6次フェルマー型代数曲面からのファイバリングのモノドロミー準同型について

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VI-10 6次フェルマー型代数曲面からのファイバリングのモノドロミー準同型について

阿原一志*

Monodromy Homomorphism of a Fibering from Fermat Type Algebraic Surface of Degree 6

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Abstract

Synopsis : Let $F_6 \subset \mathbf{CP}^3$ be the Fermat type algebraic surface of degree 6. Let $f: F_6 \to \mathbf{CP}^1$ be a holomorphic map defined in the section 1. It is known that f is a fibration of Riemann surfaces with degeneration, but values of the monodromy homomorphism aren't calculated. In this paper, using the software *Monomie*, we calculate the monodromy for each singular fiber concretely. It is worthwhile to calculate the monodromy for a family of non-hyperelliptic Riemann surfaces of high genera.

Key Words : degeneration of Riemann surfaces, Fermat type surface

1 Introduction and preparations

1.1

Let $F_6 \subset \mathbf{CP}^3$ be the Fermat type algebraic surface of degree 6 defined by

$F_6 := \{[z_0 : z_1 : z_2 : z_3] \in \mathbf{CP}^3 | z_0^6 - z_1^6 - z_2^6 + z_3^6 = 0\}$.

This definition is due to the formulation in [M1]. The fibration $f: F_6 \to \mathbf{CP}^1$ is defined by the following.

$$f[z_0 : z_1 : z_2 : z_3] := \begin{cases} [z_0 - z_1 : z_2 - z_3], \\ \quad \text{if } z_0 \neq z_1 \text{ or } z_2 \neq z_3 \\ [z_2^5 : z_0^5], \\ \quad \text{otherwize} \end{cases}$$

This definition is in [M1].

In this paper, using the software *Monomie*, we calculate the monodromy for each singular fiber concretely. Matsumoto and Montesinos show in [MM] that the topological type of a neighborhood of a singular fiber is determined by the topological monodromy around the singular fiber. It follows that concrete calculations of the monodromy homomorphism give the topological type of F_6. These calculations are more worthwhile for researchers of degeneration of Riemann surfaces. Because there are few calculations of monodromies for fibrations of non-hyperelliptic Riemann surfaces of high genera. In [A] there are calculations for a fibration in the case of degree 5 in the similar way, but the calculations in this paper are much simpler in spite of the more complicated case.

In order to calculate the monodromy homomorphism, we use numerical computing about polynomials with 2 variables of high degrees. In this paper, we use the software *Monomie*. Using this software, for any equation $F(t, x) = 0$, we can get values of continuous function $x = x(t)$ locally such that $F(t, x(t)) = 0$

1.2

From [M1] and [A], F_6 and f have the following properties.

Proposition 1.1

(1) *F_6 is an algebraic manifold of complex dimension 2. f is a holomorphic map.*

(2) (*general fibers*) *We regard* $\mathbf{CP}^1$ *as* $\mathbf{C} \cup \{\infty\}$. *Let*

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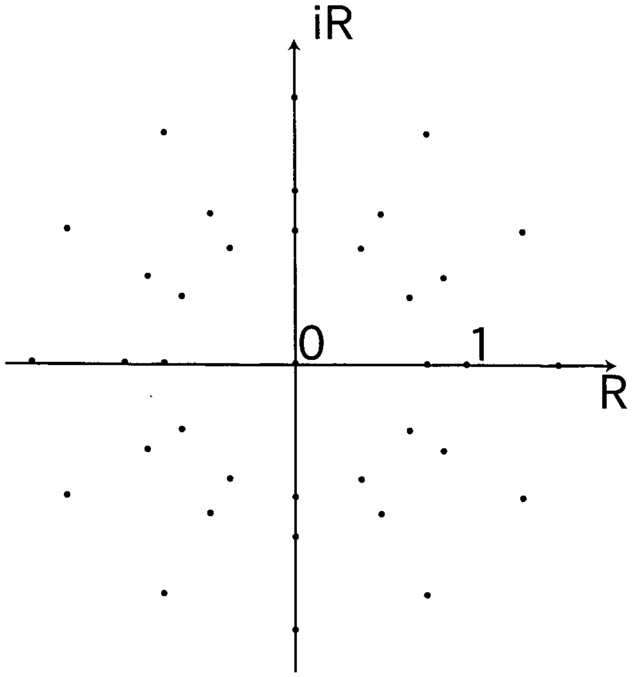


Figure 1.1

$\mathbf{CP}^1_{gen} := \mathbf{CP}^1 \backslash [\{0, \infty\} \cup \{\sigma | \sigma^{12} = 1\}$
$\cup \{\sigma | \sigma^{12} = \alpha^{\pm 1}\}]$.

(See Figure 1.1.) For any $\sigma \in \mathbf{CP}^1_{gen}$*,* $f^{-1}(\sigma)$ *is a closed Riemann surface of genus 6. Here* $\alpha = (\cos(\pi/5)/\cos(3\pi/5))^{10}$.

(3) *(singular fibers) If* $\sigma \in \{0, \infty\}$ *then* $f^{-1}(\sigma)$ *is homeomorphic to the bouquet of 5* S^2*'s. If* $\sigma^{12} = 1$ *then* $f^{-1}(\sigma)$ *is homeomorphic to a singular fiber with 4 disjoint vanishing cycles as in Figure 1.2(1). If* $\sigma^{12} = \alpha^{\pm 1}$ *then* $f^{-1}(\sigma)$ *is homeomorphic to a singular fiber with 2 disjoint vanishing cycles as in Figure 1.2(2).*

(4) *For any* $\sigma \in \mathbf{C}$*, let*

$$g_\sigma(t, x) := \frac{1 - t^6 - x^6 + (x + (t-1))^6}{t-1} \in \mathbf{C}[t, x].$$

Then we have

$f^{-1}(\sigma) \cap \{z_0 \neq 0\} = \{(t, x) | g_\sigma(t, x) = 0\}$.

Here $t = z_1/z_0$, $x = z_2/z_0$.

(5) *Let* $h_\sigma : f^{-1}(\sigma) \to \mathbf{CP}^1$ *be defined by*

$h_\sigma[z_0 : z_1 : z_2 : z_3] := [z_0 : z_1]$.

If $\sigma \in \mathbf{CP}^1{}_{gen}$ *then* h_σ *is a 5-fold branched covering over* $\mathbf{CP}^1$. *(5-fold means that the sheet number of the covering is 5.)*

(6) *If* $\sigma \in \mathbf{CP}^1{}_{gen}$ *then the set of all branch loci of* h_σ *is identified with* $\{t | L_\sigma(t) = 0\}$*. Here*

$\ell_j(t) = \sigma^6 (t-1)^5/(1-\omega^j) + (t^5 + t^4 + \cdots + 1)$

$(j = 1, 2, 3, 4,\ \omega = \exp(2\pi\sqrt{-1}/5))$

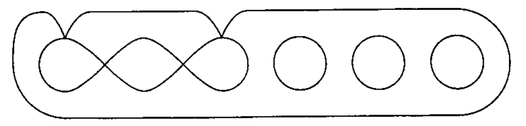


Figure 1.2(1)

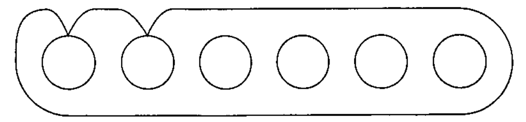


Figure 1.2(2)

$L_\sigma(t) := \prod_{j=1}^{4} \ell_j(t)$

(7) *If and only if* $\sigma \in \mathbf{CP}^1_{gen}$ *then the equation* $L_\sigma(t) = 0$ *has no multiple solution.*

Proof :

we can see (1) (2) (4) (5) (6) (7) from the description in the section 2 of [A]. (3) is shown in [M1].

1.3

We shall introduce the monodromy homomorphism ρ. The monodromy homomorphism is given by a map

$\rho : \pi_1(\mathbf{CP}^1_{gen}, \sigma_0) \to \mathcal{M}(f^{-1}(\sigma_0))$.

Here $\sigma_0 \in \mathbf{CP}^1_{gen}$ is a base point, $\mathcal{M}(f^{-1}(\sigma_0))$ is the mapping class group of $f^{-1}(\sigma_0)$ defined by

$\mathcal{M}(f^{-1}(\sigma_0)) := \mathrm{Diffeo}_+(f^{-1}(\sigma_0))/\mathrm{isotopy}$.

ρ satisfies that for any closed path γ from/to σ_0, the following two fiber bundles are equivalent.

$f : f^{-1}(Im(\gamma)) \to Im(\gamma) \approx S^1$

$[0, 1] \times f^{-1}(\sigma_0)/(0, x) \sim (1, \rho(\gamma)x)$

$\to S^1 = [0, 1]/0 \sim 1$

It is known that such map ρ is well-defined and a homomorphism.

2 Identification between Σ_6 and $f^{-1}(\sigma_0)$ in case $\sigma_0 = 5$

2.1

In the sequel, let $\sigma_0 = 5$ be the base point of $\mathbf{CP}^1_{gen}$. From Proposition 1.1 (2), $f^{-1}(5)$ is a Riemann surface of genus 6. In this section we construct a closed Riemann surface Σ_6 by gluing edges of polygons, and identify Σ_6 to $f^{-1}(5)$. And we see that there is a non-trivial action of $\mathbf{Z}/10\mathbf{Z}$ on Σ_6.

2.2

Let $D_i (i = 1, 2, 3, 4, 5)$ be five copies of the regular

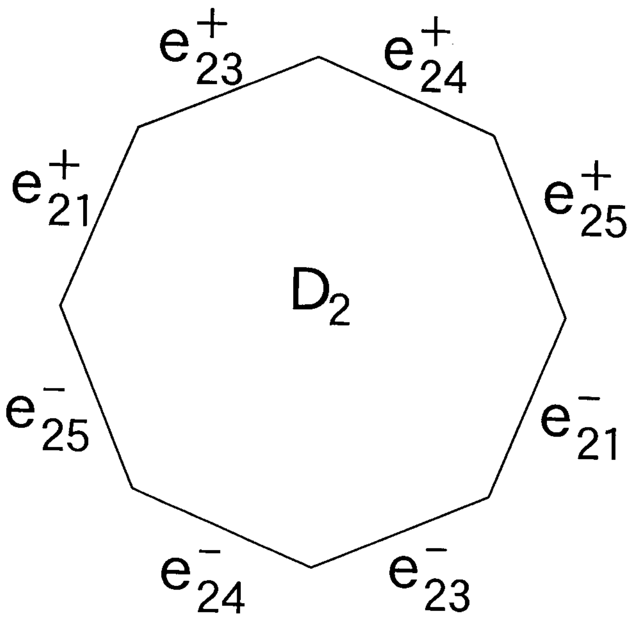


Figure 2.1

octagon. Let

$e^{+}_{i1}, \cdots, e^{+}_{i(i-1)}, e^{+}_{i(i+1)}, \cdots, e^{+}_{i5},$

$e^{-}_{i1}, \cdots, e^{-}_{i(i-1)}, e^{-}_{i(i+1)}, \cdots, e^{-}_{i5}$

be the edges of D_i in the clockwise order. In Figure 2.1, we see an example of D_2. On each edge, we fix an orientation determined from the orientation of the octagon. we construct $\Sigma_6 := \bigcup_{i=1}^{5} D_i$ by gluing the edges e^{+}_{ij} and $-e^{+}_{ji}$, e^{-}_{ij} and $-e^{-}_{ji}$. Clearly Σ_6 is a closed surface. We know that there are 5 faces, 20 edges, 5 vertices on Σ_6. It follows that the genus of Σ_6 is 6.

2.3

We define a non-trivial $\mathbf{Z}/10\mathbf{Z}$ action on Σ_6 in the following way.

Lemma 2.1

Let $\nu = (13524)$ be the permutation of degree 5.

(*1*) *There exist homeomorphisms $\tilde{\nu}_i : D_i \to D_{\nu(i)}$ satisfying the following condition.*

$$\tilde{\nu}_i(e^{\pm}_{ij}) = \begin{cases} e^{\mp}_{\tilde{\nu}(i)\nu(j)} & \text{if } (i, j) = (1, 2), (2, 1), \\ & (1, 3), (3, 1), \\ & (2, 3), (3, 2), \\ & (4, 5), (5, 4), \\ e^{\pm}_{\tilde{\nu}(i)\nu(j)} & \text{otherwise} \end{cases}$$

(*2*) *$\tilde{\nu} := \bigcup_{i=1}^{5} \tilde{\nu}_i : \Sigma_6 \to \Sigma_6$ is a well-defined homeomorphism.*

(*3*) *The order of $\tilde{\nu}$ is* 10.

Proof :

(1) It is easy to show that the sequences of edges are compatible to $\tilde{\nu}_i$. For example, the edges of D_2 are

$e^{+}_{21}, e^{+}_{23}, e^{+}_{24}, e^{+}_{25}, e^{-}_{21}, e^{-}_{23}, e^{-}_{24}, e^{-}_{25}$

in this turn. If we apply $\tilde{\nu}_i$ to these edges, we get

$e^{-}_{43}, e^{-}_{45}, e^{+}_{41}, e^{+}_{42}, e^{+}_{43}, e^{+}_{45}, e^{-}_{41}, e^{-}_{42},$

respectively. This sequence is that of edges of D_4.

(2) From the definition, $\tilde{\nu}_i(e^{+}_{ij}) = e^{\pm}_{\tilde{\nu}(i)\nu(j)}$, and $\tilde{\nu}_j(e^{+}_{ji}) = e^{\pm}_{\tilde{\nu}(j)\nu(i)}$. The definition of $\tilde{\nu}$ has a symmetry with respect to i and j. So the above two signatures coincide. It follows that $\tilde{\nu}$ and gluing are compatible.

(3) For each edge e, the sequence $\{\tilde{\nu}^n(e)\}$ $(n=0, 1, 2, \cdots)$ has a period of length 10. For example, $\{\tilde{\nu}^n(e^{+}_{12})\}$ is as follows.

$e^{+}_{12}, e^{-}_{34}, e^{-}_{51}, e^{-}_{23}, e^{+}_{45}, e^{-}_{12}, e^{+}_{34}, e^{+}_{51}, e^{+}_{23}, e^{-}_{45}, e^{+}_{12}, \cdots$

It follows that the order of this map is 10.

2.4

From Proposition 1.1(6), in the case $\sigma_0 = 5$, $h_5 : f^{-1}(5) \to \mathbf{CP}^1$ is a 5-fold branched covering and the set of branch loci is $\{t | L_5(t) = 0\}$. Let $t_0 = 1$ be a base point and we construct a graph from the monodromy permutations of h_5 in the following way.

Let $\{p_1, p_2, p_3, p_4, p_5\}$ be $h_5^{-1}(1)$. In fact $p_j = (1, \exp(2j\pi\sqrt{-1}/5)/\sqrt[5]{5})$. Let t be one of the branch loci. For a path $\gamma : [0, 1] \to \mathbf{CP}^1$ such that $\gamma(0) = 1$, and $\gamma(1) = t$, there exist 5 continuous functions $p_i : [0, 1] \to f^{-1}(5)$ $(i = 1, 2, \cdots, 5)$ such that

$h_5^{-1}(\gamma(s)) = \{p_i(s) | i = 1, 2, \cdots, 5\}.$

Since $\gamma(1)$ is a branch locus, so there are two integers i, and j such that $p_i(1) = p_j(1)$. (Remark that the order of any monodromy permutation is 2. See [A, Lemma 2.5(3)].) In this case we draw a edge (of graph) as in Figure 2.2(1). Using *Monomie*, we draw edges for all branch loci, and we get Figure 2.2(2). Remark that this graph is embedded in $\mathbf{CP}^1$. Remark again that *Monomie* calculates values approximately, but these are mathematically correct results with estimations of accidental errors.

2.5

Let G be the graph as is Figure 2.2(2). Let $\tilde{G}$ be the lifting of G to $f^{-1}(5)$ by h_5. Let $N(\tilde{G})$ be a neighborhood of $\tilde{G}$ in $f^{-1}(5)$. A restriction of h_5

$h_5 : f^{-1}(5) \backslash \tilde{G} \to \mathbf{CP}^1 \backslash G$

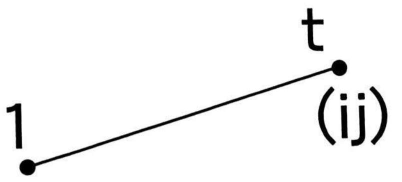


Figure 2.2(1)

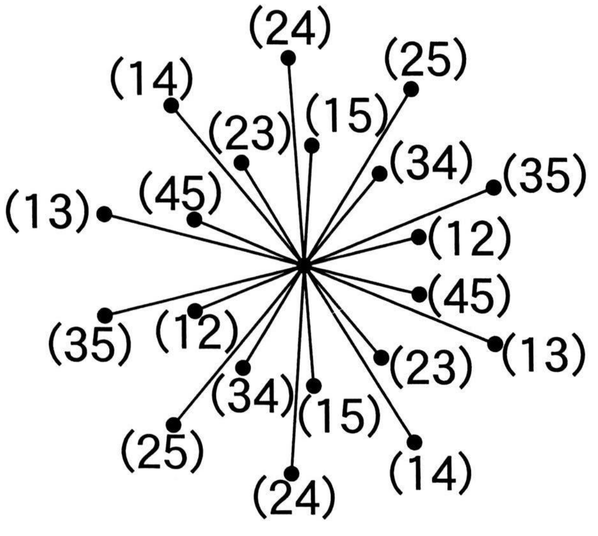


Figure 2.2(2)

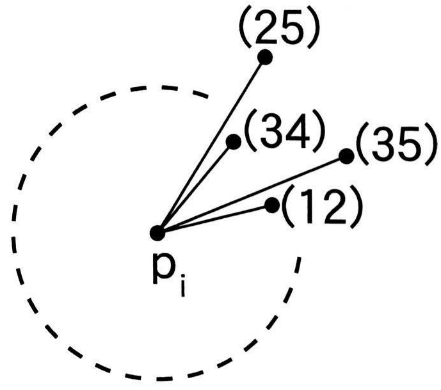


Figure 2.3(1)

is a trivial bundle, so $f^{-1}(5)\backslash N(\tilde{G})$ is a disjoint union of 5 disks. Because $h_5^{-1}(1)=\{p_1, p_2, p_3, p_4, p_5\}$, p_1, p_2, p_3, p_4, p_5 are vertices of $\tilde{G}$. In $\tilde{G}$ the legnth 1 neighborhood of p_i is as Figure 2.3(1). If a permutation (jk) on a vertex satisfies that $j \neq i$, $k \neq i$ then near the vertex $N(\tilde{G})$ is as in Figure 2.3(2). If $k=i$ then it is as in Figure 2.3(3). Hence $N(\tilde{G})$ is a surface obtained by gluing edges of the 5 figures in Figure 2.4. We observe Figure 2.4 carefully, and we have that $N(\tilde{G})$ obtained by Figure 2.4 is identified naturally to $\Sigma_6\backslash$(neighbors of vertices}). Remark that Σ_6 has 5 vertices and that $f^{-1}(5)\backslash N(\tilde{G})$ is a disjoint union of 5 disks. It follows that we can identify $f^{-1}(5)$ to Σ_6.

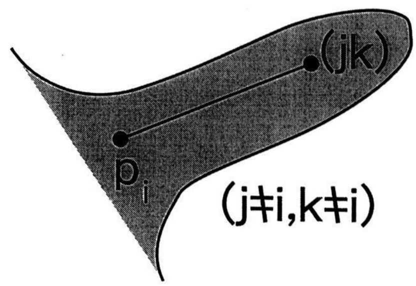


Figure 2.3(2)

3 Braid of branch loci and Dehn twist

3.1

From Proposition 1.1(6), for any $\sigma \in \mathbf{CP}^1_{gen}$, $h_\sigma : f^{-1}(\sigma) \to \mathbf{CP}^1$ is a 5-fold branched covering with 20 branch loci. It follows that for any closed path γ in $\mathbf{CP}^1_{gen}$, we can regard the homotopy of the set of branch loci as a braid on $\mathbf{CP}^1$.

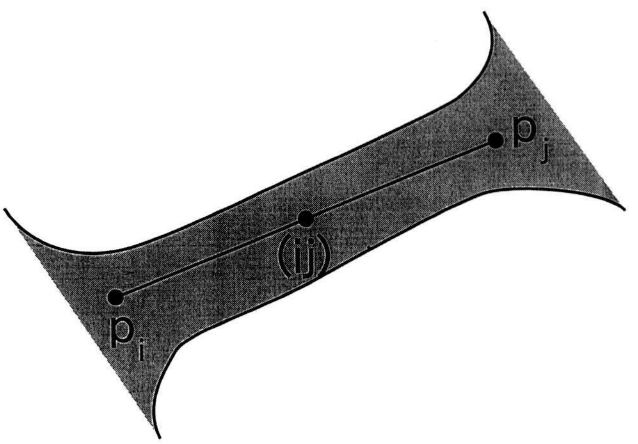


Figure 2.3(3)

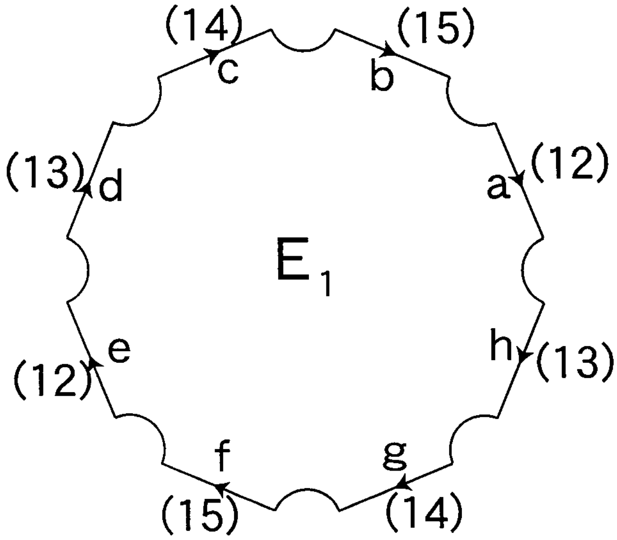


Figure 2.4(1)

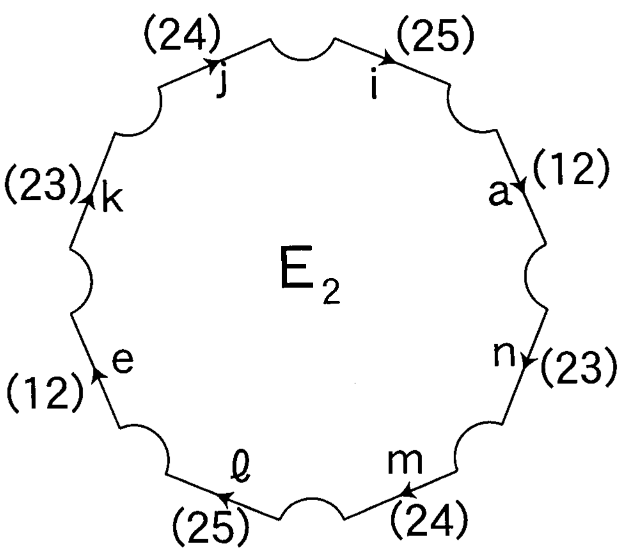
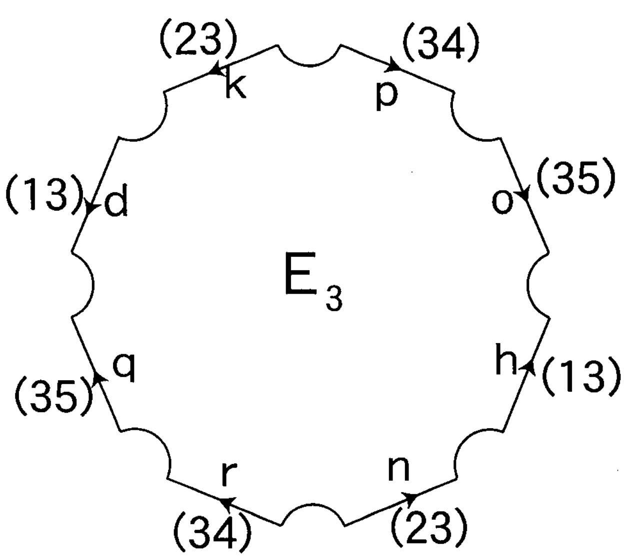


Figure 2.4(2)



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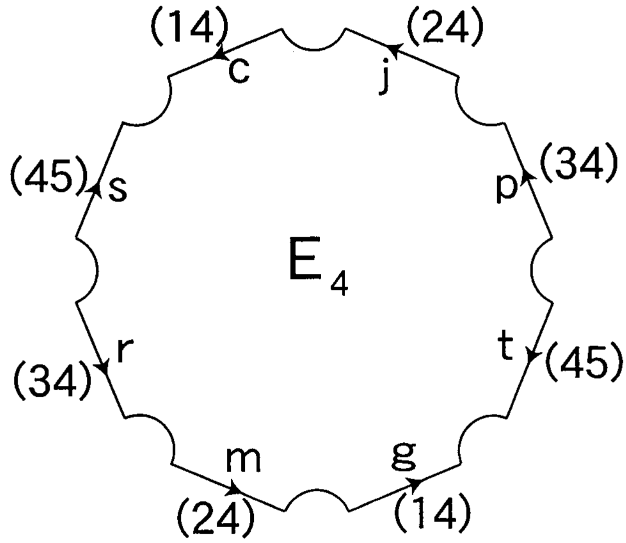

Figure 2.4(4)

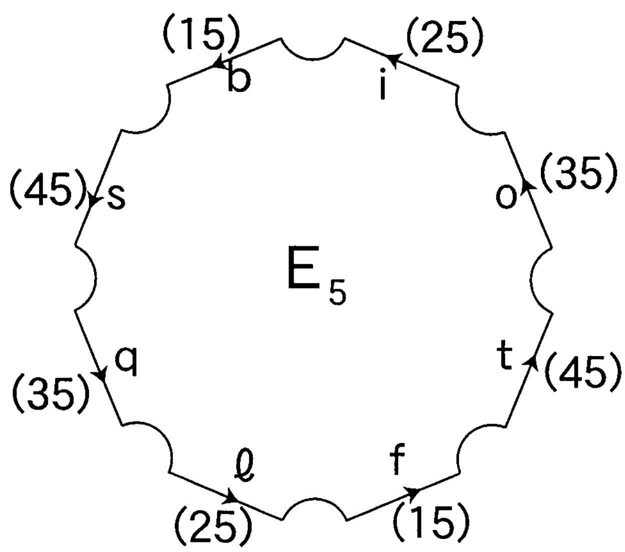

Figure 2.4(5)

Definition 3.1

Let σ_0 be a base point. For any closed curve $\gamma : [0, 1] \to \mathbf{CP}^1_{gen}$, there exist 20 continuous paths $b_i : [0, 1] \to \mathbf{CP}^1$ $(i=1, 2, \cdots, 20)$ such that $\{b_i(s) \mid i=1, 2, \cdots, 20\}$ is the set of branch loci of $h_{\gamma(s)}$. Here we see that $\{b_i(0) \mid i=1, 2, \cdots, 20\}$ and $\{b_i(1) \mid i=1, 2, \cdots, 20\}$ are identified as sets and that $\{b_i \mid i=1, 2, \cdots, 20\}$ determines a braid $B(\gamma)$ on $\mathbf{CP}^1$.

In[A ; section 5], we give a method to calculate the monodromy $\rho(\gamma)$ from the braid $B(\gamma)$. But this is very complicated. In this section we show that when the braid is an *exchange* then the monodromy is a Dehn twist, and we show the results on the monodromy homomorphism.

3.2

Definition 3.2

Suppose that a braid B on $\mathbf{CP}^1$ is represented by continuous paths $\{b_i : [0, 1] \to \mathbf{CP}^1\}$. B is an (anti-clockwise) exchange if and only if there exists a 2-disk D embedded in $\mathbf{CP}^1$ and there exist indices j, k $(j \neq k)$ such that

(1) If and only if $i=j$ or $i=k$ then $Im(b_i) \subset D$

(2) Outside of D, the restriction of the braid B is trivial. And inside of D, b_j and b_k move anti-clockwisely and exchange to each other. See Figure 3.1.

3.3

In this subsection, we discuss a relation between the braid of branch loci and Dehn twists generally. Let Λ be a parameter space. Let $\pi(\sigma) : M(\sigma) \to \mathbf{CP}^1$ be a family of branched covering with a parameter $\sigma \in \Lambda$. Let $\sigma_0 \in \Lambda$ be a base point and let γ be a closed path from/to σ_0 in Λ. Suppose that $\{b_i : [0, 1] \to \mathbf{CP}^1\}$ represents the braid $B(\gamma)$ and suppose that $B(\gamma)$ is an anti-clockwise exchange. Let a 2-disk D and let 2 indices j, k be as in Definition 3.2. Then we have the following.

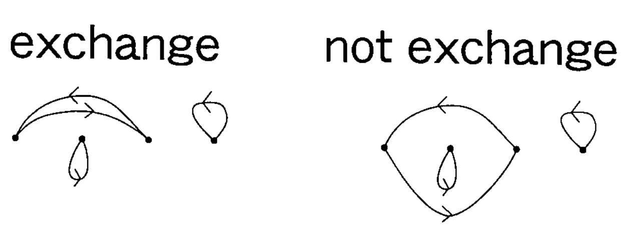

Figure 3.1

Proposition 3.3

If the orders of the monodromy permutations of the branch loci $b_j(0)$ and $b_k(0)$ are both 2, then the inverse image of D by $\pi(\sigma_0)$ is a disjoint union of some disks and some annuli, and $\rho(\gamma)$ is the product of positive Dehn twists of boundaries of annuli.

Proof :

It is sufficient to show this proposition in the case $\pi(\sigma)$ is a double branched covering with two branch loci for any σ. Let p, q be the branch loci of $\pi(\sigma_0)$. Let $M' := \pi(\sigma_0)^{-1}(\mathrm{D})$. See 3.2(1). Here o_1, o_2 are the inverse image of the point o, and r_1, r_2 are the inverse image of the point r. It is clear that M' is homeomorphic to an annulus.

Let β be a path on D as in Figure 3.2(2). Then we have two liftings $\tilde{\beta}_1$, $\tilde{\beta}_2$ as in Figure 3.2(3). We want to have the image of the monodromy $\rho(\gamma)$, that is $\rho(\gamma)(\tilde{\beta}_1)$ and $\rho(\gamma)(\tilde{\beta}_2)$. If we map these two by $\pi(\sigma_0)$ then they must be the same and they are as in Figure 3.2(4). (Because $B(\gamma)$ is an exchange.) Hence we have $\rho(\gamma)(\tilde{\beta}_1)$ and $\rho(\gamma)(\tilde{\beta}_2)$ as in Figure 3.2(5). It follows that $\rho(\gamma)$ is a positive Dehn twist of a boundary of M'.

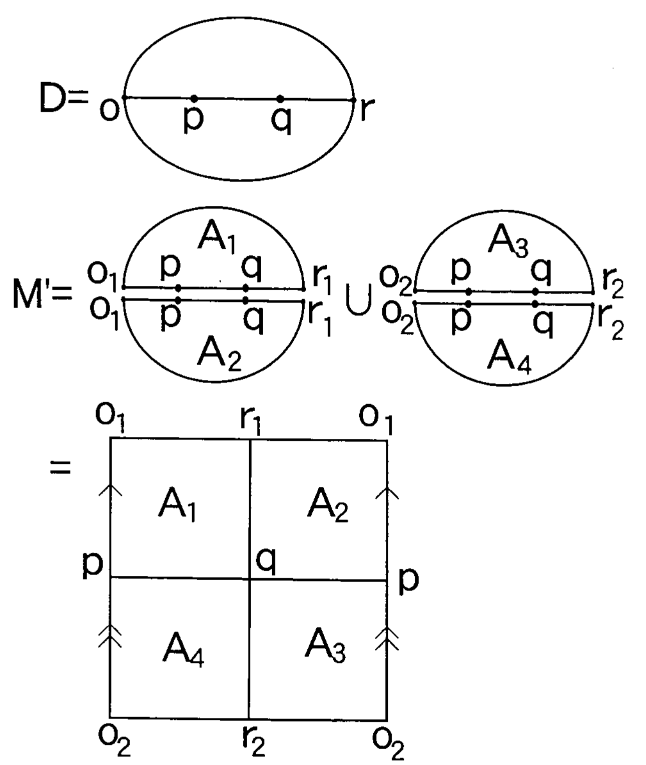

Figure 3.2(1)

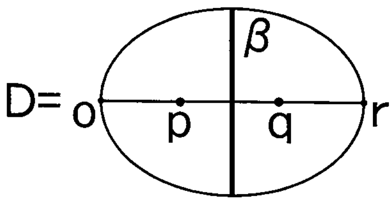

Figure 3.2(2)

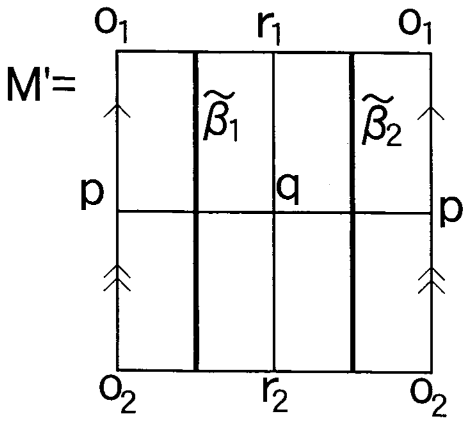

Figure 3.2(3)

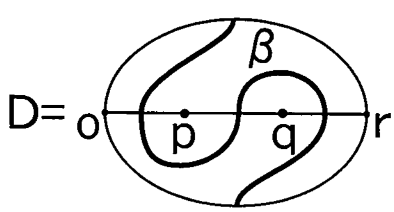

Figure 3.2(4)

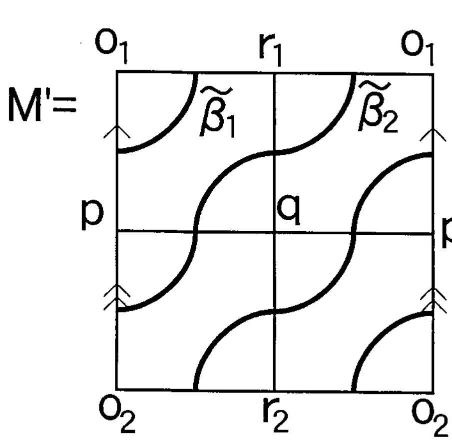

Figure 3.2(5)

3.4

Let γ_1, γ_2, γ_3 be three closed paths on $\mathbf{CP}^1_{gen}$ as in Figure 3.3. Using the software *Monomie*, we have that $B(\gamma_i)$ $(i=1, 2, 3)$ are as in Figure 3.4(1) (2) (3) respectively. From Proposition 3.3, we have the followings.

Theorem 3.4

For paths as in Figure 3.3, we have :

(1) $\rho(\gamma_1)$ is the product of positive Dehn twists of c_1 and c_2 in Figure 3.5(1).

(2) $\rho(\gamma_2)$ is the product of positive Dehn twists of c_3, c_4, c_5, and c_6 in Figure 3.5(2).

(3) $\rho(\gamma_3)$ is the product of positive Dehn twists of c_7

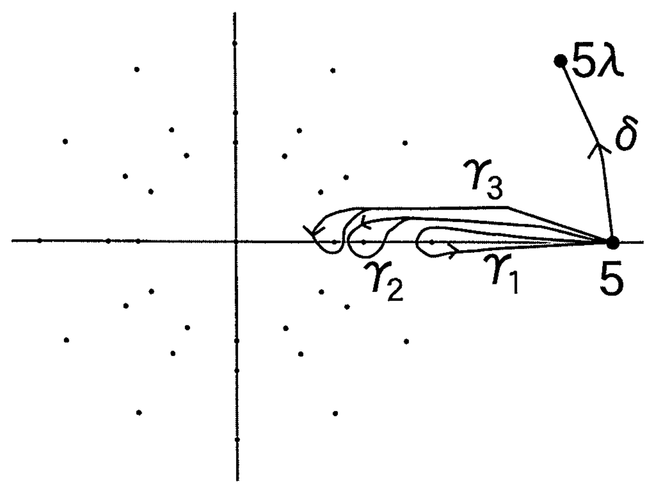

Figure 3.3

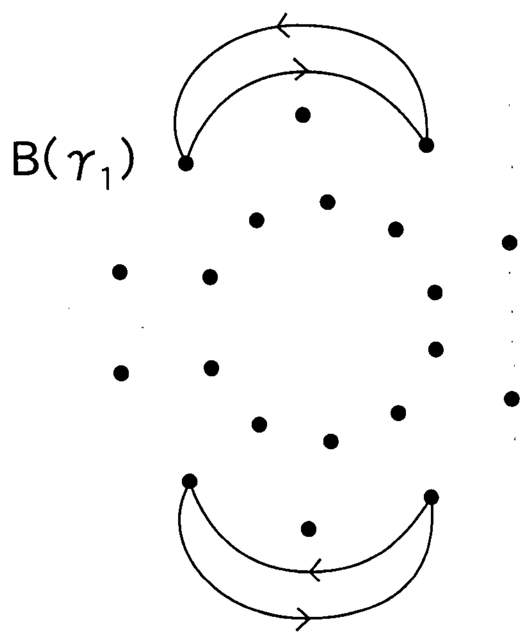

Figure 3.4(1)

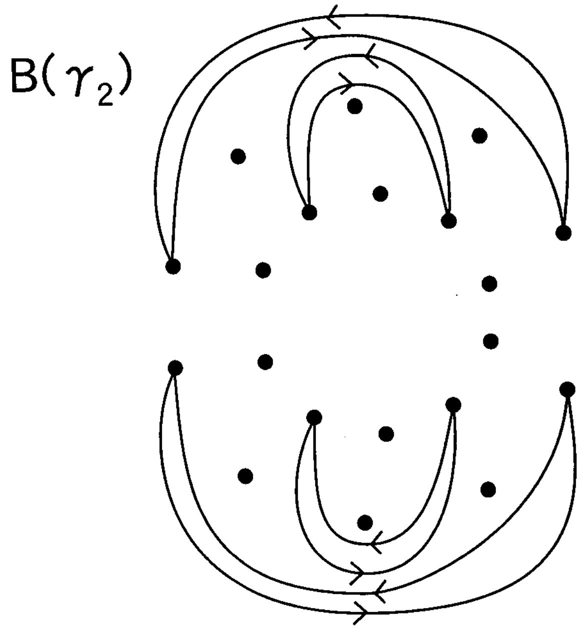

Figure 3.4(2)

and c_8 in Figure 3.5(3).

3.5

From Proposition 1.1(6), the set of branch loci of h_σ is $\{t | L_\sigma(t)=0\}$. Using the following Proposition, we can calculate the value $\rho(\gamma) \in \mathcal{M}$ of the monodromy homomorphism for any $\gamma \in \pi_1(\mathbf{CP}^1_{gen}, \sigma_0)$. In the sequel, for a path $\gamma: [0, 1] \to \mathbf{C}$ and a complex number μ, the path $\mu\gamma$ is defined by $(\mu\gamma)(s) := \mu \cdot \gamma(s)$.

Proposition 3.5

(1) $L_\sigma(t) = L_{\lambda\sigma}(t)$, *where* $\lambda = \exp(\pi\sqrt{-1}/6)$.

(2) Let a path δ *be as in Figure 3.3. Then the braid* $B(\delta)$ *is as Figure 3.6. From (1), we can identify* $f^{-1}(5)$ *and* $f^{-1}(5\lambda)$ *naturally, because they have the same branch loci set. Via this identification, we have* $\rho(\delta) = \tilde{\nu}^{-1}$. *Here* $\tilde{\nu}$ *is as in Lemma 2.1.*

(3) For any integer k,

$\rho(\delta^k \cdot \lambda^k \gamma_i \cdot (\delta^k)^{-1}) = \tilde{\nu}^{-k} \rho(\gamma_i) \tilde{\nu}^k$.

Here $\delta^k := \delta \cdot \lambda\delta \cdots\cdots \lambda^{k-1}\delta$.

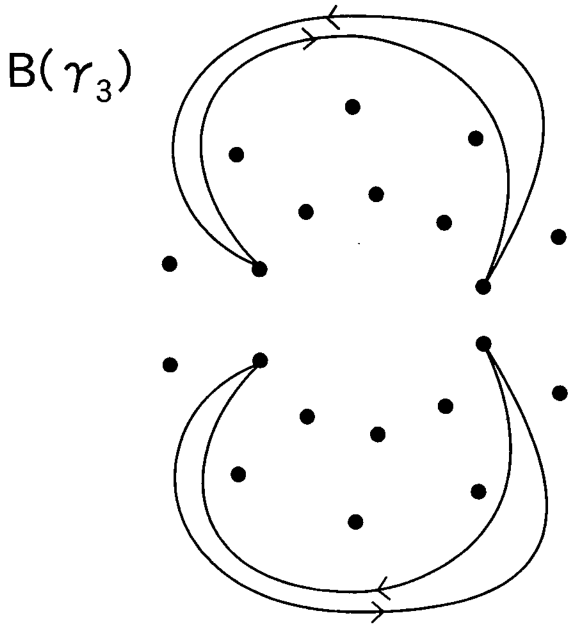

Figure 3.4(3)

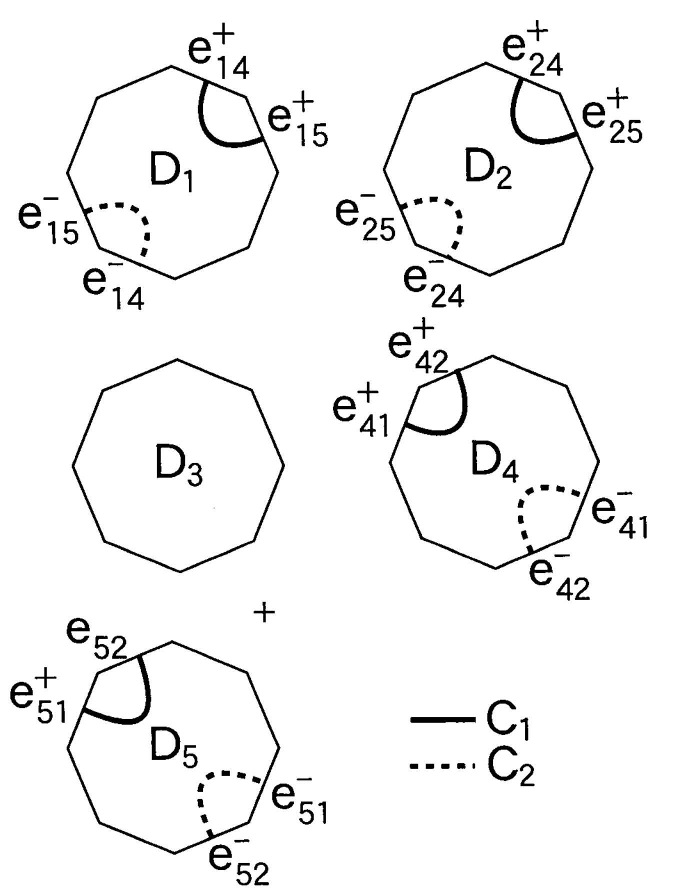

Figure 3.5(1)

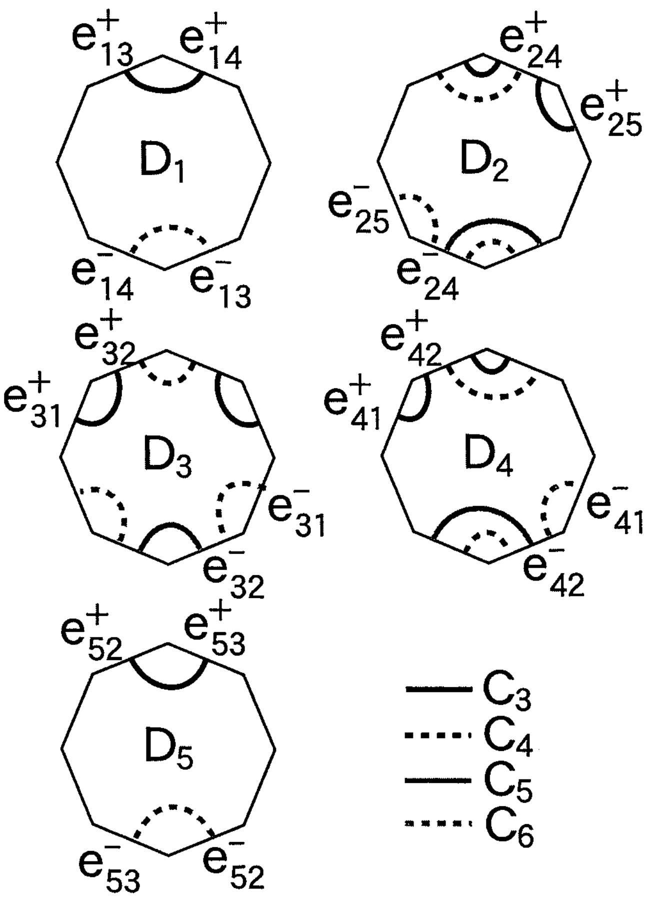

Figure 3.5(2)

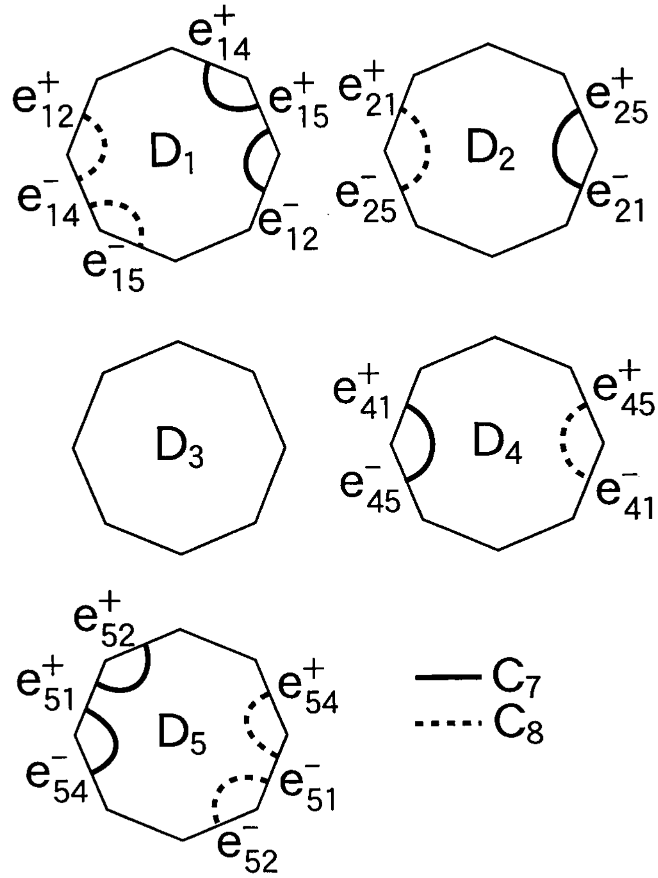

Figure 3.5(3)

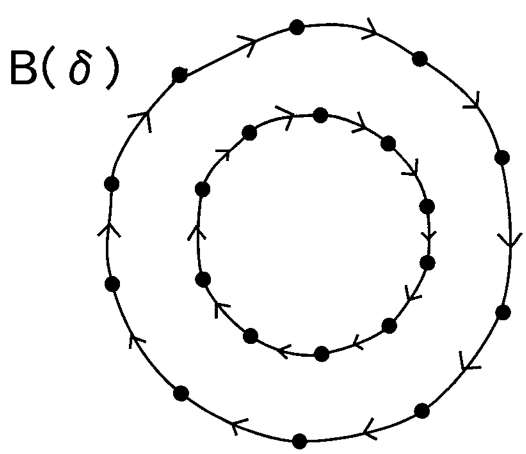

Figure 3.6

Proof :

(1) For $\omega=\exp(2\pi\sqrt{-1}/5)$, $(1-\omega^{j})^{5}=-(1-\omega^{-j})^{5}$. It follows that

$$\frac{\sigma^{6}(t-1)^{5}}{(1-\omega^{j})}=\frac{(\lambda\sigma)^{6}(t-1)^{5}}{(1-\omega^{5-j})}$$

and we have the conclusion.

(2) To obtain the braid in Figure 3.6, we only use *Monomie* again. See Figure 2.2(2). This braid in Figure 3.6 acts on Figure 2.2(2) by a permutation of indices. The permutataion is ν^{-1}, where ν is defined in Lemma 2.1. It follows that the action by $B(\delta)$ is identified with $\tilde{\nu}^{-1}$.

(3) From (1), we have $B(\gamma)=B(\lambda\gamma)$ for any path γ. Hence we have $\rho(\delta^{k})=\tilde{\nu}^{-k}$, $\rho(\lambda^{k}\gamma_{i})=\rho(\gamma_{i})$, and $\rho((\delta^{k})^{-1})=\tilde{\nu}^{k}$. This completes the proof.

4 Acknowlegement

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5 References

[A] Ahara, K. On the Topology of Fermat Type Surface of Degree 5 and the Numerical Analysis of Algebraic Curves, Tokyo Journal of Math., **16** (1993), 321-340.

[M1] Matsumoto Y., Private notebooks on Fermat type surface.

[M2] Matsumoto Y., Calculation of a splitting of a singular fiber. (in Japanese), preprint. (2000)

[MM] Matsumoto Y., Montesinos J. M., Pseudoperiodic homeomorphisms and degeneration of Riemann surfaces, Bull Amer. Math. Soc. **30** (1994), 70-75.