Double-Mirror Illusion -A New Class of 3D Illusion That Creates Anomalous U-Turn and Anomalous Translation Simultaneously-

Double-Mirror Illusion: A New Class of 3D Illusion That Creates Anomalous U-Turn and Anomalous Translation Simultaneously[1]

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Abstract

This paper presents a new class of 3D optical illusion in which two types of mirror illusions occur simultaneously. The author previously reported the left–right reversal illusion, in which an object exchanges its left and right sides in a mirror, and the translation illusion, in which an object facing toward a mirror translates into the mirror instead of turning around. We will show that these two illusions can be created by a single object. If we place the object in front of two vertical mirrors meeting at the right angle, the object exchanges its left and right sides in one mirror and translates into the other mirror. As one variation of this double-mirror illusion, we create objects whose mirror reflections together with the original objects form a circular sequence oriented clockwise or counterclockwise uniformly. The mathematics behind these illusions and the origin of their robustness are also discussed.

1. Introduction

Depth illusions created by 3D objects cover a wide range of anomalous perceptions, in each of which we feel something impossible is happening. The term "impossible objects" was initially introduced to mean imaginary 3D structures which are evoked in our mind when we see "impossible figures" but which cannot exist as real physical objects (Penrose and Penrose 1958, Unruh 2001). However, several tricks were found to create real 3D objects from impossible figures. They include the "discontinuity trick" in which an object has discontinuity at which it looks continuous (Gregory 1970), the "curved-surface trick" in which an object has curved surfaces that look planar (Ernst 1992, Elber 2011), and the "non-rectangularity trick" in which an object has arbitrary angles that look rectangular (Sugihara 1986). Now the term "impossible objects" has shifted to mean real physical objects whose behaviors appear to be impossible due to optical illusion. The realizations of impossible figures belong to the impossible objects (Sugihara 2020b).

Several other classes of impossible objects were subsequently found after that. The class of "impossible motion" objects create optical illusions in such a way that the object looks normal and familiar but the inserted motion looks impossible (Sugihara 2014). The class of "reverse-perspective art" objects create misunderstanding of their near parts and far parts so that the viewer perceives unexpected deformation of the object when the viewer moves (Wade and Hughes 1999). The class of

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"ambiguous cylinder" objects (Sugihara 2015a) creates unexpected changes of appearance in a mirror, so that the viewer feels that the object is replaced with another.

The ambiguous cylinders have been extended to various directions. The typical extensions include the height-reversal objects which reverse the heights in a mirror (Sugihara 2015b), the partly invisible objects which becomes partially invisible in a mirror (Sugihara 2016a), the topology-disturbing objects which change not only the shape but also the way of connection in a mirror (Sugihara 2018), the reflexively-fused objects which create a meaningful shape when the original object and its mirror image are fused (Sugihara and Moriguchi 2018), and the ambiguous tiling in which a tiling pattern changes to another tiling pattern in a mirror (Sugihara 2020a). In all of those extensions, the shape of the object changes drastically in a mirror. They are summarized in the review article (Sugihara 2020b).

There are two special extensions in which the shapes do not change but their postures change in an inconsistent manner. One is the class producing left–right reversal illusions in which the left part and the right part of an object exchange position with each other in the mirror (Sugihara 2016b, 2022). The other is the class producing translation illusions, in which an object facing toward a mirror translates into the mirror instead of turning in the opposite direction (Sugihara 2023).

In this paper, we present a new type of impossible objects that creates both the left–right reversal illusion and the translation illusion. Suppose that we place two vertical mirrors behind such an object. Then we can perceive the left–right reversal illusion in one mirror and the translation illusion in the other mirror. We observe the behavior of our new illusion in comparison with the two previous illusions (Section 2), discuss the mathematics behind this illusion (Section 3), give one variation of this illusion (Section 4), and discuss the robustness of this illusion (Section 5) before making concluding remarks (Section 6).

2. Observation

A 3D object and its mirror image are mutually plane-symmetric with respect to the mirror surface. However, we sometimes perceive mirror images that appear to defy this mirror-reflection rule due to optical illusion. Let us start with normal objects.

Let S be a 3D solid object of which one part is identified as the head and another part is identified as the tail. A typical example of such an object is an arrow. We refer to the line connecting the center of the head and the center of the tail as the *head–tail axis*. We consider the head–tail axis as a directed line whose direction is chosen as either from the tail to the head or from the head to the tail, but once the direction is chosen, it is fixed to the object.

Suppose that the object S is placed so that its head–tail axis is parallel to the mirror surface, as shown in Figure 1(a), where the head–tail axis is represented by a dash–dot line. Then, from the point of view of optics, the mirror image of the head–tail axis is a translation of the original head–tail axis. If we perceive this relation between the original head–tail axis and its mirror image, we say that we perceive a *normal translation*.

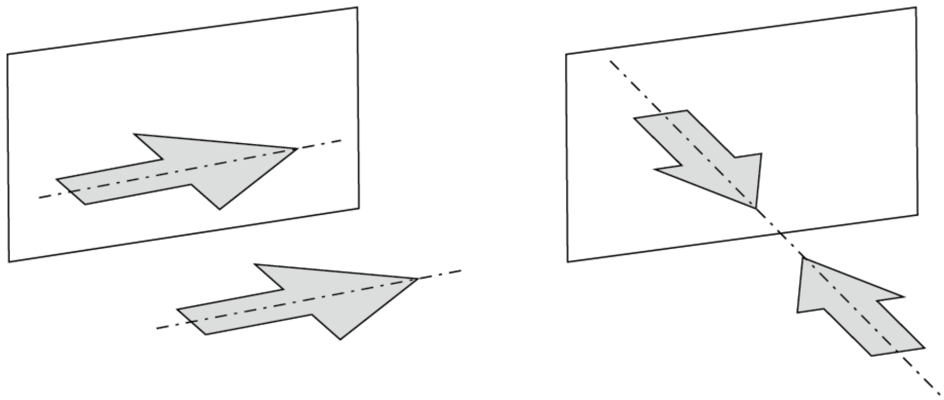


Figure 1. Normal behaviors of the head–tail axis reflected in a mirror.

Next, suppose that the object is placed so that its head–tail axis is perpendicular to the mirror surface, as shown in Figure 1(b). Then, the mirror image of the head–tail axis turns to the opposite direction of the original head–tail axis. If we perceive this relation between the original head–tail axis and its mirror image, we say that we perceive a *normal U-turn*.

The normal translation and the normal U-turn are not necessarily perceived for real objects because of optical illusion. Actually, there exists the left–right reversal illusion, in which an object facing toward the right changes its direction to the left in a mirror standing behind the object (Sugihara 2016b, 2022). An example will be shown soon in Figure 3. Also, there exists the translation illusion, in which an object facing toward a mirror translates to its mirror image instead of changing its direction to the opposite one (Sugihara 2023); an example will be shown soon in Figure 4. We say that we perceive an *anomalous U-turn* if the head–tail axis that is parallel to the mirror surface appears to change its direction to the opposite one in the mirror image. Similarly, we say that we perceive an *anomalous translation* if the head–tail axis that is perpendicular to the mirror surface appears to translate into the mirror instead of changing its direction to the opposite one.

Note that we use the term "anomalous U-turn" instead of "left–right reversal." This is because the concepts of left and right depend on the observer's orientation, while the term "U-turn" describes the behavior of the head–tail axis independently of the direction that the observer is facing.

Before presenting our new illusion, let us review the two mirror illusions. Figure 2 shows the plan view (i.e., the scene viewed from exactly above) of our setting to create mirror illusions. The object S is placed on a desk surface and two mirrors, M1 and M2, are placed vertically behind the object in such a way that they meet at a right angle. Hence, the object creates three mirror images: image S1 created by the mirror M1, image S2 created by the mirror M2, and image S3 created through two reflections by the two mirrors. We are interested in how S1 and S2 look. We assume that the object S has a head–tail axis (represented by the dash–dot line in Figure 2) and that the axis is parallel to M1 and perpendicular to M2. We consider three viewing directions, represented by the arrows A, B, and C in Figure 2, where the direction A is nearly parallel to the mirror M2 and slanted downward, the direction B is nearly parallel to the mirror M1 and slanted downward, and the direction C is horizontal and meets both mirrors M1 and M2 at 45 degrees.

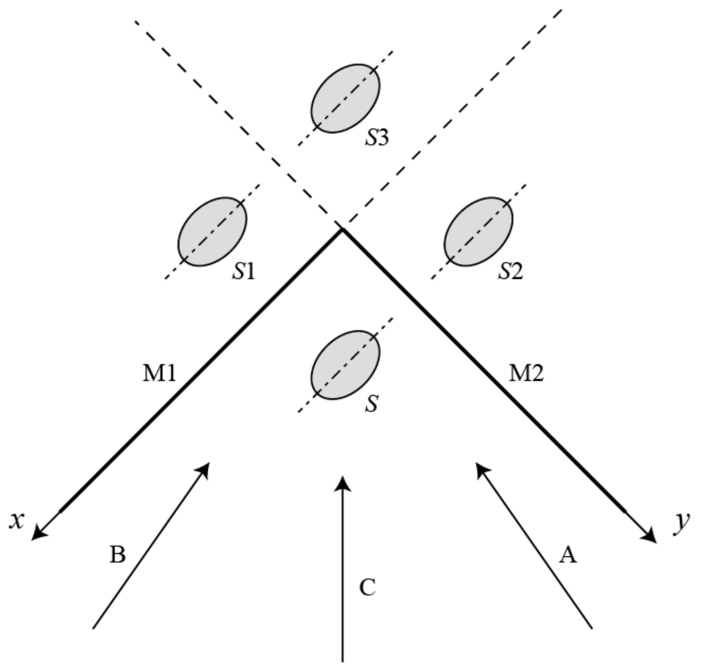


Figure 2. Plan view of the scene composed of an object and two mirrors, and three viewing directions.

Figure 3 shows the behavior of the left–right reversal illusion. We chose a left–right reversing arrow as the object S, and view this scene in the direction A. The object appears to be an arrow facing toward the right. We see the object reflected by the mirror M1, but the mirror image appears to be an arrow facing in the opposite direction; that is, the head–tail axis makes a U-turn in the mirror M1. Thus, we perceive an anomalous U-turn in M1. On the other hand, the image S2 created by the mirror M2 is natural because the head–tail axis, which is perpendicular to the mirror, makes a U-turn in the mirror. Thus, we perceive a normal U-turn in M2. This is a typical behavior of a left–right reversing object (Sugihara 2016b).

Figure 3. Behavior of a left–right reversing arrow.

Figure 4 shows the behavior of the translation illusion for the scene viewed in the direction B. The object appears to be a gull in flight, and hence has a natural head–tail axis. The axis is facing toward the mirror M2, that is, perpendicular to M2. However, the head–tail axis of the mirror image S2 created by M2 is in the same direction as S. Thus, we perceive an anomalous translation in M2. On the other hand, in the mirror image S1 created by M1, the head–tail axis is in the same direction as the original object S. Thus, we perceive a normal translation in M1. This is a typical behavior of a translation illusion object (Sugihara 2023).

Figure 4. Behavior of a translation-illusion gull.

Figure 5 shows an example of our new illusion, showing the scene for the view direction C in Figure 2. The object appears to be a fish swimming toward the mirror M2. Thus, it has a head–tail axis that is parallel to M1 and perpendicular to M2. However, in the mirror image S1, the head–tail axis is in the opposite direction, and hence we perceive an anomalous U-turn in M1. Also, in the mirror image S2, the head–tail axis does not change direction, and hence we perceive an anomalous translation in M2. Thus, the object creates anomalous perceptions in both of the mirror images. Hence, our new illusion object creates both the left–right reversal illusion and the translation illusion simultaneously. Let us name this visual effect as the *double-mirror illusion*.

Figure 5. Object creating an anomalous U-turn in the left mirror and an anomalous translation in the right mirror.

It should be noted that the view directions in Figures 3, 4, and 5 are not optimal. The object in Figure 3 was designed so that the left–right reversal illusion occurs most clearly when it is viewed in the direction perpendicular to the mirror M1. However, we view it in the slanted direction A shown in Figure 2. This is because we also want to see the mirror image S2 created by the right mirror M2. We can still perceive the illusion because it is robust in the sense that it does not disappear when the viewpoint is slightly changed, although the appearance of the object is somewhat deformed. Similarly, the object in Figure 4 was designed so that the translation illusion occurs most clearly when viewed in the direction perpendicular to the mirror M2. However, we still view it in the direction B shown in Figure 2, with some slight deformation in the shape.

The view direction in Figure 5 is again not optimal. The object was designed so that the illusion occurs most clearly when it is viewed in the direction at an angle of 45 degrees from both mirrors M1 and M2. If we viewed the object in the direction C in the setting of Figure 2, we would be observing in the ideal view direction, but in that case, the camera and a photographer would also be reflected near the third object image S3, which would be distracting. Therefore, we slightly increase the angle between the two mirrors, making the reflected images of the camera and the photographer invisible. This results in the view direction slightly deviating from the optimal direction, but we can still perceive the illusion because of the robust nature of this illusion object.

3. Mathematics behind the Illusion

It is known that the left–right reversal illusion occurs if the object is line-symmetric with respect to the vertical axis (Sugihara 2016b, 2022). However, that property holds only when the mirror is oriented exactly perpendicular to the viewer. In our present case, the mirror M1 is rotated toward the right by 45 degrees, and hence the above property does not hold. On the other hand, the translation illusion occurs if the head–tail axis is perpendicular to the mirror and the object is plane-symmetric with respect to a plane parallel to the mirror (Sugihara 2023). This property does not depend on the view direction, and hence can be applied to our present situation. From these observations, we can understand that both illusions occur at the same time only when the object is viewed in the direction 45 degrees from both of the mirrors.

This property can be explained intuitively with Figure 6. Suppose that the object S having a head–tail axis is placed so that the axis is parallel to the mirror M1 and perpendicular to the mirror M2. Let us assume that S is a translation illusion object, and hence it is plane-symmetric with respect to a plane parallel to the mirror M2; the plane of symmetry is represented by the broken line L in Figure 6. Let K represent the image of the object S viewed in the direction C, which is horizontal and 45 degrees from both mirrors M1 and M2. Let R(K) represent the image obtained by rotating the image K by 180 degrees around a vertical line; that is, R(K) is the image obtained when we reverse the left and right parts of K.

As shown in Figure 6, let C1 be the view direction perpendicular to C from the right. Because C and C1 are plane-symmetric with respect to L, the image of the object S viewed in the direction C1 coincides with R(K). This image is reflected by the mirror M2 and consequently reversed again, resulting in RR(K), which we can view in the direction C2. Because RR(K) = K, we perceive an anomalous translation of S in the mirror M2.

Next, let C3 be the view direction perpendicular to C from the left. Viewing the object along this direction is equivalent to viewing the silhouette from the other side of C1, and hence the image coincides with the reverse of R(K), so we get RR(K). This image is reflected by the mirror M1 and hence reversed again, resulting in RRR(K), which we can view along the direction C4. Because RRR(K) = R(K), we perceive an anomalous U-turn in the mirror M1. Thus, we perceive both an anomalous U-turn and an anomalous translation, in M1 and M2, respectively.

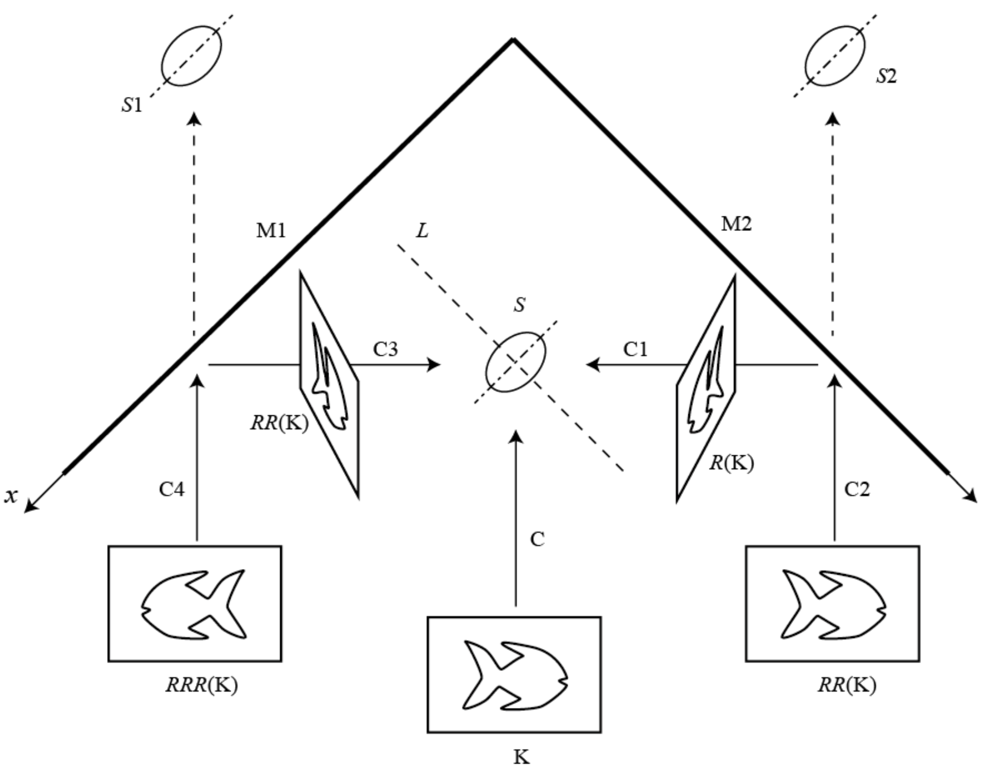


Figure 6. Intuitive explanation of the new illusion.

We fix the *xyz* Cartesian coordinate system in such a way that the origin is at the point of intersection of a desk surface and the two mirrors M1 and M2, the *x* and *y* axes are the intersections of the desk surface and M1 and M2, respectively, and the *z* axis is the intersection of the two mirrors. Figure 2 shows the *x* and *y* axes. Figure 7 shows how we can construct the object from two shapes which we want to view in the directions C and C1. The top part represents the plan view (the scene viewed along the *z* axis) and the bottom part represents the side view viewed along the *y* axis. As shown in this figure, the desired shape of the object viewed in the direction C is projected by parallel projection onto the *xz* plane. Let the resulting silhouette curve be denoted in parametric representation of the closed curve as $P(t) = (P_x(t), 0, P_z(t))$, $0 \leq t \leq 1$, where t is a parameter and $P(0) = P(1)$. The y coordinate of $P(t)$ being 0 implies that the curve is on the *xz* plane. Similarly, let $Q(t) = (Q_x(t), 0, Q_z(t))$, $0 \leq t \leq 1$, represent the silhouette curve of the desired shape of the object projected onto the *xz* plane by parallel projection along C1.

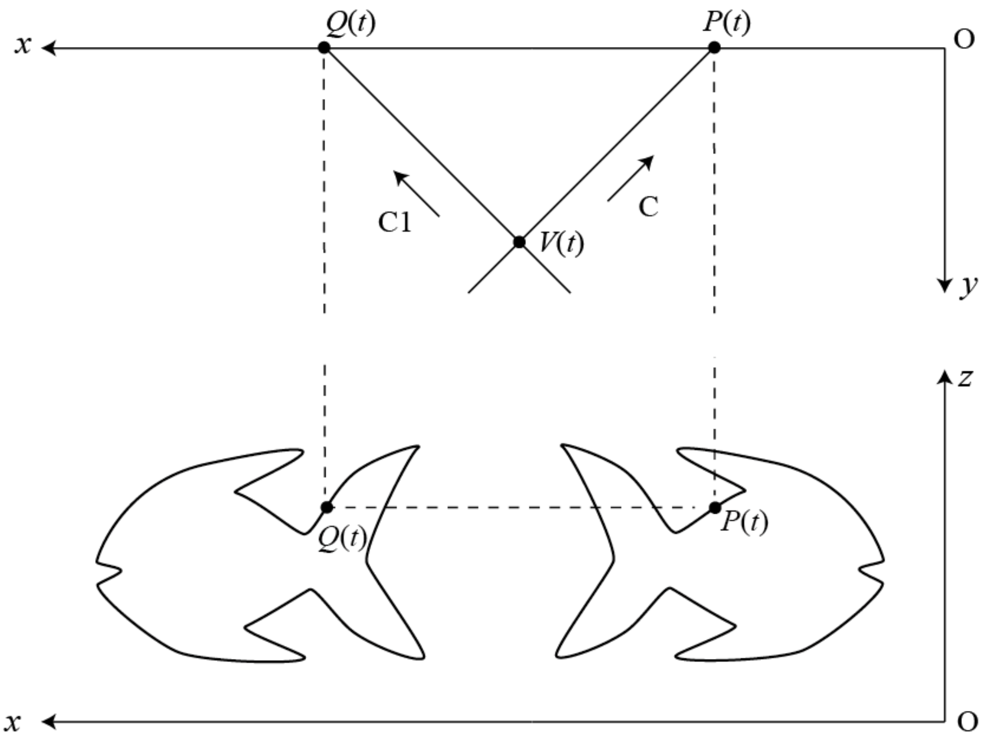


Figure 7. Construction of the new illusion object.

We have to note that the symbol $P(t)$ has two meanings. The first meaning is the whole curve defined when t moves from 0 to 1, and the second meaning is a point defined for a specific value of t. In order to avoid confusion, we write "curve $P(t)$" for the first meaning, and "point $P(t)$" for the second meaning.

We assume that curves $P(t)$ and $Q(t)$ each have no self-intersection. We assume that points $P(t)$ and $Q(t)$ each move along their curve so that the represented shape is to the right of the curve as t increases. In other words, the curves $P(t)$ and $Q(t)$ bound their shapes in a clockwise manner. Recall that the view directions C and C1 are horizontal and meet the xz plane at 45 degrees. Because we want an object that creates an anomalous translation in the mirror M1, we need the following condition to be satisfied.

Condition 1. The two curves $P(t)$ and $Q(t)$ are line-symmetric with respect to a vertical line in the xz plane.

Note that, as we have seen in Figure 6, this condition is equivalent to the case where we want an object that creates an anomalous U-turn in the mirror M1. Hence, an object satisfying Condition 1 can create both an anomalous U-turn and an anomalous translation.

Next, we consider under what conditions such an object exists. We have freedom in choosing the parameter t. Indeed, we can choose the start point $P(0)$ arbitrarily. If $u = \varphi(t)$ is a monotone increasing function of t such that $\varphi(0) = 0$ and $\varphi(1) = 1$, then $P(t)$ and $P(u)$ represent the same curve. We assume that we can adjust the parameter t to satisfy the following condition.

Condition 2. $P_z(t) = Q_z(t)$ holds for all $0 \leq t \leq 1$.

Condition 2 implies that points $P(t)$ and $Q(t)$ are in the same horizontal plane. If Condition 2 is satisfied, we can construct a space curve, say $V(t)$, $0 \leq t \leq 1$, that has both desired appearances $P(t)$ and $Q(t)$ in the following way (Sugihara 2016b). For an arbitrary value of t, two points $P(t)$ and $Q(t)$ are on a common horizontal line. Let $V(t)$ be the point of intersection of the line passing through point $P(t)$ parallel to the view direction C and the line passing through point $Q(t)$ parallel to the view direction C1, as shown in Figure 7. Collecting the points $V(t)$ for all values of $0 \leq t \leq 1$, we obtain the space curve $V(t)$, $0 \leq t \leq 1$. This is the space curve that we want.

If both Conditions 1 and 2 are satisfied, we can construct the space curve $V(t)$ that creates both the anomalous U-turn and the anomalous translation. Moreover, the resulting curve is plane-symmetric with respect to a plane parallel to the yz plane, because the two curves $P(t)$ and $Q(t)$ are line-symmetric with respect to a vertical line. The object in Figure 5 was constructed first by constructing the space curve $V(t)$, next by stretching a smooth surface over $V(t)$, and finally by adding a uniform thickness by translating the surface in the direction parallel to the y axis. For stretching the smooth surface over $V(t)$, we first construct a Delaunay triangular mesh (Okabe et al., 2000) and then apply the Laplacian-smoothing scheme (Farin 1990). Thus, what we have to do is to find a curve $P(t)$ such that it and its left–right reverse version $Q(t)$ satisfy Condition 2.

Figures 8 and 9 show two more examples of objects constructed in this way. Figure 8 is a flying bird object which creates an anomalous U-turn in the left mirror and an anomalous translation in the right mirror. This object was constructed by applying our method twice, once for the body and right wing, and once for the body and left wing. We combined the resulting objects to obtain the object shown in Figure 8.

Figure 8. Flying bird object that creates both an anomalous U-turn and anomalous translation.

Figure 9 shows a cat object that also creates an anomalous U-turn in the left mirror and an anomalous translation in the right mirror.

Figure 9. Cat object that creates an anomalous U-turn in the left mirror and an anomalous translation in the right mirror.

4. Self-Rotation Illusion

The double-mirror illusion can be modified to create a circular sequence with uniform orientation. An example is shown in Figure 10. A bear is placed in front of two mirrors so that it is nearly parallel to the right mirror and nearly perpendicular to the left mirror. However, it is not strictly parallel or strictly perpendicular; it is rotated by 22.5 degrees (half of 45 degrees). As a result, the bear and its three mirror images form a circular sequence of bears oriented clockwise. More precisely, the original bear and its left mirror image are walking in nearly the same direction, nearly creating an anomalous translation, but because the original object turned forward a little left, the left mirror image bends 45 degrees toward the right. Similarly, the original bear and its right mirror image are walking in nearly opposite directions, nearly creating an anomalous near U-turn. However, it is not an exact U-turn, but a sharp turn (by 180 − 45 = 135 degrees) to the right. Consequently, the original bear and its three mirror images together form a clockwise-rotating sequence of walking bears in the order of original object S, left-reflected image S1, twice-reflected image S3, and right-reflected S2 represented in Figure 2. Thus, we obtain a self-rotation illusion.

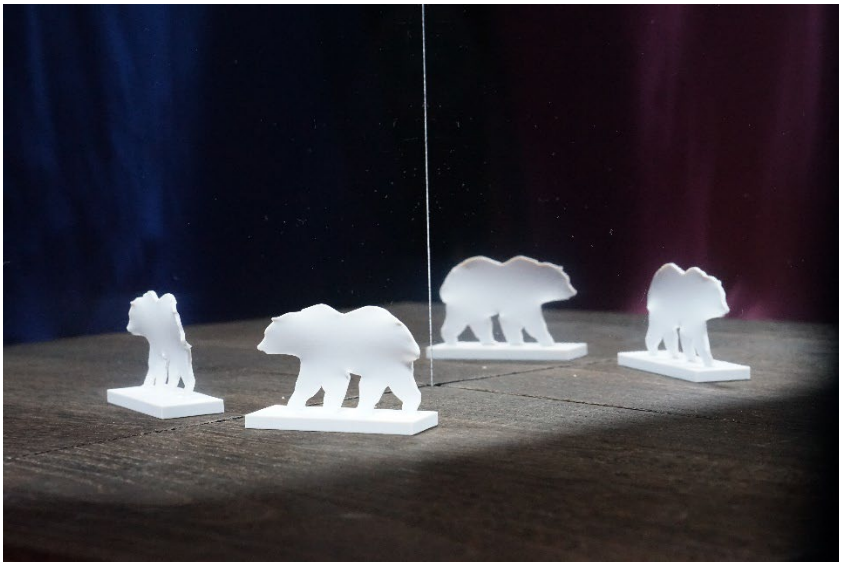


Figure 10. Self-rotation illusion created by a single object.

Note that this is not exactly the same object as a double-mirror illusion object. Indeed, if we place a double-mirror illusion object in this orientation, the apparent shape would be distorted. We computed a new object for this specific orientation in the following way. We first considered a vertical plane perpendicular to the left mirror and fixed the shape of a bear to it. Next, we rotated the plane by 22.5 degrees around a vertical axis and constructed an ambiguous object that realized the shape of the bear and its mirror image by the general method for designing an ambiguous object (Sugihara 2015a). The resulting object could create this type of self-rotation illusion automatically.

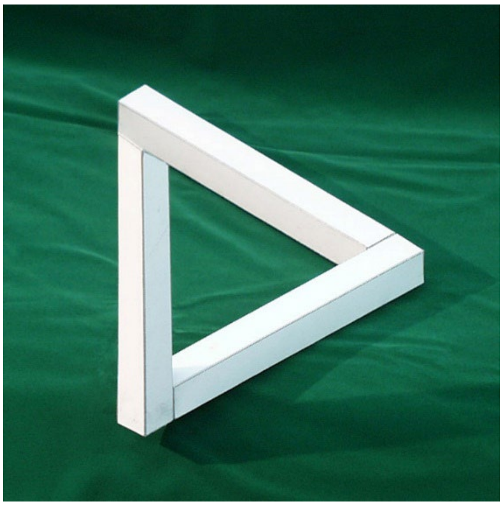
5. Robustness against Changes in Viewpoint

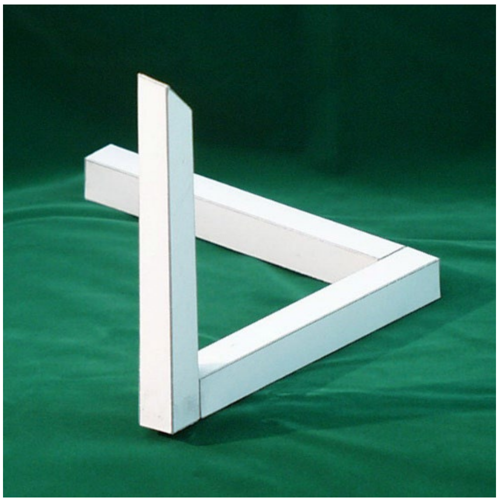
From a mathematical point of view, the illusion is guaranteed only when we view an object in the horizontal direction C in Figure 2. In many cases, however, the illusion is robust in the sense that we can enjoy it from a wide range of viewing directions. Indeed, the scene is viewed from a little above in Figures 9 and 10, but still the illusion occurs.

In some cases, we can observe an illusion even if we use a higher viewpoint to look at the object. The dog object in Figure 11 creates the double-mirror illusion even from a high viewpoint.

Figure 11. Dog object that creates an anomalous U-turn and an anomalous translation.

This robustness is remarkable in comparison with the 3D realizations of impossible figures. For example, as shown in Figure 12(a), the Penrose triangle can be realized as a 3D solid object using the discontinuity trick (Gregory 1970). However, this illusion is sensitive to viewpoint. If we change the viewpoint even slightly, the trick becomes visible, as shown in Figure 12(b), and thus the illusion disappears. A 3D realization of this kind is valid only for a unique viewpoint, and the use of the unique viewpoint is nothing but projecting the 3D object onto a 2D plane, and hence it is almost equivalent to the original 2D representation. This type of viewpoint sensitivity might be considered unavoidable.





(a) (b)

Figure 12. 3D realization of the Penrose triangle using the discontinuity trick.

Our robust illusion seems to break the mold, in that the impossible 3D world can be represented by existing 3D objects without a strong restriction on the viewpoint. This robustness can be understood from a mathematical point of view in the following way. The 3D realization of the Penrose triangle creates a sense of impossibility only when discontinuous parts align accidentally from a special viewpoint. In other words, the origin of the illusion is a viewpoint-dependent characteristic of objects. Recall that the translation illusion object is plane-symmetric with respect to a plane parallel to the mirror surface. This symmetry is a property of the object itself and does not depend on the viewpoint from which we observe the object. Hence, the translation illusion does not disappear when we shift the viewpoint. That is, the original object and its mirror image have the same appearance and orientation regardless of where we observe them from. Moreover, some objects give almost the same appearance when the viewpoint is changed. This happens in the case of the dog object in Figure 11. The origin of the translation illusion is the symmetry inherent to the object itself, while the origin of the illusion of the 3D realization of an impossible figure is viewpoint dependent.

To observe this visual effect by an example, let us take the object in Figure 5 again. This object was designed so that the illusion is guaranteed if we see it in the horizontal direction. Figure 13 shows three appearances of the same scene seen from different heights; the viewpoint is a little high in (a), higher in (b) and the highest in (c). The apparent shapes are gradually broken from (a) to (c); we may recognize the fish in (a) but not in (c). This is natural because the viewpoint moves gradually from the correct horizontal position.

However, one remarkable point is that the right two shapes (the object and the right-mirror reflected image) look the same, and the left two shapes (the left-mirror reflected image and the twice reflected image) look the same no matter where the viewpoint is. This is the mathematical consequence of the plane symmetry of the object.

Note that the object and its left-right reversed version (second from the right and the leftmost image) do not necessarily give the same shape. This is because the object is curved and we are looking at the mutually other sides, so that the exact left-right reversal is not guaranteed even though the object is plane symmetric.

(a) (b)

(c)

Figure 13. The scene in Figure 5 seen from different heights

In this way, the translation illusion is mathematically guaranteed even when we change the viewpoint. Whether the appearance does not change much when we change the viewpoint, on the other hand, depends on the individual object shape. At present, it is not easy to estimate the robustness of each object before creating it, but we have a higher probability of creating a robust illusion in the case of a translation illusion or a double-mirror illusion than in the case of a 3D realization of an impossible figure.

Some self-rotation illusion objects are also robust against the viewpoint. Two examples are shown in Figures 14 and 15. Figure 14 shows a rooster and its three mirror reflections seen from a high viewpoint. They form a counterclockwise sequence of rotations, and the appearance of the rooster is fairly stable.

Figure 15 shows a woman wearing a kimono. The object and its three mirror reflections together form a counterclockwise sequence of women performing a Japanese traditional Bon folk dance. This is shown from above, but the illusion is still achieved.

Figure 14. Rooster object creating the self-rotation illusion.

Figure 15. Woman object that creates a counterclockwise sequence of women performing a Bon folk dance.

It is the author's empirical observation that the robustness of the illusion becomes stronger when we use living-creatures such as humans and animals than when we use abstract figures such as circles and squares. Living creatures are deformable, and hence the variations of the shapes belonging to each creature are much wider than those belonging to each abstract figure. Therefore, when we see a living-creature object, we feel more stably that we are looking at the same object even though the shape changes due to fluctuation of the viewpoint. This must be another factor that makes the illusion robust. Therefore, the actual robustness seems to come from the combination of the geometric symmetry and the perceptual stability, which is one of our issues to study in future.

6. Concluding Remarks

We showed a new class of illusions, named the double-mirror illusion, in which both an anomalous U-turn and an anomalous translation occur, and the self-rotation illusion in which the object and its mirror images form an oriented circular sequence. We presented the conditions under which this class of objects exists, presented a method for constructing them, and also discussed the robustness from a mathematical point of view.

The illusion is remarkable in its robustness against changes in viewpoint. The robust nature can be partly understood from the symmetry of the objects. However, the actual robustness depends strongly on the individual objects. Characterizing the reasons of the robustness more precisely is our new research topic.

Acknowledgments

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