonical commutation relation for a type I field, i. e. a field which contains only the positive frequency part:

$$[\phi(t,\vec{x})^\mu,\ \bar{\phi}(t,\vec{y})^\nu] = \delta^{\mu\nu}\delta(\vec{x}-\vec{y}). \tag{10}$$

By taking Fourier transform of eq. (8), we get the Bogoliubov transformations which relate the fields $\phi(t, \vec{x})$ and $\bar{\phi}(t, \vec{x})$ to $\xi(t, \vec{x})$ and $\bar{\xi}(t, \vec{x})$:

$$\phi(t,\vec{x})^\mu = \frac{1}{(2\pi)^3} \int d\vec{y} \int d\vec{k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} B^{-1}[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\mu\nu} \xi(t,\vec{y})^\nu, \tag{11a}$$

$$\bar{\phi}(t,\vec{x})^\mu = \frac{1}{(2\pi)^3} \int d\vec{y} \int d\vec{k} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \bar{\xi}(t,\vec{y})^\nu B[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\nu\mu}. \tag{11b}$$

The inverse transformations to eq. (11) are given by

$$\xi(t,\vec{x})^\mu = \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} B[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\mu\nu}\phi(t,\vec{y})^\nu, \tag{12a}$$

$$\bar{\xi}(t,\vec{x})^\mu = \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\bar{\phi}(t,\vec{y})^\nu B^{-1}[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\nu\mu}. \tag{12b}$$

The space-time depedence of $n(t,\ \vec{x};\ \vec{k})$ gives rise to the noncommutativity of the Bogoliubov transformation (11) and the derivations with respect to space-time coordinates:

$$\begin{aligned}\partial_\alpha\phi(t,\vec{x})^\mu &+ \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\partial_\alpha n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})M_0^{\mu\nu}\phi(t,\vec{y})^\nu \\ &= \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}B^{-1}[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\mu\nu}\partial_\alpha\xi(t,\vec{y})^\nu, \end{aligned}\tag{13a}$$

$$\begin{aligned}\partial_\alpha\bar{\phi}(t,\vec{x})^\mu &- \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\partial_\alpha n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})\bar{\phi}(t,\vec{y})^\nu M_0^{\nu\mu} \\ &= \frac{1}{(2\pi)^3}\int d\vec{y}\int d\vec{k} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\partial_\alpha\bar{\xi}(t,\vec{y})^\nu B[n(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})]^{\nu\mu}, \end{aligned}\tag{13b}$$

where

$$M_0 = \begin{pmatrix}\sigma & -1 \\ 1 & -\sigma\end{pmatrix}$$

Here the derivative operator ∂_α is defined by

$$\partial_\alpha = \begin{cases} \partial/\partial x_i & \alpha = i = 1,2,3 \\ \partial/\partial t & \alpha = 0 \end{cases} \tag{14}$$

and the following notation is also used:

$$\partial_i n(t, \frac{1}{2}(\vec{x} + \vec{y}); \vec{k}) = \frac{\partial}{\partial X_i} n(t, \vec{X}; \vec{k})\Big|_{\vec{X}=\frac{1}{2}(\vec{x}+\vec{y})} . \tag{15}$$

The noncommutativity of the derivations and local (space-time dependent) symmetry transformation is common story in the gauge theory formalism. This entice us to introduce gauge fields associated with the space-time dependent thermal Bogoliubov transformation (11).

Let us consider new fields $A_\alpha(t, \vec{x}; \vec{k})$ and assume that these A_α are transformed like

$$A_\alpha(t, \vec{x}; \vec{k}) \to A_\alpha(t, \vec{x}; \vec{k}) - i\partial_\alpha n(t, \vec{x}; \vec{k}) \tag{16}$$

under the thermal Bogoliubov transformation which transforms

$$\xi(t, \vec{x})^\mu \to \phi(t, \vec{x})^\mu . \tag{17}$$

By using A_α thus introduced, we define the operator D_α by

$$D_\alpha f(t,\vec{x})^\mu \equiv \partial_\alpha f(t,\vec{x})^\mu$$
$$-\frac{i}{(2\pi)^3}\int d\vec{y}\int d\vec{k}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}A_\alpha(t,\frac{1}{2}(\vec{x}+\vec{y});\vec{k})M_0^{\mu\nu}f(t,\vec{y})^\nu. \quad (18)$$

Then, from eq. (13) and eq. (16), the operator D_α thus defined has the simple transformation property

$$D_\alpha\xi(t,\vec{x})^\mu \to D_\alpha\phi(t,\vec{x})^\mu \quad (19)$$

under the transformation (11), so that the opeator D_α in (18) gives the covariant derivative with respect to the termal Bogoliubov transformation (11).

The minimal coupling of the gauge field A_α to the field ϕ can be derived in the usual way, that is, by replacing ∂_α with D_α in the Lagrangian which is invariant under the global transformation.

For the simplest example, we take a free Schrödinger field which has the TFD Larangian,

$$\hat{\mathcal{L}}_0 = \int d\vec{x}\,\{i\bar{\phi}(t,\vec{x})^\mu\partial_t\phi(t,\vec{x})^\mu - \frac{1}{2m}(\partial_i\bar{\phi}(t,\vec{x})^\mu)(\partial_i\phi(t,\vec{x})^\mu)\}. \quad (20)$$

Clearly $\hat{\mathcal{L}}_0$ is invariant under a global (with n_k not depending on the space -time coordiates) thermal Bogoliubov transformation. We can turn this invariance into one under local (with n_k depending on the space-time coordinate) transformation (11) by replacing ∂_α with D_α to get

$$
\begin{aligned}
\hat{\mathcal{L}} &= \int d\vec{x}\, \{ i\bar{\phi}(t,\vec{x})^{\mu} D_t \phi(t,\vec{x})^{\mu} - \frac{1}{2m}(D_i\bar{\phi}(t,\vec{x})^{\mu})(D_i\phi(t,\vec{x})^{\mu}) \} \\
&= \hat{\mathcal{L}}_0 - i\int d\vec{x} \int d\vec{k} \rho_Q(t,\vec{x};\vec{k}) A_t(t,\vec{x};\vec{k}) \\
&\quad -i\int d\vec{x} \int d\vec{k} j_{Qi}(t,\vec{x};\vec{k}) A_i(t,\vec{x};\vec{k}),.
\end{aligned} \tag{21}
$$

with

$$
\rho_Q = \frac{1}{(2\pi)^3}\int d\vec{\xi} e^{i\vec{k}\cdot\vec{\xi}} \bar{\phi}(t,\vec{x}+\frac{1}{2}\vec{\xi})^{\mu} M_0^{\mu\nu} \phi(t,\vec{x}-\frac{1}{2}\vec{\xi})^{\nu}, \tag{22a}
$$

$$
\begin{aligned}
j_{Qi} &= \frac{1}{(2\pi)^3}\int d\vec{\xi} e^{i\vec{k}\cdot\vec{\xi}} \frac{i}{2m} \{ \partial_i \bar{\phi}(t,\vec{x}+\frac{1}{2}\vec{\xi})^{\mu} M_0^{\mu\nu} \phi(t,\vec{x}-\frac{1}{2}\vec{\xi})^{\nu}, \\
&\qquad - \bar{\phi}(t,\vec{x}+\frac{1}{2}\vec{\xi})^{\mu} M_0^{\mu\nu} \partial_i \phi(t,\vec{x}-\frac{1}{2}\vec{\xi})^{\nu} \}.
\end{aligned} \tag{22b}
$$

We note that, in the r. h. s. of (21), the term quadratic in A_α vanishes because of the identity

$$
M_0^2 = 0. \tag{23}
$$

The second and the third terms of the r. h. s. of eq. (21) gives the minimal coupling of the gauge field to the field ϕ.

Thus we have been able to introduce the gauge field associated with the thermal Bogoliubov transformation and to construct its minimal interaction along a conventional way.

We remain with the question of the physical interpretation of this gauge field.

Here we note only that the Umezawa's thermal energy [7] $Q(t)$ is written by using ρ_Q in eq. (22a) as

$$Q(t) = i\int d\vec{x}\int d\vec{k}\partial_t n(t,\vec{x};\vec{k})\rho_Q(t,\vec{x};\vec{k}).$$

This suggests the possibility that we may be able to interpret our gauge field as some thermodynamical force. This must be a very interesting problem left in the future.

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