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O-106 Asymptotic behavior of eigenfunctions corresponding to positive eigenvalues of the Schrödinger-type operator on a spherically symmetric Riemannian manifold

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Abstract

The aim of this paper is to obtain the growth order of non-trivial solutions of the equation $-\triangle f+q(\mathrm{x})f=\lambda f\ (\lambda>0)$ on a spherically symmetric Riemannian manifold. We find that the square integral of the solution in a shell-like region is bounded from below by a function which is determined by the metric of the manifold. Four theorems and two corollaries are set forth corresponding to various situations. But they are proved in a unified manner which is reffered to as the abstract theory.

§0 Introduction

The problem "When does the Schrödinger operator $-\triangle+q(x)$ possess no positive eigenvalues?" or more precise problem such as "How is the growth order of the solution?" has been studied by many authors. The same is said to the more general operators, e. g. $-\Sigma\,\partial_i a_{ij}(x)\partial_j+q(x)$ or $-\Sigma(\partial_i+\sqrt{-1}b_i(x))a_{ij}(x)(\partial_j+\sqrt{-1}b_j(x))+q(x)$ (cf. [3], [4], [7] etc.).

However, it seems that there are very few literatures which refer explicitly to the conditions that give rise to the absence of positive eigenvalues of the Schrödinger operator considered on non-Euclidean manifolds. This article deals with problems of this type and gives several interesting results, though in the special case that the metric of the manifold M is "spherically symmetric".

In this paper, we consider the local problem in the neighbourhood of infinity. That is, the solution of $-\triangle f+q(x)f=\lambda f$ is simply assumed not to vanish identically in any neighbourhood of infinity, with no boundary conditions imposed. And then, we try to get the lower bound to the integral of $|f(x)|^2$ in the "ball of the radius R", which the Riemannin structure and the behavior of $p(x)$ affect only in the neighbourhood of infinity.

In view of the results obtained, it is easy to derive the absence of positive eigenvalues of any selfadjoint realization of $-\triangle+q(x)$ on a manifold a part of which coincides with ours, provided a global condition is added which excludes the existence of a solution with compact support. Therefore we do not comment upon it in each case.

Every result contains in particular the case that M is a part of a Euclidean space. In that case, the results coincide with well-known ones or relax their assumptions slightly.

Another aim of this paper is to simplify the proofs. All the theorems are proved by a unified technique, which in particular gives considerably simple proofs for the well-known theorems in the case of the Euclidean spaces. The common parts of the processes are described in an abstract manner so that the flow of reasoning may become clear.

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It should be noted that a part of our methods originates mostly from S. Agmom's works, especially from [1]. Furthermore, the abstract part of our theory is an immediate development of K. Masuda's work [6]. In [6] he considered the differential equation $u-\mu(r)u+A_0(r)u+A_1(r)u=0$ (cf. (2.1) of this paper). He gave two alternative estimates on the norm of u at infinity by calculating the derivative of the function $F(r)=\|u\|^2+(A_0(r)u, u)$, both implying well-known results on the absence of positive eigenvalues of the Schrödinger operator. He also dealt with the more general equation $(d/dr+B_0(r))^2u+B_1(r)du/dr+A_0(r)u+A_1(r)u=0$ and applied it to seccond order elliptic equations with variable coefficients.

Now, let M be an n-dimensional Riemannian manifold ($n\geqq 2$) of the structure

$$M=(r_0, \infty)\times S^{n-1}=\{(r, \omega) \mid r\in(r_0, \infty), \omega\in S^{n-1}\}$$

with the metric

$$ds^2=dr^2+\rho(r)^2 d\bar{s}^2$$

where $d\bar{s}$ is the ordinary line element of n-1 dimensional unit sphere S^{n-1} and $\rho(r)$ is a non-negative twice continuously differentiable function. Then, the Laplace-Beltrami operator on M is expressed as

$$\triangle=\frac{1}{\rho^{n-1}}\frac{\partial}{\partial r}\left(\rho^{n-1}\frac{\partial}{\partial r}\right)+\frac{1}{\rho^2}\Lambda$$

where Λ is that on S^{n-1}

We use the following notations throughout this paper: x denotes the point of M, $q(x)$ is a measurable real-valued function* defined on M, λ is a positive constant and $f(x)$ is the real-valued solution* of the Schrödinger equation

$$-\triangle f+qf=\lambda f \qquad \text{on } M \tag{0.1}$$

which belongs to $H^2_{\mathrm{loc}}(M)$ and does not vanish identically in any neighbourhood of infinity. Moreover, we denote by a topside or superior dot the (ordinary or partial) derivative with respect to r and by superior -1 the reciprocal number. Further, the expression "($r\to\infty$, unif.)" should be read as "uniformly on S^{n-1} as $r\to\infty$".

§1. Theorems.

(A) Main theorem.

In what follows, we assume $n\geqq 2$ unless otherwise is stated.

Assumption (ρ, 0). $\rho\in C^2([r_0, \infty])$ while $\rho(r)$ is monotone increasing and diverging.

Assumption (ρ, 1). $\rho^{-1}\dot{\rho}=o(1)$, $\dot{\rho}^{-1}\ddot{\rho}=o(1)$ $\quad(r\to\infty)$.

Assumption (ρ, 1). q is decomposed into the sum of two bounded and continuous functions q_1 and q_2 with the properties

(i) $q_1\in C^1(M)$ and there exists a positive function $e(r)$ such that

$$e(r)=o(1) \quad (r\to\infty),$$

$$q_1\leqq e(r),\ \dot{q}_1\leqq\rho^{-1}\dot{\rho}\ e(r) \qquad (\text{on } M).$$

(ii) $q_2=o(\rho^{-1}\dot{\rho})$ $\qquad(r\to\infty$, unif.).

Theorem 1. *Let Assumptions* (ρ, 0) (ρ, 1) (q, 1) *be satisfied. For an arbitrary positive constant* ε, *there exist a positive constant* C, *a constant* $\tilde{C}$ *and an* r_1 ($\geqq r_0$) *such that*

*With little modification, we can treat complex-valued ones.

$$\int_{r_0<r<R}|f(x)|^2dx\leqq C\int_{r_0}^{R}\rho(r)^{-\varepsilon}dr+\tilde{C} \qquad (R\geq r_1)$$

holds. ($dx=\rho(r)^{n-1}drd\omega$ *is the volume element of* M).

Remark. If M is a part of a Euclidean space ($\rho(r)=r$), this theorem almost implies the results of T. Kato [5; Theorem 1 with $\mu=0$], S. Agmon [2; Theorem 4] and F. Odeh [9].

We like to know how the conditions are written in the case of a surface of revolution imbedded in $\mathbf{R}^{n+1}$.

Corollary. *Let* M *be the surface obtained by rotating the graph of a* C^2*-function* $t=t(\rho)$ ($\rho\geqq\rho_0\geqq0$) *around the t-axis in* $\mathbf{R}^{n+1}$. *i. e.*

$$M=((t(\rho),\ \rho\omega)\in\mathbf{R}^{n+1}\,|\,\rho\geqq\rho_0,\ \omega\in\mathbf{S}^{n-1}).$$

If $t(\rho)$ *and bounded continuous real-valued functions* $q_1(\rho,\omega)$ *and* $q_2(\rho,\omega)$ *satisfy the following conditions*

(i) $$\frac{t'\,t''}{(1+t'^2)^{3/2}}=o(1) \qquad (\rho\to\infty),$$

(ii) $q_1\in C^1$ *and there exists a positive function* $e(\rho)$ *such that*

$$e(\rho)=o(1) \qquad (r\to\infty),$$

$$q_1\leqq e(\rho),\quad \partial q_1/\partial\rho\leqq\rho^{-1}e(\rho) \qquad (on\ M),$$

(iii) $$q_2=o\left(\frac{1}{\rho\sqrt{1+t'^2}}\right) \qquad (\rho\to\infty,\ \text{unif.}),$$

then, for $q=q_1+q_2$ *and for any* $\varepsilon>0$, *we can find positive constants* C *and* ρ_1 ($\geqq\rho_0$) *such that*

$$\int_{\rho_0<\rho<P}|f(\rho,\omega)|^2d\sigma\geqq C\int_{\rho_1}^{P}\rho^{-\varepsilon}\sqrt{1+t'(\rho)^2}\ d\rho \quad (\geqq CP^{1-\varepsilon}) \quad (P\geqq\rho_1)$$

holds, where $d\sigma$ *is the surface element.*

(B) Two dimensional case; elimination of $\ddot{\rho}$.

If in particular $n=2$, we can remove the smallness requirement on $\ddot{\rho}(r)$ in Assumption (ρ, 1). Namely:

Assumption (ρ, 2). $\int_{r_0}^{\infty}\rho(r)^{-1}dr=\infty.$

Assumption (q, 2). $q\in C^1(M)$ and q is bounded. Moreover, there is a positive function $e(r)$ such that

$$e(r)=o(1) \qquad (r\to\infty),$$

$$q\leqq e(r),\quad \dot{q}\leqq\rho^{-1}\dot{\rho}e(r) \qquad (\text{on } M).$$

Theorem 2. *If* $n=2$ *and if Assumption* (ρ, 0)(ρ, 2)(q, 2) *are satisfied, then we can find positive constants* C *and* r_1 ($\geqq r_0$) *such that*

$$\int_{r_0<r<R}|f(x)|^2dx\geqq C\int_{r_0}^{R}\rho(r)^{-1}dr \qquad (R\geqq r_1)$$

holds.

The following corollary may arouse our special interest, because we find there very weak restrictions on the shape of a surface except for the axial symmetry in order to guarantee the absence of positive eigenvaluse of Schrödinger-type operators on the surface.

Corollary. *Let* M *be the two dimensional surface obtained by rotating the graph of a* C^2*-function* $t=t(\rho)$ ($\rho\in[\rho_0,\infty)$), $\rho_0\geqq0$ *around the t-axis in* $\mathbf{R}^3$. *If a* C^1*-function* q *is bounded and if there exists a positive function* $e(\rho)$ *such that*

$$e(\rho)=o(1) \qquad (\rho\to\infty),$$
$$q\leqq e(\rho), \quad \partial q/\partial\rho\leqq\rho^{-1}\dot{\rho}e(\rho) \qquad (on\ M),$$

then there exist positive constants C *and* ρ_1 *such that*

$$\int_{\rho_0>\rho>\mathrm{P}}|f(\rho,\ \omega)|^2d\sigma\geqq C\int_{\rho_0}^{P}\rho^{-1}\sqrt{1+t'(\rho)^2}\ d\rho \qquad (\mathrm{P}\geqq\rho_1)$$

holds, (the second member being $\geqq C\log \mathrm{P}$). *Here* $d\sigma$ *means the surface element of* M.

(C) Linear lower bound.

The ε appearing in Theorem 1 can be removed under some stronger conditions. S. Agmon [2; Theorem 2] and T. Kato [5; Theorem 1a] showed estimates of the type

$$\int_{r_0}^{R}|f(x)|^2dx\geqq CR \qquad (R\geqq r_1)$$

in the case when $M=\{x\in R^n|\ \ |x|\geqq r_0\}$, $\rho(r)=r$. We have also similar results as follows.

Assumption(ρ. 3). There exists a $\delta>0$ such that

$$\dot{\rho}=O(\rho^{1-\delta}), \quad \ddot{\rho}=O(\rho^{-\delta}\dot{\rho}) \qquad (r\to\infty).$$

Assumption(q, 3). q is decomposed into the sum of two bounded continuous functions q_1 and q_2 with the properties

(i) $q_1\in C^1(M)$, $q_1\leqq c\rho^{-\delta}$, $\dot{q}_1\leqq c\rho^{-1-\delta}\dot{\rho}$ (on M),

(ii) $q_2=O(\rho^{-1-\delta}\dot{\rho})$ ($r\to\infty$, unif.)

for some C, $\delta>0$

Theorem 3. *Under Assumptions* (ρ, 0) (ρ, 3) (q, 3), *we can find positive constants* C *and* r_1 ($\geqq r_0$) *with which*

$$\int_{r_0<r<R}|f(x)|^2dx\geqq CR \qquad (R\geqq r_1)$$

holds.

In the case $\rho(r)=r$ and $M=\{x\in\mathbf{R}^n|\ |x|\geqq r_0\}$, Assumption ($q$, 3) and Theorem 3 imply Agmon's result. Next, let us concern ourselves in conditions of integral type.

Assumption(ρ, 4). $\rho^{-1}\dot{\rho}\in L^2(r_0,\ \infty)$, $\dot{\rho}^{-1}\ddot{\rho}\in L^1(r_0,\ \infty)$.

Assumption(q. 4). $q\in C(M)$ and, setting $q^*(r)=\sup_{\omega\in\mathbf{R}^{n-1}}|q(r,\ \omega)|$, we have

(i) $q^*\in L^1((r_0,\ \infty))$,

(ii) $\displaystyle\limsup_{r\to\infty}\frac{1}{\sqrt{\lambda}}\left\{\frac{|n-1|\ |n-3|}{4}\rho^{-1}\dot{\rho}+\frac{n-1}{2}\dot{\rho}^{-1}|\ddot{\rho}|+\rho\dot{\rho}^{-1}q^*\right\}<2.$

Theorem 4. *Under Assumptions* (ρ, 0) (ρ, 4) (q, 4), *we can find positive constants* C *and* r_1($\geqq r_0$) *for which we have*

$$\int_{r_0<r<R}|f(x)|^2dx\geqq CR \qquad (R\geqq r_1).$$

In the case of the Euclidean spaces $\rho(r)=r$, $M=(x\in\mathbf{R}^n|\ \ |x|\geqq r_0\}$, this theorem almost corresponds to Kato's result.

Remark. If $\ddot{\rho}$ is positive or negative definite, then the fact $\rho^{-1}\dot{\rho}\in L^2((r_0,\ \infty))$ and Assumption (ρ, 1) imply $\rho^{-1}\ddot{\rho}\in L^1((r_0,\ \infty))$, because the integration by parts shows

$$\int_{r_0}^{r}\frac{\ddot{\rho}(s)}{\rho(s)}ds=\frac{\dot{\rho}(r)}{\rho(r)}-\frac{\dot{\rho}(r_0)}{\rho(r_0)}+\int_{r_0}^{r}\frac{\dot{\rho}(s)^2}{\rho(s)^2}ds,$$

while $\rho(r)^{-1}\dot{\rho}(r)\to 0$.

§2. Abstract differential equation.

In order to carry out the proofs systematically, we prefer to describe the process of estimation in an abstract manner. That is, we interpret (0.1) as an equation for a vector-valued function and closely examine a subsidiary function to obtain the estimate of the solution.

Let H be a Hilbert space and D_0 its linear subset. $A_0=A_0(r)$ and $A_1=A_1(r)$ are assumed to be linear operators defined for each value of r. Further, we assume that for each value of r the domains of $A_0(r)$ and $A_1(r)$ contain D_0. We denote in the sequel by $(\ ,\ )$ and $\|\ \|$ the inner product and the norm of H respectively, and by a dot the derivative with respect to r. Moreover, instead of writing as "for almost every $r \geqq r_0$", we say simply "$(r \geqq r_0)$".

We shall consider a differential equation in H and want to get the estimate of the norm of the solution. To this end, let us state several conditions and definitions.

Condition 0.

(i) $(A_0(r)v, w)=(v, A_0(r)w)$ for every $v, w \in D_0$.

(ii) For each $v \in D_0$, $A_0(r)v$ is strongly differentiable at almost every r (being $\dot{A}_0(r)v$ its derivative).

Definition 1. Let $p(r)$, $a(r)$ and $\varphi(r)>0$ be real-valued C^2-functions. For every $v \in D_0$ and $w \in$ H, we set

$$B(v, w)=(\{(\varphi A_0)^{\cdot}+p\varphi(A_0+A_1)+(a\varphi)^{\cdot}\}v, v) \\ +(\dot{\varphi}-p\varphi)\|w\|^2+(\{2a\varphi-2\varphi A_1-(p\varphi)^{\cdot}\}v, w)$$

Condition 1. There exist a nonnegative function ψ and a number r_2 $(\geqq r_0)$ such that

(i) $\int_{r_0}^{\infty}\psi(r)dr=\infty$,

(ii) $B(v, w)\geq \psi\|v\|^2 \qquad (r\geqq r_2)$
for every $v \in D_0$, $w \in$ H.

Condition 2. We can find a nonnegative function $b(r)$ and a number r_2 $(\geqq r_0)$* such that for an arbitrary $v \in D_0$,

(i) $(\{a(r)-A_1(r)\}v, v) \geqq -b(r)\|v\|^2 \qquad (r\geqq r_2)$,

(ii) $\int_{r_0}^{\infty}e^{P(r)}b(r)dr<\infty$, where $P(r)=\int p(r)dr$.

Condition 3. Let K be an arbitrary positive constant and put

$$\zeta(r)=\int_r^R\frac{e^{P(r)}}{\varphi(r)}\int_r^{\infty}\varphi(s)\ \exp\{-K\int_r^s e^{-P(t)}\ dt\}dsdr.$$

Then we have

$$\lim_{R\to\infty}\zeta(R)=\infty, \qquad \lim_{R\to\infty}e^{-P(R)}\zeta(R)=\infty.$$

Condition 4. There exist a number r_2 and a function $\eta(r) \in L^1_{\mathrm{loc}}((r_2, \infty))$ such that for every $v \in D_0$ and $w \in$ H,

$$B(v, w)\geqq \eta(r)\ \dot{\varphi}\{\|w\|^2+(A_0v, v)-p(v, w)+a\|v\|^2\} \qquad (r\geqq r_2)$$

holds.

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*We can choose the same value of r_2 as in Condition 1.

Condition 4′. There exist a number r_2 and a function $\chi(r) \in L^1((r_2, \infty))$ such that for any $v \in \mathrm{D}_0$ and $w \in \mathrm{H}$,

$$B(v, w) \geqq (\dot\varphi - \varphi\chi)\{\|w\|^2 + (A_0 v, v) - p(v, w) + a\|v\|^2\} \qquad (r \geqq r_2)$$

holds.

Now we consider the following differential equation in H:

$$\ddot u + A_0(r)u + A_1(r)u = 0,$$

and study the solution $u(r)$ which does not vanish identically in any neighbourhood of infinity. We mean by "u is the solution" that (i) $u(r) \in \mathrm{D}_0$ a. e., (ii) $u(r)$ exists a. e. in the strong sense belonging to $L^2_{\mathrm{loc}}((r_0, \infty);\ \mathrm{H})$ and enjoying (2. 1), (iii) u, $\dot u$ are the indefinite integrals of $\dot u$, $\ddot u$ respectively in the strong sense, (iv) $(A_0(r)u, u)$ is absolutely continuous and satisfies $\frac{d}{dr}(A_0 u, u) = (\dot A_0 u, u) + 2\mathrm{Re}(A_0 u, \dot u)$ a. e.

Definition 2. For the solution u, we set

$$F(r) = \|\dot u\|^2 + (A_0 u, u) - p(\dot u, u) + a\|u\|^2,$$

where $p(r)$ and $a(r)$ are the the functions appearing in Definition 1. (Note that $F(r)$ is an absolutely continuous function).

We are now in a position to describe several estimates for $F(r)$ and $u(r)$.

Lemma 1. *If Condition* 0 *and* 1 *are satisfied, we have*

$$(\varphi F)^{\cdot} = B(u, \dot u) \geqq \psi\|u\|^2 \qquad (r \geqq r_2).$$

Proof. Differentiating φF by r and considering (2. 1), we have

$$\begin{aligned}(\varphi F)^{\cdot} &= \varphi\{2(\ddot u + A_0 u, \dot u) + (\dot A_0 u, u)\} + \dot\varphi\{\|\dot u\|^2 + (A_0 u, u)\} \\ &\quad - (P\varphi)^{\cdot}(u, \dot u) - p\varphi\|\dot u\|^2 - p\varphi(\ddot u, u) + 2a\varphi(u, \dot u) + (a\varphi)^{\cdot}\|u\|^2 \\ &= -2\varphi(A_1 u, \dot u) + (\{(\varphi A_0)^{\cdot} + (a\varphi)^{\cdot}\}u, u) + (\dot\varphi - p\varphi)\|\dot u\|^2 \\ &\quad + \{2a\varphi - (p\varphi)^{\cdot}\}(u, \dot u) + p\varphi((A_0 + A_1)u, u) \\ &= (\{(\varphi A_0)^{\cdot} + p\varphi(A_0 + A_1) + (a\varphi)^{\cdot}\}u, u) \\ &\quad + (\dot\varphi - p\varphi)\|u\|^2 + (\{2a\varphi - 2\varphi A_1 - (p\varphi)^{\cdot}\}u, u) \\ &= B(u, \dot u).\end{aligned}$$

Therefore, by Condition 1, we obtain

$$(\varphi F)^{\cdot} \geqq \psi\|u\|^2.$$

Lemma 2. *Under Conditions* 0, 1, 2 *and* 3, *we can find an* r_3 $(\geqq r_2)$ *such that*

$$F(r_3) > 0.$$

Proof. Suppose by way of contradiction that $F(r) \leqq 0$ for almost every r $(\geqq r_2)$. Then, since F and $\dot F$ belong to $L^1_{\mathrm{loc}}((r_0, \infty))$, it follows from Lemma 1 that

$$\begin{aligned}-\varphi(r)\, F(r) &= -\varphi(t)F(t) + \int_r^t (\varphi(s)\, F(s))^{\cdot}\, ds \\ &\geqq \int_r^t \psi(s)\|u(s)\|^2\, ds.\end{aligned}$$

The last member is an increasing function of t, while the first one does not depend on t. Hence, letting $t \to \infty$, we obtain

$$-\varphi(r)\, F(r) \geqq \int_r^\infty \psi(s)\|u(s)\|^2\, ds$$

together with the finiteness of the right member.

Now, let I be an interval in which $u(r)$ does not vanish, and for $r \in I$, set

$$g(r)=\log\|u(r)\|^2.$$

Then,

$$\dot g=2(\dot u,\ u)/\|u\|^2,$$
$$\ddot g=\{2(\ddot u,\ u)+2\|\dot u\|^2\}/\|u\|^2-4(\dot u,\ u)^2/\|u\|^4,$$

and the Schwarz inequlity shows

$$\begin{aligned}\ddot g&\geqq\{2(\ddot u,\ u)-2\|\dot u\|^2\}/\|u\|^2\\&=-2e^{-g}\{F+p(\dot u,\ u)-a\|u\|^2+(A_1u,\ u)\}\\&=-2e^{-g}F-p\dot g+2a-2e^{-g}(A_1u,\ u).\end{aligned}$$

Hence,

$$\begin{aligned}(e^P\dot g)^{\cdot}&=e^P(\ddot g+p\dot g)\\&\geqq-2e^{P-g}F+2e^P\{a-(A_1u,\ u)/\|u\|^2\}.\end{aligned}$$

Therefore, from Condition 2 and the assumption $F\leqq 0$, we see

$$(e^P\dot g)^{\cdot}\geqq-2e^Pb,$$

and hence, there exists a positive constant K such that

$$\begin{aligned}e^{P(t)}\dot g(t)&\geqq e^{P(r_2)}\dot g(r_2)-2\int_{r_1}^{t}e^{P(r)}b(r)dr\\&\geqq-K\end{aligned}$$

holds. Consequently, we obtain

$$g(s)-g(r)\ \geqq-K\int_r^s e^{-P(t)}dt.$$

Since the right-hand side is finite for any $s>r>r_0$, $g(s)$ never goes to $-\infty$ at a finite s. That means

$$u(r)\neq 0\quad\text{throughout the interval } (r_2,\ \infty).$$

Moreover, from (2.2) and (2.3) we have

$$\begin{aligned}(e^{P(r)}\dot g(r))^{\cdot}&\geqq\frac{2e^{P(r)}}{\varphi(r)}\int_r^\infty\psi(s)e^{g(s)-g(r)}ds+\text{summable function}\\&\geqq\frac{2e^{P(r)}}{\varphi(r)}\int_r^\infty\psi(s)\exp\Big(-K\int_r^s e^{-P(t)}dt\Big)ds+\text{summable function}\end{aligned}$$

which yields

$$\begin{aligned}e^{P(R)}\dot g(R)&\geqq2\int_{r_1}^{R}\frac{e^{P(r)}}{\varphi(r)}\int_r^\infty\psi(s)\exp\Big(-K\int_r^s e^{-P(t)}dt\Big)dsdr+\text{const.}\\&=2\zeta(R)+\text{const.}\end{aligned}$$

Therefore, Condition 3 shows

$$\dot g(R)=e^{-P(R)}(2\zeta(R)+\text{const.})\geqq e^{-P(R)}\zeta(R)\to\infty\qquad(R\to\infty),$$

and hence

$$\|u(r)\|=e^{g(r)/2}\to\infty\qquad(r\to\infty),$$

which contradicts the fact that

$$\int_{r_0}^\infty\psi(s)\|u(s)\|^2\,ds<\infty\quad\text{while}\quad\int_{r_0}^\infty\psi(s)ds=\infty,$$

as were found in (2.2) and Condition 1. Thus, Lemma 2 is established.

Lemma 3. *under Condition* 0, 1, 2 *and* 3, *we can find a positive constant C satisfying*

$$F(r)\geqq C\varphi(r)^{-1}\qquad(r\geqq r_3).$$

Proof. Lemma 1 shows $(\varphi F)^{\cdot}\geqq 0$ for $r\geqq r_2$. Therefore, $\varphi>0$ and $F(r_3)>0$ (by Lemma 2) give

$$\varphi(r)F(r)\geqq\varphi(r_3)\ F(r_3)=C>0.$$

Lemma 4. *Under Conditions* 0, 1, 2, 3 *and* 4, *there exists a positive constant* C *such that*

$$F(r)\geqq C\varphi(r)^{-1}\exp\{\int\eta(r)\varphi(r)^{-1}\dot{\varphi}(r)dr\}\qquad(r\geqq r_3)$$

holds. In particular, if $\eta(r)$ *is a constant* η, *we have*

$$F(r)\geqq C\varphi(r)^{\eta-1}\qquad(r\geqq r_3).$$

Proof. Since Condition 4 reads

$$(\varphi F)^{\cdot}\geqq\eta\dot{\varphi}F=\eta\varphi^{-1}\dot{\varphi}(\varphi F),$$

the fact that $\varphi(r)F(r)>0$ $(r\geqq r_3)$ (by Lemma 2) shows

$$\varphi(r)F(r)\geqq\varphi(r_3)F(r_3)\exp\{\int_{r_3}^{r}\eta(s)\varphi(s)^{-1}\dot{\varphi}(s)ds\}\qquad(r\geqq r_3).$$

Lemma 4′. *Under Conditions* 0, 1, 2, 3 *and* 4′, *we can find a positive constant* C *such that*

$$F(r)\geqq C\qquad(r\geqq r_3).$$

Proof. Condition 4′ means just

$$(\varphi F)^{\cdot}\geqq(\varphi^{-1}\dot{\varphi}-\chi)\varphi F,$$

by Lemma 1. Hence, from Lemma 2 it follows that

$$(\varphi F)^{-1}(\varphi F)^{\cdot}\geqq\varphi^{-1}\dot{\varphi}-\chi$$

for $r\geqq r_3$. Integrating both sides from r_3 to r, we obtain

$$F(r)\geqq F(r_3)\exp\{-\int_{r_3}^{r}\chi(s)ds\}\geqq C,$$

which proves Lemma 4′.

We thus have had estimates for $F(r)$, but our final need is those for the solution $u(r)$. To this end, we note a lemma which is a modification of Lemma 2 of Agmon [2]. Let us choose a C^2-function $\sigma_R(r)$ possessing the following properties where r_4 $(\geqq r_3)$ is chosen arbitrarily:

(i) $0\leqq\sigma_R(r)\leqq 1\qquad(r_0\leqq r<\infty)$,

(ii) $\sigma_R(r)=0\qquad(r_0\leqq r\leqq r_4)$,

(iii) $\sigma_R(r)=1\qquad(r_4+1\leqq r\leqq R-1)$,

(iv) $\sigma(r)=0\qquad(R\leqq r)$,

(v) the value of $\sigma_R(r)$ does not depend on R in the interval $r_0\leqq r\leqq r_4+1$,

(vi) in the interval $R-1\leqq r\leqq R$, the graph of $\sigma_R(r)$ does not change its shape but for translation. (Note that $\sup_{r_0<r<R}|\ddot{\sigma}_R|$ does not depend on R).

With this function $\sigma_R(r)$, we claim

Lemma 5. *If Condition* 0 *is satisfied.* u *and* F *enjoy the following inequality with certain positive constants* C_1, C_2, C_3 *and* C_4.

$$\int_{r_0}^{R}\ddot{\sigma}_R\|u\|^2dr$$
$$+\int_{r_0}^{R}\sigma_R\{C_1p^2\|u\|^2+C_2a\|u\|^2+C_3(A_0u,\ u)+C_4(A_1u,\ u)\}\ dr$$
$$\geqq\int_{r_0}^{R}\sigma_RFdr.$$

Proof. The integration by parts and the equation (2.1) show

$$\frac{1}{2}\int_{r_0}^{R}\ddot{\sigma}_R\|u\|^2dr=\frac{1}{2}\int_{r_0}^{R}\sigma_R\frac{d^2}{dr^2}\|u\|^2dr$$

$$=\int_{r0}^{R}\sigma_R\{\|\dot u\|^2+(\ddot u,\ u)\}\,dr$$

$$=\int_{r0}^{R}\sigma_R\{\|\dot u\|^2-(A_0\,u,\ u)-(A_1\,u,\ u)\}\,dr.$$

Therefore,

$$\int_{r0}^{R}\sigma_R\|\dot u\|^2dr=\int_{r0}^{R}\left\{\frac{1}{2}\ddot\sigma_R\|u\|^2+\sigma_R(A_0\,u,\ u)+\sigma_R(A_1\,u,\ u)\right\}dr. \tag{2.4}$$

On the other hand, $F=\|\dot u\|^2-p(\dot u,\ u)+a\|u\|^2+(A_0\,u,\ u)$ implies

$$F\leqq\|\dot u\|^2+\|\dot u\|^2+\frac{1}{4}p^2\|u\|^2+a\|u\|^2+(A_0\,u,\ u).$$

Hence, by (2.4) we have

$$\int_{r0}^{R}\sigma_R\,Fdr\leqq 2\int_{r0}^{R}\left\{\frac{1}{2}\ddot\sigma_R\|u\|^2+\sigma_R(A_0\,u,\ u)+\sigma_R(A_1\,u,\ u)\right\}dr$$

$$+\int_{r0}^{R}\sigma_R\left\{\frac{1}{4}p^2\|u\|^2+a\|u\|^2+(A_0\,u,\ u)\right\}dr$$

$$=\int_{r0}^{R}\ddot\sigma_R\|u\|^2dr+\int_{r0}^{R}\sigma_R\left\{\frac{1}{4}p^2\|u\|^2+a\|u\|^2\right.$$

$$\left.+3(A_0\,u,\ u)+2(A_1\,u,\ u)\right\}dr.$$

Thus the lemma is established.

§3. Proofs of the theorems.

In every case, we may set

$$q=q_1+q_2.$$

(For example, we consider as $q_2=0$ in Theorem 2, etc.). If we put

$$u=u(x)=u(r,\ \omega)=\rho^{(n-1)/2}f(x)$$

for the solution $f(x)$ of (0.1), a straightforward calculation shows that u satisfies

$$\ddot u+\rho^{-2}\Lambda u+(\lambda-q)u-n_1\rho^{-2}\dot\rho^2u-n_2\rho^{-1}\ddot\rho\ u=0, \tag{3.1}$$

on M, where a dot stands for $\partial/\partial r$ and

$$n_1=\frac{(n-1)(n-3)}{4},\qquad n_2=\frac{n-1}{2}.$$

Now, let

$$\mathrm{H}=L^2(\mathbf{S}^{n-1})\quad\text{with}\quad (v,\ w)=\int_{\mathbf{S}^{n-1}}v(\omega)\overline{w(\omega)}d\omega,$$

and set

$$\mathrm{D}_0=H^2(\mathbf{S}^{n-1}).$$

Our aim is to reduce the problems to the abstract theory described in §2. At first, we note that the fact $f\in H^2{}_{\mathrm{loc}}(M)$ implies that $u(r)$ satisfies the conditions (i) ～ (iii) on the solution.

If we set

$$A_0=\rho^{-2}\Lambda+\lambda-q_1\qquad\text{with } \mathrm{D}(A_0)=\mathrm{D}_0, \tag{3.2}$$

$$A_1=-q_2-n_1\rho^{-2}\dot\rho^2-n_2\rho^{-1}\ddot\rho\qquad\text{with } \mathrm{D}(A_1)=\mathrm{H},$$

then Condition 0 and the condition (iv) on the solution is trivial and (3.1) is rewritten as

$$\ddot u+A_0u+A_2u=0,$$

where $u=u(r,\ \cdot)$ is the solution in the sense of §2.

For the proof of the theorems, we choose as

$$\varphi(r)=\rho(r)^{\alpha}, \qquad p(r)=\beta\rho(r)^{-1}\dot{\rho}(r) \tag{3.3}$$

where α and β are constants chosen appropriately in each case. (In fact, $p(r)$ is set to be 0 so far as this paper is concerned. But we prefer to leave $p(r)$ for general convenience). The function $a(r)$ should also be determined later. By substituting these functions into the definition of $B(v, w)$, one immediately verifies the following formula.

Proposition 1.

$$\begin{aligned} B(v, w)=&(\{(\alpha+\beta-2)\rho^{\alpha-3}\dot{\rho}\Lambda+(\alpha+\beta)\rho^{\alpha-1}\dot{\rho}(\lambda-q_1)-\rho^{\alpha}\dot{q}_1 \\ &-\beta\rho^{\alpha-1}\dot{\rho}(q_2+n_1\rho^{-2}\dot{\rho}^2+n_2\rho^{-1}\ddot{\rho})+(\rho^{\alpha}a)^{\cdot}\}v, v) \\ &+(\alpha-\beta)\rho^{\alpha-1}\dot{\rho}\|w\|^2 \\ &+\rho^{\alpha}(\{2a+2q_2+(2n_1-\beta(\alpha-1))\rho^{-2}\dot{\rho}^2+(2n_2-\beta)\rho^{-1}\ddot{\rho}\}v, w). \end{aligned}$$

Although the choices of α, β, $p(r)$ and $a(r)$ will be different by each theorem, we shall extensively choose as

$$\psi(r)=\text{const. } \rho(r)^{\alpha-1}\dot{\rho}(r)$$

which, as we now show, realizes Condition 3 by itself.

Proposition 2. *If $\alpha > \beta$ and if Assumption $(\rho, 0)$ is satisfied, then Condition* 3 *applies with* $\psi=\text{const. } \rho^{\alpha-1}\dot{\rho}$.

Proof. Since $e^{P(r)}=\rho(r)^{\beta}$, we see that

$$\text{const. } \zeta(R)=\int_{r0}^{R}\rho(r)^{\beta-\alpha}\int_{r}^{\infty}\rho(s)^{\alpha-1}\dot{\rho}(s)\exp\{-K\int_{r}^{s}\rho(t)^{-\beta}dt\}\,dsdr.$$

Putting $Q(r)=\int_{r0}^{r}\rho(t)^{-\beta}dt$ we have

$$\begin{aligned} \text{const. } \zeta(R)&=\int_{r0}^{R}\rho(r)^{\beta-\alpha}\int_{r}^{\infty}\rho(s)^{\alpha-1}\dot{\rho}(s)\exp\{-K(Q(s)-Q(r))\}\,dsdr \\ &\geqq\int_{r0}^{R}\rho(s)^{\alpha-1}\dot{\rho}(s)e^{-KQ(s)}\int_{r0}^{s}\rho(r)^{\beta-\alpha}e^{KQ(r)}drds \\ &=\int_{r0}^{R}\rho(s)^{\alpha-1}\dot{\rho}(s)e^{-KQ(s)}\int_{r0}^{s}\rho(r)^{2\beta-\alpha}\rho(r)^{-\beta}e^{QK(r)}drds \\ &\geqq\begin{cases}\int_{r0}^{R}\rho(s)^{2\beta-1}\dot{\rho}(s)e^{-KQ(s)}\int_{r0}^{s}\rho(r)^{-\beta}e^{KQ(r)}drds & (\text{if } 2\beta\leqq\alpha) \\ \text{const. } \int_{r0}^{R}\rho(s)^{\alpha-1}\dot{\rho}(s)e^{-KQ(s)}\int_{r0}^{s}\rho(r)^{-\beta}e^{KQ(r)}drds & (\text{if } 2\beta\geqq\alpha)\end{cases} \\ &=\begin{cases}\text{const. }\int_{r0}^{R}\rho(s)^{2\beta-1}\dot{\rho}(s)\,(1-e^{-KQ(s)})ds \\ \text{const. }\int_{r0}^{R}\rho(s)^{\alpha-1}\dot{\rho}(s)\,(1-e^{-KQ(s)})ds\end{cases} \\ &\geqq\begin{cases}\text{const. } \rho(R)^{2\beta}+\text{const.} \\ \text{const. } \rho(r)^{\alpha}+\text{const.}\end{cases} \end{aligned}$$

Hence, if $\beta < \alpha$,

$$\lim_{R\to\infty}\zeta(r)=\infty, \qquad \lim_{R\to\infty}e^{-P(R)}\zeta(R)=\lim_{R\to\infty}\rho(R)^{-\beta}\zeta(R)=\infty.$$

Thus, Proposition 2 is proved.

Proof of Therem 1. Let us fix an arbitrary (small) $\varepsilon > 0$ and put

$$\alpha=\varepsilon,\ \beta=0,\ a(r)=\varepsilon^2\sqrt{\lambda}\,\rho^{-1}\dot\rho$$

and substitute them into Proposition 1 which reads

$$B(v,\ w)=\Big(\varepsilon\rho^{\varepsilon-1}\dot\rho\Big\{\frac{1}{\varepsilon}(\varepsilon-2)\rho^{-2}\Lambda+\lambda-q_1-\frac{1}{\varepsilon}\rho\dot\rho^{-1}\dot q_1$$
$$+\varepsilon\sqrt{\lambda}\,[(\varepsilon-1)\rho^{-1}\dot\rho+\dot\rho^{-1}\ddot\rho]\Big\}v,\ v\Big)+\varepsilon\rho^{\varepsilon-1}\dot\rho\|w\|^2$$
$$+\Big(2\varepsilon\rho^{\varepsilon-1}\dot\rho\{\varepsilon\sqrt{\lambda}+\frac{1}{\varepsilon}\rho\dot\rho^{-1}q_2+\frac{n_1}{\varepsilon}\rho^{-1}\dot\rho+\frac{n_2}{\varepsilon}\dot\rho^{-1}\ddot\rho\}v,\ w\Big) \tag{3.4}$$

Now, considering Assumptions $(\rho,\ 1)$ $(q,\ 1)$, we can find an r_2 such that each of the following inequalities

$$\begin{cases} q_1\leqq\varepsilon\lambda, \quad \frac{1}{\varepsilon}\rho\dot\rho^{-1}q\leqq\varepsilon\lambda, \quad (\varepsilon-1)\rho^{-1}\dot\rho+\dot\rho^{-1}\ddot\rho\geqq-\lambda \\ \frac{1}{\varepsilon}\rho\dot\rho^{-1}|q_2|\leqq\frac{\varepsilon\sqrt{\lambda}}{3}, \quad \frac{|n_1|}{\varepsilon}\rho^{-1}\dot\rho\leqq\frac{\varepsilon\sqrt{\lambda}}{3}, \quad \frac{|n_2|}{\varepsilon}\dot\rho^{-1}|\ddot\rho|\leqq\frac{\varepsilon\sqrt{\lambda}}{3} \end{cases} \tag{3.5}$$

holds for $r\geqq r_2$. Then, we observe in virtue of $(\Lambda v,\ v)\leqq 0$ that

the first inner product $\geqq\varepsilon\rho^{\varepsilon-1}\dot\rho(1-3\varepsilon)\lambda\|v\|^2$,

the last inner product $\geqq-\varepsilon\rho^{\varepsilon-1}\dot\rho\cdot 2\varepsilon(\|w\|^2+\lambda\|v\|^2)$

where we have used the inequality

$$2\|v\|\,\|w\|\ \leqq\ \|w\|^2/\sqrt{\lambda}+\sqrt{\lambda}\ \|v\|^2. \tag{3.6}$$

Therefore,

$$B(v,\ w)\geqq\varepsilon\rho^{\varepsilon-1}\dot\rho(1-5\varepsilon)\,(\|w\|^2+\lambda\|v\|^2) \qquad (r\geqq r_2)$$

that is, Condition 1 holds with $\varphi=\text{const.}\ \rho^{\varepsilon-1}\dot\rho$. Next, being

$$a-A_1=\varepsilon\rho^{-1}\dot\rho\ (\varepsilon\sqrt{\lambda}+\frac{1}{\varepsilon}\rho\dot\rho^{-1}q_2+\frac{1}{\varepsilon}n_1\rho^{-1}\dot\rho+\frac{1}{\varepsilon}n_2\dot\rho^{-1}\ddot\rho)$$

$$\geqq\varepsilon\rho^{-1}\dot\rho\ (\varepsilon\sqrt{\lambda}-\frac{1}{3}\varepsilon\sqrt{\lambda}-\frac{1}{3}\varepsilon\sqrt{\lambda}-\frac{1}{3}\varepsilon\sqrt{\lambda})=0$$

in virtue of (3.5), Condition 2 is satisfied by choosing $b=0$.
Condition 3 was already shown (Proposition 2). Hence, by dint of Lemma 3, we can conclude that

$$F(r)\geqq C\rho(r)^{-\varepsilon} \qquad (r\geqq r_3).$$

Now, Lemma 5 and (3.2) show that

$$\sup_{r_0<r<R}\{|\ddot\sigma_R|+C_3\sigma_R(\lambda+|q_1|)+C_4\sigma_R|q_2|\}\int_{r_0}^{R}\|u\|^2dr$$
$$\geqq C\int_{r_0}^{R}\sigma_R F(r)dr.$$

But we know that $\sup|\ddot\sigma_R|$ and $\sup\sigma_R$ do not depend on R. Hence we have

$$\int_{r_0}^{R}\|u\|^2dr\geqq\text{ const. }\int_{r_4+1}^{R-1}F(r)dr\geqq\text{const. }\int_{r_4+1}^{R-1}\rho(r)^{-\varepsilon}dr$$
$$\geqq C\int_{r_0}^{R}\rho(r)^{-\varepsilon}dr+\tilde C$$

for some C and $\tilde C$. On the other hand, from $u=\rho^{(n-1)/2}$, we have

$$\int_{r_0<r<R}|f(x)|^2dx=\int_{r_0}^{R}\int_{S^{n-1}}|f|^2\rho^{n-1}d\omega dr=\int_{r_0}^{R}\|u\|^2dr.$$

This proves the theorem.

Proof of Corollary to Theorem 1. Put

$$r=\int_{\rho_0}^{\rho}\sqrt{1+t'(\rho)^2}\,d\rho.$$

Then, r is the length along the meridian which corresponds to the r in Theorem 1. The operation $\partial/\partial r$, denoted by a dot, is equal to $\dot\rho\ \partial/\partial\rho$. Therefore,

$$\dot\rho=\frac{1}{\sqrt{1+t'^2}},\qquad \dot t=\dot\rho t'=\frac{t'}{\sqrt{1+t'^2}}$$

$$\dot\rho^{-1}\ddot\rho=\frac{d}{d\rho}\dot\rho=-t't''(1+t'^2)^{-3/2}.$$

From these formulas and the assumption (i) it follows that

$$0\leqq\dot\rho\leqq 1,\quad \rho^{-1}\dot\rho=O(\rho^{-1}),\quad \dot\rho^{-1}\ddot\rho=o(1).$$

On the other hand, writing $e(\rho)$ as $e(r)$, we have

$$\dot q_1=\dot\rho\ \partial q_1/\partial\rho\leqq\rho^{-1}\dot\rho\ e(r),$$
$$\rho\dot\rho^{-1}q_2=\rho\sqrt{1+t'^2}\,q_2=o(1).$$

Hence from Theorem 1 (letting $r=R$ when $\rho=P$), we have

$$\int_{r_0<r<R}|f(x)|\geqq C\int_{r_0}^{R}\rho^{-\epsilon}dr\qquad (R\geqq r_1=r(\rho_1))$$

(ρ_1 being some constant $\geqq\rho_0$). Consequently,

$$\int_{\rho_0<\rho<P}|f(\rho,\ \omega)|^2d\sigma\geqq C\int_{\rho_0}^{P}\rho^{-\epsilon}\sqrt{1+t'(\rho)^2}\,d\rho\qquad (P\geqq\rho_1),$$

which proves Corollary.

Proof of Theorem 2. We introduce a variable τ by

$$\tau=\int_{r_0}^{r}\rho(r)^{-1}dr,$$

and set $u=f$. Then we have

$$\ddot u+\Lambda u+\rho^2(\lambda-q)u=0\qquad (0<\tau<\infty)$$

where the dot means $d/d\tau$.*

Let us now set

$$A_0=\Lambda+\lambda\rho^2-\rho^2q,\quad A_1=0,$$
$$\varphi(r)=1,\quad p(r)=0,\quad a(r)=0,$$

and substitute them into Definition 1 in §2. Then we have

$$B(v,\ w)=(\dot A_0 v,\ v)$$
$$=2\rho\dot\rho(\lambda-q-\frac{1}{2}\rho\dot\rho^{-1}\dot q)\|v\|^2$$
$$\geqq 2\rho\dot\rho(\lambda-\varepsilon)\|v\|^2$$

by Assumption (q, 2). Hence, Condition 1 applies with

$$\psi(\tau)=2(\lambda-\varepsilon)\rho(\tau)\dot\rho(\tau).$$

Condition 2 is trivial with $b=0$. Condition 3 is a conclusion of

$$\zeta(T)=\int_0^T\int_\tau^\infty \text{const.}\ \rho(s)\dot\rho(s)e^{-K(s-\tau)}ds\,d\tau$$

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* If $n\neq 2$, it seems difficult to find τ increasing till infinity and u satisfying an equation free of $\dot\rho$ and $\ddot\rho$.

$$\geqq \int_0^T \text{const. } \rho(s)\dot{\rho}(s)e^{-Ks} \int_0^s e^{K\tau} d\tau$$

$$\to \infty \qquad (T \to \infty).$$

Hence, it follows from Lemma 3 that

$$F(\tau) \geqq \exists C \qquad (\tau \geqq \exists \tau_3)$$

where $F(\tau) = \|v\|^2 + (A_0 u, u)$. Therefore, one observes from Lemma 5 that

$$\int_0^T \ddot{\sigma}_T \|u\|^2 d\tau + C_3 \int_0^T \sigma_T((\lambda - q)u, u)\rho^2 d\tau \geqq \int_{\tau_4+1}^{T-1} F d\tau$$

for some τ_4. (We write as σ_T instead of σ_R). Hence,

$$C_3(\lambda + \sup_{0<\tau<T} |q(\tau)| + \sup_{T-1<\tau<T} \left| \frac{\ddot{\sigma}_T(\tau)}{\rho(T-1)^2} \right|) \int_0^T \|u\|^2 \rho^2 d\tau$$

$$\geqq CT - \text{const.}$$

which leads to the desired inequality by putting $T = \int_{r_0}^R \rho(r)^{-1} dr$, because $d\tau = \rho(r)^{-1} dr$, $d\sigma = \rho(r) dr d\omega$. Theorem 2 is thus established.

Proof of Corollary to Theorem 2. Putting

$$r = \int_{\rho_0}^{\rho} \sqrt{1 + t'(\rho)^2}\, d\rho, \qquad R = \int_{\rho_0}^{P} \sqrt{1 + t'(\rho)^2}\, d\rho$$

and writing $\rho = \rho(r)$, we have $\rho \dot{\rho}^{-1} \dot{q} = \rho\, \partial q / \partial \rho \leqq e(\rho)$ besides

$$\int_{r_0}^R \rho(r)^{-1} dr = \int_{\rho_0}^{\rho} \frac{\sqrt{1 + t'(\rho)^2}}{\rho} d\rho \geqq C \log \mathrm{P} \to \infty.$$

Thus, writing as $e(r) = e(\rho)$, we affirm Assumption $(q, 2)$. Accordingly, Theorem 2 shows for $\mathrm{P} \geqq \rho_1$ that

$$\int_{\rho_0<\rho<\rho} |f(\rho, \omega)|^2\, d\sigma = \int_{r_0<r<R} |f(x)|^2\, dx$$

$$\geqq C \int_{r_0}^R \rho(r)^{-1} dr$$

$$= C \int_{\rho_0}^P \rho^{-1} \sqrt{1 + t'(\rho)^2}\, d\rho,$$

which proves the corollary.

Proof of Theorem 3. Put $\varphi = \rho$ (i. e. $\alpha = 1$), $p = 0$ (i. e. $\beta = 0$) and $a = 0$. Further, let q_+, q_- be the positive and the negative parts of q_1 respectively. Then, from Proposition 1 and Assumptions $(\rho, 3)$ $(q, 3)$, we observe for sufficiently large r that

$$B(v, w) \geqq \dot{\rho}((\rho^{-2} \Lambda + \lambda + q_- - q_+ - \rho \dot{\rho}^{-1} \dot{q}_1) v, v) + \dot{\rho} \|w\|^2$$

$$+ 2\dot{\rho}((\rho \dot{\rho}^{-1} q_2 + n_1 \rho^{-1} \dot{\rho} + n_2 \dot{\rho}^{-1} \ddot{\rho}) v, w)$$

$$\geqq \dot{\rho}((\lambda + q_- - c\lambda \rho^{-\delta}) v, v) + \dot{\rho} \|w\|^2 - 2c\sqrt{\lambda}\, \rho^{-\delta} \|v\| \|w\|$$

$$\geqq \dot{\rho}((\lambda + q_- - 2c\lambda \rho^{-\delta}) v, v) + \dot{\rho}(1 - c\rho^{-\delta})) \|w\|^2$$

$$\geqq (1 - 2c\rho^{-\delta}) \dot{\rho} \{\|w\|^2 + ((\lambda + q_-) v, v)\}$$

$$\geqq (1 - 2c\rho^{-\delta}) \dot{\rho} \{\|w\|^2 + \rho^{-2}(\Lambda v, v) + ((\lambda - q_1) v, v)$$

$$- p(v, w) + a\|v\|^2\}.$$

These calculations show that Condition 1 is fulfiled with $\psi = \text{const. } \dot{\rho}$ and Condition 4 with $\eta = 1 - 2c\rho^{-\delta}$. Condition 2 is a consequence of the summability of $A_1 = -q_2 - n_1 \rho^{-2} \dot{\rho}^2 - n_2 \rho^{-1} \ddot{\rho} = O(\rho^{-1-\delta} \dot{\rho})$. Condition 3 was already shown. Hence, we can apply Lemma 4 obtaining

$$F(r) \geqq \text{const. } \rho(r)^{-1} \exp\left\{\int_r^r (1-2c\rho(s)^{-\delta})\rho(s)^{-1}\dot{\rho}(s)dr\right\}$$
$$= \text{const. } \exp\{-\rho(r)^{-\delta}\} \geqq C > 0 \qquad (r \geqq r_3)$$

by an appropriate choice of C. Accordingly, Lemma 5 yields

$$\int_{r0}^{R} \|u\|^2 dr \geqq CR \qquad (R \geqq {}^{\exists}r_1)$$

as before, which leads to the desired inequality. Proof is completed.

Proof of Theorem 4. Set $\alpha=2$, $\beta=0$, $q_0=q_1=0$, $q_2=q$, $a=0$ and substitute them into Proposition 1. Then we have

$$B(v, w) = 2\rho\dot{\rho}\lambda\|v\|^2 + 2\rho\dot{\rho}\|w\|^2 + 2\rho^2(\{q+n_1\rho^{-2}\dot{\rho}^2+n_2\rho^{-1}\ddot{\rho}\}v, w)$$
$$\geqq 2\rho\dot{\rho}(\|w\|^2+\lambda\|v\|^2)$$
$$-2\rho\dot{\rho}(\rho\dot{\rho}^{-1}q^* + |n_1|\rho^{-1}\dot{\rho} + |n_2|\dot{\rho}^{-1}|\ddot{\rho}|)\|\lambda^{1/4}v\|\|\lambda^{-1/4}w\|$$
$$\geqq 2\rho\dot{\rho}(\|w\|^2+\lambda\|v\|^2)$$
$$-\frac{\rho\dot{\rho}^{-1}q^* + |n_1|\rho^{-1}\dot{\rho} + |n_2|\dot{\rho}^{-1}|\ddot{\rho}|}{\sqrt{\lambda}}\rho\dot{\rho}(\|w\|^2+\lambda\|v\|^2).$$

Hence, from (ii) of Assumption (q, 4), it follows that

$$B(v, w) \geqq C\rho\dot{\rho}\|v\|^2,$$

which implies Condition 1. On the other hand, from $A_0=\rho^{-2}A+\lambda$ and $p=a=0$, we see that

$$B(v, w) \geqq \{2\rho\dot{\rho} - \rho^2(q^* + |n_1|\rho^{-2}\dot{\rho}^2 + |n_2|\rho^{-1}|\ddot{\rho}|)/\sqrt{\lambda}\} \times$$
$$\times (\|w\|^2 + (A_0 v, v) - p(v, w) + a\|v\|^2).$$

Hence, from Assumptions (ρ, 4) (q, 4) we have

$$\chi(r) \equiv (q^* + |n_1|\rho^{-2}\dot{\rho}^2 + |n_2|\rho^{-1}|\ddot{\rho}|)/\sqrt{\lambda} \quad \in L^1((r_0, \infty)).$$

Consequently, we can apply Lemma 4′ obtaining

$$F(r) \geqq C \qquad (r \geqq r_3).$$

which, by virtue of Lemma 5, implies

$$\int_{r0}^{R} \|u\|^2 dr \geqq CR \qquad (R \geqq {}^{\exists}r_1).$$

This is nothing but the desired inequality. Theorem 4 is proved.

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Added in proof. Recently, T. Tayoshi [9] investigated the spectrum of second order elliptic differential operators considered on noncompact Riemannian manifolds. His assumptions on the metric and the coefficients of the operators are very general. But, in so for as the spherically symmetric case and the Laplacian are concerned, our results are seemed to be a little sharper.