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Anomalous Mirror Symmetry Generated by Optical Illusion

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Abstract

This paper introduces a new concept of mirror symmetry, called "anomalous mirror symmetry", which is physically impossible but can be perceived by human vision systems because of optical illusion. This symmetry is characterized geometrically and a method for creating cylindrical surfaces that create this symmetry is constructed. Examples of solid objects constructed by a 3D printer are also shown.

Keywords: . ambiguous cylinder, mirror symmetry, visual illusion, visual perception.

1 Introduction

An ordinary planar mirror generates a plane symmetric image of an object, that is, for an arbitrary point of an object and its mirror image, the line segment connecting them crosses the mirror plane at the midpoint in the right angle. Although the concept of the mirror symmetry has been generalized in many directions in mathematics and physics [2, 5], the original mirror symmetry with respect to a planar mirror is simple and well understood from a mathematical point of view.

When we see the object and its mirror image, however, what we perceive does not necessarily obey this physical law, because what we perceive is the result of image processing in our brains, and hence optical illusion arises [3, 14]. In particular it was discovered recently that there are cylindrical objects whose mirror images appear to be quite different from what we usually expect [12, 13]. Those cylinders seem to defy the law of optical physics. This phenomenon is called ambiguous cylinder illusion.

In this paper we propose a new concept of symmetry, which we call "anomalous mirror symmetry" ("anomalous symmetry" for short). The anomalous symmetry is a modification of mirror symmetry, but it does not obey the law of optical physics. Hence it cannot be realized in a purely physical manner.

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However, we show that anomalous symmetry can be generated if we employ ambiguous cylinder illusion. We characterize the class of cylinders whose mirror images generate anomalous symmetry and moreover the same anomaly is preserved even if the objects are rotated around a vertical axis.

From a mathematical point of view, we solve a problem of finding a pair of mirror symmetric space curves whose projections coincide with a pair of given point symmetric planar curves. The problem of finding symmetric space curves from given planar curves have been studied in many setting. For example, Hong et al. [4] showed that both the symmetric space curves and the camera poses can be reconstructed from their perspective images. Sawada et al. [10, 11] constructed an algorithm for finding a pair of mirror symmetric space curves and the associated mirror from a pair of planar curves. The present problem is similar to them in the sense that we construct symmetric space curves, but simpler because the mirror pose is given.

The paper is organized in the following way. In Section 2, we review the nature of mirror symmetry and introduce a concept of anomalous symmetry. In Section 3, we review a method to generate ambiguous cylinders. In Section 4, we apply this method to design solids that produce the anomalous symmetry. Section 5 gives examples of anomalously symmetric objects and Section 6 gives concluding remarks together with future work.

2 Mirror Symmetry and Anomalous Symmetry

As shown in Fig. 1, we consider an (x, y, z) Cartesian coordinate systems, in which the xy plane is horizontal and hence the z axis is vertical.
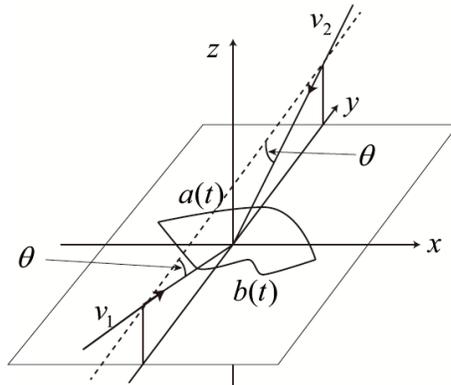

Figure 1: Pair of horizontal curves and a pair of view directions.

Let O be an object and O' be its mirror image with respect to a mirror placed in the xz plane. Then, any point (x, y, z) on O is mapped to the point $(x, -y, z)$. Thus, the y coordinate is reversed while the x and z coordinates

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are kept unchanged. Hence, we call O' the *y-reversal* mirror image. We extend the meaning of the term "y-reversal" for any translated version of O'. In other words, we refer to "y-reversal" to represent the posture of O' with respect to O, and do not care about the position of O'. Similarly we name the x-reversal mirror image for the image generated by a mirror in the yz plane, and its translated version.

We are interested in a class of objects whose y-reversal mirror images appear to be x-reversal.

An example is shown in Fig. 2. There is a cylinder whose section appears to be an arrow, and a vertical mirror stands behind it. However, the mirror image of the cylinder is anomalous in the sense that the left part and the right part appear to be reversed. Let us consider an (x, y, z) coordinate system such that the mirror is contained in the xz plane. Then, the mirror image of the cylinder should be y-reversal, but the actual mirror image in Fig. 2 appears to be x-reversal.
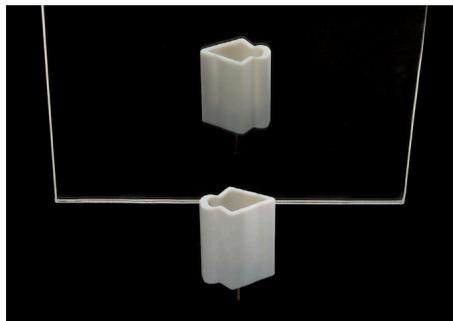

Figure 2: Ambiguous cylinder "Left-Right Reversal Mirror".

If we rotate this cylinder around a vertical axis, the apparent shape of the section changes as shown in Fig. 3. Note that in all postures of the cylinder, the corresponding mirror images are x-reversal instead of y-reversal.

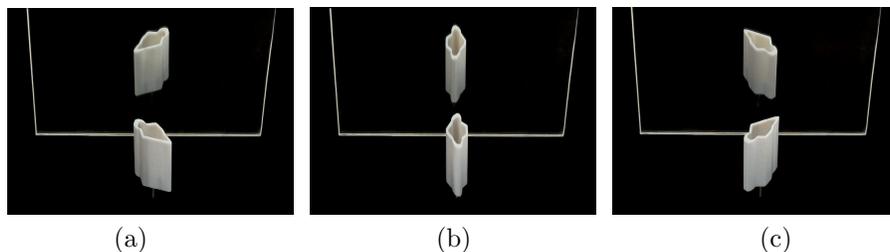(a)   (b)   (c)

Figure 3: Different views of the same cylinder as in Fig. 2 rotated around a vertical axis.

Note that the concept of anomalous symmetry is mathematically equivalent to line symmetry with respect to a vertical line. However, we use the term "anomalous mirror" symmetry in order to emphasize that it is physically impossible by a planar mirror.

It might be interesting to note that one of typical errors made by human subjects in predicting motion and orientation in the mirror image is similar to what we call anomalous symmetry. For example, Savardi et al. [9] and Bianchi et al. [1] made experiments in which they showed motion of an object to subjects and asked them to predict the motion and orientation in its mirror image. They reported that a typical error in estimation is line symmetric with respect to a vertical line instead of mirror symmetric.

The goal of this paper is to characterize the class of cylinders with this anomalous symmetry. For this purpose we review geometric aspects of the class of optical illusion called ambiguous cylinders in the next section.

3 Ambiguous Cylinders

Let $a(t)$ and $b(t)$ be two real-valued continuous functions defined for $-1 \leq t \leq 1$, and let us define three-dimensional vectors,

$$\boldsymbol{a}(t) = (t, a(t), 0), \quad \boldsymbol{b}(t) = (t, b(t), 0),$$

where t is a parameter. The vectors $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ represent planar curves on the xy plane, and they are monotone in the x direction.

As shown in Fig. 1, suppose that we see those curves along two view directions $\boldsymbol{v}_1 = (0, \cos\theta, -\sin\theta)$ and $\boldsymbol{v}_2 = (0, -\cos\theta, -\sin\theta)$. They are parallel to the yz plane, and are slanted downward by the same angle θ, $0 < \theta < \pi/2$, with respect to the horizontal plane in mutually opposite directions in the y axis. This means that the two viewpoints are both at infinity. This is a good approximation of actual perspective projection if the viewpoint is sufficiently far from the objects.

We are interested in finding a space curve that coincides with $\boldsymbol{a}(t)$ when it is seen along $\boldsymbol{v}_1$ and that coincides with $\boldsymbol{b}(t)$ when it is seen along $\boldsymbol{v}_2$. This space curve can be determined uniquely in the following way. For any $t \in [-1, 1]$, $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ are in the plane $x = t$, which is parallel to the yz plane. Hence the line passing through $\boldsymbol{a}(t)$ and parallel to $\boldsymbol{v}_1$ and the line passing through $\boldsymbol{b}(t)$ and parallel to $\boldsymbol{v}_2$ are both in this plane, and consequently they have a point of intersection. Let this point of intersection be denoted by $\boldsymbol{c}(t) = (t, c_1(t), c_2(t))$. As t moves in $-1 \leq t \leq 1$, $\boldsymbol{c}(t)$ traces a space curve. This is the curve that we wanted to get.

The curve $\boldsymbol{c}(t)$ is just a space curve, and it is not necessarily be perceived as the planar curves $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ when it is seen in the corresponding view directions.

However, the human brains prefer right angles in interpreting retinal images as three-dimensional objects; this property is typically observed in Ames's room illusion [3], and is usually understood as a priori constraint for right angles [6–8].

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We can use this nature of the brains to guide viewers to perceive the planar curves $\boldsymbol{a}(t)$ and $\boldsymbol{b}(c)$ when they see the space curve $\boldsymbol{c}(t)$ in the following way.

Let l be a vertical line segment, that is, l is parallel to the z axis. Suppose that we move l in a space in such a way that its orientation is fixed and its upper terminal point traces along the curve $\boldsymbol{c}(t)$. Let the surface swept by l be S.

As shown in Fig. 4(a), the resulting surface S has the same vertical length as the length of l. In this case we are apt to interpret the upper edge $\boldsymbol{c}(t)$ as the intersection of S and a plane perpendicular to l as shown in Fig. 4(b) because of the preference of right angles; we usually do not consider it as the intersection of S and a curved surface as shown in Fig. 4(c) although it is mathematically possible.
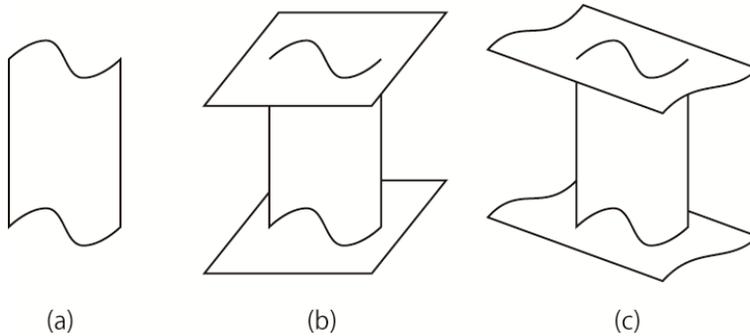

(a)   (b)   (c)

Figure 4: Swept surface and its interpretations: (a) surface swept by a vertical line segment; (b) plane-cut interpretation of the upper edge; (c) another interpretation which is mathematically possible but empirically unnatural.

Therefore, we can expect that when the surface S is seen along the view directions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, the upper edge $\boldsymbol{c}(t)$ appears to be the planar curves $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$, respectively. Moreover, if we place a vertical mirror parallel to the xz plane, the two views of S are obtained simultaneously, the direct view corresponds to $\boldsymbol{v}_1$ and the mirror image corresponds to $\boldsymbol{v}_2$. For the solid in Fig. 2, we applied the above method twice to the upper curve and the lower curve, resulting in the closed cylinder. This is the mathematical trick to generate ambiguous cylinder illusion.

4 Creation of Anomalous Mirror Symmetry

Our goal is to construct a method for designing the ambiguous cylinders that create anomalous symmetry. For this purpose we replace the pair of x-monotone curves $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ with a pair of closed curves.

Let $a_1(t)$ and $a_2(t)$ be two continuous functions defined in $[-1, 1]$ such that $a_1(-1) = a_2(-1)$ and $a_1(1) = a_2(1)$. Hence $\boldsymbol{a}_1(t) = (t, a_1(t), 0)$ and $\boldsymbol{a}_2(t) =$

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$(t, a_2(t), 0)$ altogether form a closed curve on the xy plane. Similarly, let $b_1(t)$ and $b_2(t)$ be two continuous functions defined in $[-1, 1]$ such that $b_1(-1) = b_2(-1)$ and $b_1(1) = b_2(1)$. Hence $\boldsymbol{b}_1(t) = (t, b_1(t), 0)$ and $\boldsymbol{b}_2(t) = (t, b_2(t), 0)$ constitute another closed curve on the xy plane. We want to construct a closed space curve that appears to be the closed curve composed of $\boldsymbol{a}_1(t)$ and $\boldsymbol{a}_2(t)$ when it is seen in the view direction $\boldsymbol{v}_1$, and that appears to be the closed curve composed of $\boldsymbol{b}_1(t)$ and $\boldsymbol{b}_2(t)$ when it is seen in the view direction $\boldsymbol{v}_2$, and it generates anomalous symmetry when we place a mirror parallel to the xz plane.

We apply the method for constructing an ambiguous cylinder twice, once to the pair $\boldsymbol{a}_1(t)$ and $\boldsymbol{b}_1(t)$, and once more to the pair $\boldsymbol{a}_2(t)$ and $\boldsymbol{b}_2(t)$. In order to create anomalous symmetry, the curves should satisfy

$$b_1(x) = -a_2(-x), \tag{1}$$
$$b_2(x) = -a_1(-x). \tag{2}$$

This is because, as shown in Fig. 5, the two appearances corresponding to the two view directions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ should be point symmetric with respect to the origin as the center of the symmetry; if they are point symmetric as shown in Fig. 5(a) and Fig. 5(b), the mirror parallel to the xz plane will generate a normal image that is y-reversal to Fig. 5(b), resulting in the curve shown in Fig. 5(c), which is x-reversal to the curve in Fig. 5(a), that is, which generates anomalous symmetry.
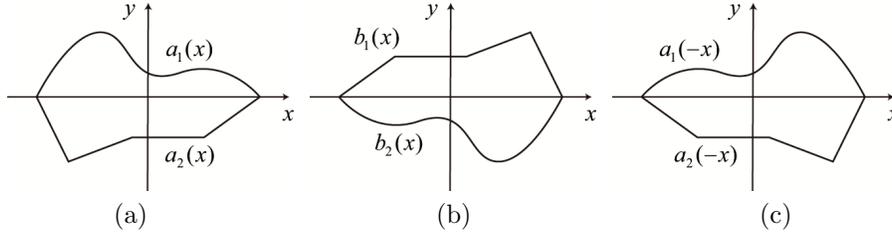

Figure 5: Pair of plane curves that creates anomalous symmetry: (a) arbitrary closed curve compound of upper and lower curves both of which are monotone in the x direction; (b) closed curve that is point symmetric to the curve (a); (c) mirror image of (b) with respect to the mirror parallel to the xz plane.

In Fig. 6 we show the intersection of the scene by a plane $x = t$ parallel to the yz plane. As shown in this figure, the space curve $\boldsymbol{c}_1(t)$ that appears $\boldsymbol{a}_1(t)$ when it is seen in the view direction $\boldsymbol{v}_1$, and that appears $\boldsymbol{b}_1(t)$ when it is seen in the view direction $\boldsymbol{v}_2$, can be represented as

$$\boldsymbol{c}_1(t) = \left(t, \frac{a_1(t) + b_1(t)}{2}, \alpha \frac{a_1(t) - b_1(t)}{2} \right), \tag{3}$$

where $\alpha = \tan\theta$. Similarly, we get

$$\boldsymbol{c}_2(t) = \left(t, \frac{a_2(t) + b_2(t)}{2}, \alpha \frac{a_2(t) - b_2(t)}{2} \right). \tag{4}$$
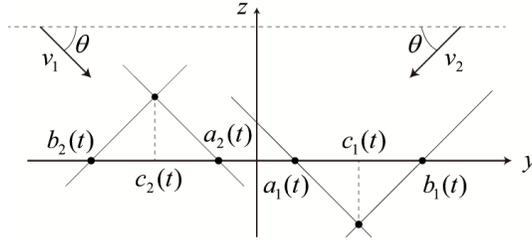
Figure 6: Construction of the space curves from the pair of planar curves on the xy plane.

Substituting eqs. (1) and (2), we get the explicit representation of the pair of space curves:

$$\boldsymbol{c}_1(t) = \left(t, \frac{a_1(t) - a_2(-t)}{2}, \alpha \frac{a_1(t) + a_2(-t)}{2} \right), \tag{5}$$

$$\boldsymbol{c}_2(t) = \left(t, \frac{a_2(t) - a_1(-t)}{2}, \alpha \frac{a_2(t) + a_1(-t)}{2} \right). \tag{6}$$

Note that we can choose a pair of functions $a_1(t)$ and $a_2(t)$ arbitrary. Once we choose them, the space curves $\boldsymbol{c}_1(t)$ and $\boldsymbol{c}_2(t)$ are determined by eqs. (5) and (6), which create anomalous symmetry.

Next, we consider the conditions such that the anomalous symmetry property is preserved even if the cylinder is rotated around the z axis. Fortunately it turns out that no additional conditions are necessary; the clossed curve given by eqs. (5) and (6) automaticallly satisfies the desired property. That is, the next proposition holds.

Proposition. The closed curve given by eqs. (5) and (6) does not destroy the anomalous symmetry property even if it is rotated around the z axis.

Proof. The rotation around the z axis does not destroy the anomalous symmetry property if the space curve is line symmetric with respect to the z axis. That is,

$$\boldsymbol{c}_1(t)|_y = -\boldsymbol{c}_2(-t)|_y, \tag{7}$$

$$\boldsymbol{c}_1(t)|_z = \boldsymbol{c}_2(-t)|_z, \tag{8}$$

where $\boldsymbol{w}|_y$ and $\boldsymbol{w}|_z$ represent the y and z coordinates of vector $\boldsymbol{w}$. Actually the conditions (7) and (8) are satisfied by $\boldsymbol{c}_1(t)$ and $\boldsymbol{c}_2(t)$ defined by (5) and (6). Thus the anomalous symmetry property is preserved when the curve is rotated around the z axis. (End of proof)

Recall that the object shown in Fig. 2 preserves the anomalous symmetry property when it is rotated around a vertical axis, as shown in Fig. 3. The above

proposition implies that this is not accidental. When we rotate such an object around a vertical axis, we can enjoy an unordinary change of the appearance of the object that always keeps anomalous symmetry. This visual effect might be used as a means for new artistic presentation.

It is interesting to note that a similar non-accidental nature is observed by Sawada et al. [10] in the context of reconstructing 3D symmetric curves from 2D projected curves.

5 Examples

We applied our method to various figures whose upper and lower boundaries are both monotone in the x direction. The following are examples.

Fig. 7 shows a shape of a fish. Let $a_1(x)$ and $a_2(x)$ be the upper boundary curve and the lower boundary curve, respectively, of this shape. We applied our method to this pair of curves, and constructed a solid object by first making a short cylinder whose edge curve consists of $c_1(t)$ and $c_2(t)$, second making surfaces which cap the top and the bottom of the cylinder, and finally filling material in it.

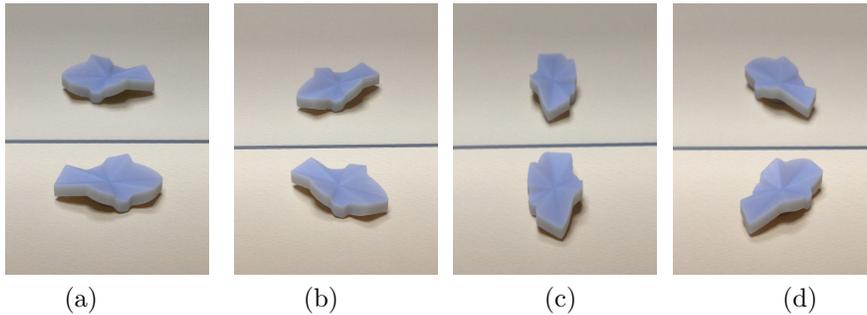Figure 7: Shape of a fish.

| (a) | (b) | (c) | (d) |

Figure 8: Fish-shape object creating anomalous mirror symmetry.

The resulting object is shown in Fig. 8. Fig. 8(a) is a picture of the resulting solid in the original posture and its mirror image. The panels (b), (c) and (d) of this figure show the same object rotated around a vertical axis clockwise step by step in this order.

Fig. 9 is a shape of a car, and Fig. 10 shows the object constructed from this shape. Fig. 11 is a shape of a ship, and Fig. 12 shows the object constructed from this shape. In both Figs. 10 and 12, (a) is the original posture, and (b), (c), (d) are the results of rotation.

Figure 9: Shape of a car.
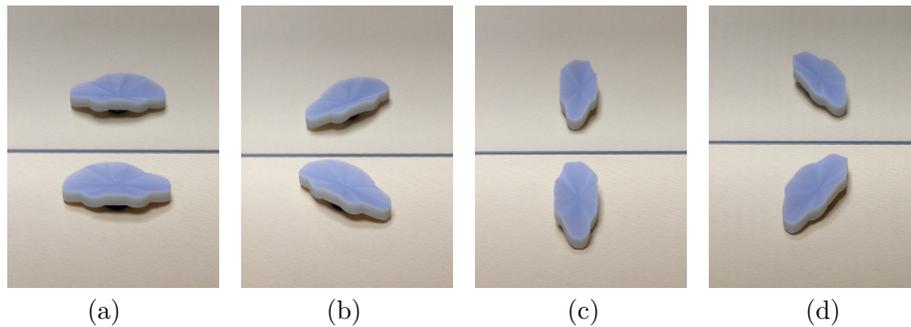

| (a) | (b) | (c) | (d) |

Figure 10: Car-shape object creating anomalous mirror symmetry.

Figure 11: Shape of a ship.

From these examples, we can observe that the mirror image of an object appears to be the x-reversal mirror image instead of the y-reversal, not only
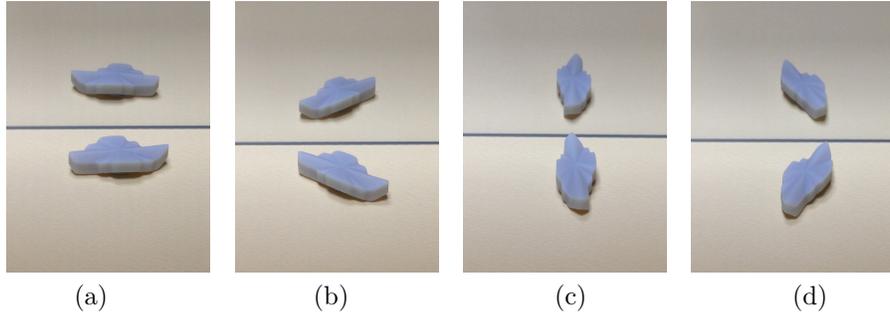
Figure 12: Ship-shape object creating anomalous mirror symmetry.

in its original posture but also in its rotated versions. Thus, we achieve the anomalous mirror symmetry.

6   Concluding Remarks

We introduced a new concept called "anomalous mirror symmetry", which is not consistent with the law of optical physics, but which can be perceived subjectively because of optical illusion. We characterized this symmetry and constructed a method for designing the class of space curves and the associated cylindrical surfaces that create anomalous symmetry.

The concept of anomalous symmetry we have introduced in this paper might be just one example of many possible new symmetries. The mirror symmetry with respect to a planar mirror is well studied from optical physics point of view. However, if we consider optical illusion, we may be able to extend the concept of mirror symmetry in many directions. The anomalous mirror symmetry introduced in this paper is such that the y-reversal image appears to be x-reversal. In general, for any pair of mirrors A and B, we might be able to define anomalous mirror symmetry in such a way that the mirror image with respect to the mirror A appears to be a mirror image with respect to the mirror B. To study those extensions is one of our future problems.

Acknowledgments

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