リーマン対称空間内の部分多様体のベクトル束による研究

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A study of submanifolds in Riemannian symmetric spaces by vector bundles

リーマン対称空間内の部分多様体のベクトル束による研究

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Chapter 1

Introduction

In this thesis, we study submanifolds of Riemannian symmetric spaces by vector bundles. We get new viewpoints in geometry of submanifolds using vector bundles. In particular, we attack three problems: holomorphic isometric embeddings of the projective line into quadrics, killing vector fields on complex hypersurfaces in the complex projective space, and isoparametric functions on Riemannian symmetric spaces.

So, this article has three parts. In Chapter 2, we explain Nagatomo's results about harmonic maps into Grassmann manifolds [24]. These are the main tools in this thesis. Then, we discuss holomorphic isometric embeddings of the projective line in quadrics in Chapter 3, killing vector fields on complex hypersurfaces in the complex projective space in Chapter 4, and isoparametric functions on Riemannian symmetric spaces in Chapter 5.

The study of harmonic maps from the complex projective line into complex quadrics has a long history and has been pursued by various authors in different ways, e.g. [4, 8, 20, 34]. Our particular standpoint is a generalization by Nagatomo [24] of the methods of Takahashi [29] and of do Carmo and Wallach [5], which can be summarized as follows: A well-known theorem by Takahashi [29, Theorem 3] proves that an isometric immersion of a Riemannian manifold in Euclidean space is an eigenvector for the Laplacian if and only if it is a minimal immersion in some Euclidean sphere. The energy density of maps is then related to the corresponding eigenvalue. A generalization of this result via vector bundles can be found in [24, Theorem 3.5]. The statement is that a smooth map f of a Riemannian manifold into the Grassmannian $Gr_p(W)$, where W is a real or complex vector space with a scalar product, is harmonic if and only if W satisfies the *zero property for the Laplacian*: for arbitrary $t \in W \subset \Gamma(f^*Q)$, $\Delta t = -At$, where $Q \to Gr_p(W)$ is the universal quotient bundle, Δ is the Laplace operator acting on sections and A is the *mean curvature operator* (defined in [24, §2]) related to the energy density of f. This viewpoint leads to a description in which a harmonic map from a Riemannian manifold into a Grassmannian is induced by a triple composed by a vector bundle, a space of sections of this bundle and a Laplace operator.

A celebrated example of such induced maps is Kodaira's embedding of an algebraic manifold into complex projective space [18], which in the aforesaid description is induced by a holomorphic line bundle and the space of holomorphic sections.

Takahashi's original result finds major application in do Carmo and Wallach [5] undertaking of the classification of minimal (isometric) immersions of spheres into spheres. A key role in do Carmo–Wallach theory is played by certain symmetric positive semi-definite linear operators [5, Proposition 1.3] interweaving minimal immersions: finding the space of the image-inequivalent operators amounts to describe the moduli space, an endeavour which is dealt successfully with representation theory.

From the generalized version of the theorem of Takahashi [24, Theorem 3.5], a generalisation of do Carmo–Wallach theory in terms of vector bundles is possible [24, Theorem 5.5]. We recall its principal features in Theorem 2.3.4 below. In essence, the theorem affirms that the harmonic induced map f of a Riemannian manifold into the Grassmannian $Gr_p(W)$ by the aforementioned harmonic triple is naturally equipped with a family of symmetric positive semi-definite operators determining the moduli space, as in the classical do Carmo–Wallach theory. Uniqueness of the associated symmetric operator reduces the moduli to a single point yielding rigidity of the induced map: this is the case of the *real standard* map of our Theorem 3.3.4.

For the second problem, totally geodesic submanifolds of a Riemannian manifold have the following distinct property. Let S be a totally geodesic submanifold of a Riemannian manifold M. For a Killing vector field X on M, the tangent part of the restriction of X to S is also a Killing vector field.

Then, we determine submanifolds satisfying such a property. In particular, we study the induced metrics with non-trivial Killing vector fields on complex hypersurfaces in the complex projective space.

For the third problem, we construct isoparametric functions on symmetric spaces of compact type systematically. The research of an isoparametric hypersurface, which is the regular level set of an isoparametric function, has a long history, going back to Levi-Civita and É.Cartan. We have a lot of literatures about isoparametric hypersurfaces of spaces of constant curvatures, which have constant principal curvatures. We denote by g the number of distinct principal curvatures. Amongst all, the research of an isoparametric hypersurface of a sphere is extensive and well-known. Substantial results are exhibited in [3], [9], [22], [26] and [27], etc. In [22], Münzner shows that $g = 1, 2, 3, 4, 6$ and in [9], a lot of isoparamertric functions on a sphere are systematically constructed by an algebraic method, which are called isoparametric functions of OT-FKM type. By contrast, we have few explicit examples of isoparametric functions on general Riemannian manifolds.

We utilize a homogeneous vector bundle and a section to construct an isoparametric function on an irreducible symmetric space, say G/K. To choose a vector bundle and a section, we consider an irreducible G-module W of spherical type. This means that the principal orbits are hyperspheres of W and so, we obtain a subgroup $H \subset G$ as a stabilizer. If the representation is restricted to a subgroup K, then K-submodules of W induce homogeneous vector bundles over G/K. Using an invariant metric on the bundle, we define a function $f : G/K \to \mathbf{R}$ as the square of the norm of the section.

If the action of H on G/K is of cohomogeneity one, then f is an isoparametric function. The mean curvature of the level hypersurface is also computed (Theorem 5.3.14). We have common description of $|df|^2$ and the mean curvature on any pairs $(G/K, W)$. As a by-product, we can specify the precise value whose inverse image of f is a minimal hypersurface in a family given by the isoparametric function. On the contrary, when we compute the principal curvature, we have distinct difference between pairs and no unified way (Theorems 5.3.17, 5.3.18, 5.3.19, 5.3.20 and 5.3.21). In those computations, the second fundamental forms of vector bundles [16] play essential roles and the theory developed by Nagatomo in [24] provides us with a unified method.

If the cohomogeneity of the action of H on G/K is greater than one, then the function $f : G/K \to \mathbf{R}$ is not an isoparametric function. However, we can construct a new isoparametric function $F : G/K \to \mathbf{R}^k$ in the sense of Wang [33] (see also [6, p.55]), where k denotes the cohomogeneity of H-action. One component of F consists of the function f. In the case that a chosen pair is $(\mathrm{Sp}(n)/\mathrm{U}(n), \mathbf{C}^{2n})$, F coincides with a moment map for an $\mathrm{Sp}(1)$-action on $\mathrm{Sp}(n)/\mathrm{U}(n)$.

Moreover, we can find a new isoparametric function $\tilde{f} : G/K \to \mathbf{R}$. The function $\tilde{f}$ has a larger symmetry than the original f. In short, a subgroup $\tilde{H} \subset G$ such that $H \subset \tilde{H}$ enters into our theory and $\tilde{f}$ is invariant under the action of $\tilde{H}$. The appearance of $\tilde{f}$ and $\tilde{H}$ is not accidental. We use other vector bundles and spaces of sections to explain in an algebraic and geometric way that the chosen section in §3 has really a hidden symmetry $\tilde{H} \subset G$. The $\tilde{H}$-action on G/K turns out to be of cohomogeneity one. The relation between $\tilde{f}$ and F makes some properties of H-action and level sets of F transparent. In particular, any submanifold in our family induced by F is *not* an equifocal submanifold in the sense of Terng-Thorbergsson [31].

In the final section, we interpret the reason that representations of spherical type are chosen. One of our aims in the present paper is to provide a geometric mean of constructing an isoparametric function on a sphere.

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Chapter 2

Preliminaries

2.1 Geometry of Grassmannian manifolds

We review geometry of Grassmannian manifolds, in order to fix notation and our convention in this paper. For proofs, see [24].

Let W be an N-dimensional vector space. In the case that W is a real vector space, we also consider the orientation of W.

Let $Gr_p(W)$ be a Grassmannian manifold of (oriented) p-planes in W and $S \to Gr_p(W)$ a tautological vector bundle. Since $S \to Gr_p(W)$ is regarded as a subbundle of a trivial vector bundle $\underline{W} \to Gr_p(W)$ of fiber W, we have an exact sequence of vector bundles:

$$0 \to S \xrightarrow{i_S} \underline{W} \xrightarrow{\pi_Q} Q \to 0.$$

The quotient bundle $Q \to Gr_p(W)$ is called the *universal quotient bundle*. The tangent bundle is identified with $S^* \otimes Q$. (More precisely, the holomorphic tangent bundle is identified with $S^* \otimes Q$ in case of complex Grassmannian.)

We fix a scalar product $(\cdot, \cdot)$ on W. On the one hand, the orthogonal projection gives a bundle surjection $\pi_S : \underline{W} \to S$. On the other hand, $Q \to Gr_p(W)$ is regarded as the orthogonal complementary bundle $S^\perp \to Gr_p(W)$ to $S \to Gr_p(W)$, and so we obtain a bundle injection $i_Q : Q \to \underline{W}$. The vector bundles $S \to Gr_p(W)$ and $Q \to Gr_p(W)$ are equipped with metrics g_S and g_Q, respectively.

We can define a connection ∇^Q on $Q \to Gr_p(W)$ using a trivialization of $\underline{W} \to Gr_p(W)$ with an orthonormal basis. If t is a section of $Q \to Gr_p(W)$, then $i_Q(t)$ is considered as a W-valued function. Then we have

$$d\left(i_Q(t)\right) = \pi_S\left(d\left(i_Q(t)\right)\right) + \pi_Q\left(d\left(i_Q(t)\right)\right).$$

The connection $\nabla^Q t = \pi_Q\left(d\left(i_Q(t)\right)\right)$ is nothing but the canonical connection. The other term in right hand side $\pi_S\left(d\left(i_Q(t)\right)\right)$ is a 1-form with values in $\mathrm{Hom}(Q, S) \cong Q^* \otimes S$

which is called *the second fundamental form* in the sense of Kobayashi [16] and denoted by J.

In a similar way, if s is a section of $S \to Gr_p(W)$, then we have

$$d\left(i_S(s)\right) = \pi_S\left(d\left(i_S(s)\right)\right) + \pi_Q\left(d\left(i_S(s)\right)\right).$$

The canonical connection is expressed as $\nabla^S s = \pi_S\left(d\left(i_S(s)\right)\right)$ and we define the second fundamental form $I = \pi_Q d i_S$, which is a 1-form with values in $\mathrm{Hom}(S, Q) \cong S^* \otimes Q$.

In the case of a complex Grassmannian, we can also consider complex analytical structures. Canonical connections give holomorphic structures to $S \to Gr_p(W)$ and $Q \to Gr_p(W)$. In particular, W can be regarded as the space of holomorphic sections of $Q \to Gr_p(W)$ by a theorem of Borel-Weil. The second fundamental form $I \in \Omega^1(\mathrm{Hom}(S, Q))$ is of type $(1, 0)$ and The second fundamental form $J \in \Omega^1(\mathrm{Hom}(Q, S))$ is of type $(0, 1)$.

Since the (holomorphic) tangent bundle is identified with $S^* \otimes Q$, we can induce a Riemannian metric g_{Gr} on a Grassmannian.

• **real case** We have

$$g_{Gr}(X, Y) = -\mathrm{trace}\, J_Y I_X = -\mathrm{trace}\, I_Y J_X,$$

where X and Y are tangent vectors.

• **complex case** Let h_{Gr} be the Hermitian metric on the holomorphic tangent bundle $T_{1,0}$ induced by Hermitian metrics g_S and g_Q. The definition yields that

$$h_{Gr}(Z, W) = -\mathrm{trace}\, J_{\overline{W}} I_Z,$$

where Z and W are $(1, 0)$-vectors. Consequently we have

$$\begin{aligned} g_{Gr}(X, Y) &= -\ \mathrm{trace}\, J_Y I_X - \mathrm{trace}\, J_X I_Y \\ &= -\ \mathrm{trace}\, I_Y J_X - \mathrm{trace}\, I_X J_Y, \end{aligned}$$

where X and Y are (real) tangent vectors.

The Levi-Civita connection D is nothing but a connection induced by ∇^S and ∇^Q.

Proposition 2.1.1. *The second fundamental forms I and J are parallel.*

For a vector $w \in W$, we have two sections $s = \pi_S(w)$ and $t = \pi_Q(w)$, each of which is sometimes called *the section corresponding to w*. Obviously, we have

Proposition 2.1.2. *If s and t are the sections corresponding to $w \in W$, then*

$$\nabla^S s = -Jt, \quad \nabla^Q t = -Is.$$

Lemma 2.1.3. *The second fundamental forms I and J satisfy*

$$g_Q(Is, t) = -g_S(s, Jt).$$

We now can easily compute $(\nabla^S)^2$ and $(\nabla^Q)^2$. If s and t are the corresponding sections to $w \in W$, then we have

$$(\nabla^S)^2 s = \nabla^S(-Jt) = -(\nabla J)(t) - J(\nabla^Q t) = JIs,$$

$$(\nabla^Q)^2 t = \nabla^S(-Is) = -(\nabla I)(s) - I(\nabla^S s) = IJt.$$

More precisely, we have

$$\nabla^S_X(\nabla^S s)(Y) = J_Y I_X s, \quad \nabla^Q_X(\nabla^Q t)(Y) = I_Y J_X t.$$

For instance, we take the trace of $(\nabla^S)^2$ to define the Laplace operator: $\Delta s = -\sum_{i=1}^n \nabla^S_{e_i}(\nabla^S s)(e_i)$. We see that sections s and t are eigensections of the Laplacian ($\Delta s = qs, \quad \Delta t = pt$, where $q = N - p$).

2.2 Totally geodesic immersions into Grassmannians

Let (G, K) be an irreducible symmetric pair of compact type, where G is a simply-connected compact Lie group and K is a closed subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebras. The standard decomposition is expressed as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.

Let $\varrho : G \to \mathrm{GL}(W)$ be an irreducible representation with an G-invariant scalar product. For simplicity, we do not distinguish a representation $\varrho : G \to \mathrm{GL}(W)$ from the representation space W. We assume that W has a non-trivial K-invariant orthogonal decomposition $W = U \oplus V$ such that $\mathfrak{m}U \subset V$ and $\mathfrak{m}V \subset U$. (Non-trivial decomposition means that neither U nor V is zero-dimensional.) Such a decomposition is called a *generalized Cartan decomposition* of W. More generally, we define

Definition 2.2.1. Let $\varrho : G \to \mathrm{GL}\, W$ be an orthogonal or unitary representation of G. The (ϱ, W) has a generalized Cartan decomposition (for the symmetric pair (G, K)) if W is decomposed into two non-zero K-modules $W = U_0 \oplus V_0$ over the same coefficient field as that of W under the restriction of the homomorphism ϱ to a subgroup K, in such a way that

$$\varrho(\mathfrak{m})U_0 \subset V_0, \quad \varrho(\mathfrak{m})V_0 \subset U_0, \quad U_0 \perp V_0,$$

and neither U_0 or V_0 is a G-module (in other words, $\varrho(\mathfrak{m})U_0 \neq \{0\}$ and $\varrho(\mathfrak{m})V_0 \neq \{0\}$). The decomposition $W = U_0 \oplus V_0$ is called a a generalized Cartan decomposition, more accurately, a real generalized Cartan decomposition or a complex generalized Cartan decomposition according to the coefficient field of W.

Assume that W has a generalized Cartan decomposition : $W = U \oplus V$. Let $\dim U = p$ and $\dim V = q$. We define an immersion $i : G/K \to Gr_p(W)$ by

$$i\,(gK) = \varrho(g)U, \quad g \in G.$$

We assume throughout this paper that a Riemannian metric on G/K is provided in such a way that the immersion $i : G/K \to Gr_p(W)$ is an *isometric* immersion. Then $i : G/K \to Gr_p(W)$ is indeed a totally geodesic immersion.

We can define two homogeneous vector bundles $G \times_K U$ and $G \times_K V$ with canonical connections, which are denoted by $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$. Frobenius reciprocity yields that W can be regarded as a finite dimensional space of sections of $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$. More precisely, $\pi_U : W \to U$ and $\pi_V : W \to V$ denote the orthogonal projections. For $w \in W$, we put

$$s([g]) := \left[g, \pi_U(g^{-1}w)\right], \quad t([g]) := \left[g, \pi_V(g^{-1}w)\right],$$

where $g \in G$ and $[g] \in G/K$. The sections $s \in \Gamma(\mathbf{U})$ and $t \in \Gamma(\mathbf{V})$ are also called the *corresponding sections to* $w \in W$.

From the construction, $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$ are pull-back bundles of the tautological bundle and the universal quotient bundle over $Gr_p(W)$, respectively. Then the pull-back connections are the same as the canonical connections. We can also pull-back the second fundamental forms I and J which are sections of $i^*T^* \otimes \mathrm{Hom}(\mathbf{U}, \mathbf{V})$ and $i^*T^* \otimes \mathrm{Hom}(\mathbf{V}, \mathbf{U})$, respectively, where T^* is the cotangent bundle of Grassmannian. Using the projection $i^*T^* \to T^*G/K$, the pull-backs of I and J are the second fundamental forms of vector bundles, and so we denote by the same symbol the pull-backs of the second fundamental forms.

Theorem 2.2.2. [24, Lemma 4.1] *A map $f : G/K \to Gr_p(W)$ is totally geodesic (i.e.$\nabla df = 0$) if and only if the second fundamental form I of vector bundles is parallel.*

Proof. Since we have a fundamental relation $\nabla I = I_{\nabla df}$, the result follows. □

We define an endomorphism $A \in \Gamma(\mathrm{End}(\mathbf{V}))$ by

$$A = \sum_{i=1}^{n} I_{e_i} J_{e_i}, \quad n = \dim G/K,$$

where $e_1, \cdots, e_n$ is an orthonormal basis of the tangent space of G/K. We call A the *mean curvature operator*. Notice that A can be defined in a similar way, even if the domain is a Riemannian manifold [24]. Then we have

Theorem 2.2.3. [24, Theorem 3.5] *Let (M, g) be an n-dimensional Riemannian manifold and $F : M \to Gr_p(W)$ a smooth map. We fix an inner product or a Hermitian inner product $(\cdot, \cdot)$ on W.*

Then, the following two conditions are equivalent.

1. *$F : M \to Gr_p(W)$ is a harmonic map.*

2. $\Delta t + At = 0$ *for an arbitrary* $t \in W$*, where the vector space* W *is regarded as a space of sections of the pull-back bundle* $F^*Q \to M$*.*

Under these conditions, we have

$$|df|^2 = -\mathrm{trace}\, A.$$

The role of the universal quotient bundle in Theorem 2.2.3 can be replaced by the tautological bundle. To do so, we define an endomorphism B of $\mathbf{U} \to G/K$ by

$$B = \sum_{i=1}^{n} J_{e_i} I_{e_i},$$

which is also called the mean curvature operator.

2.3 The generalization of the theorem of do Carmo–Wallach

In this section we give a short account of results in [24] needed to state a version of the generalization of the theorem of do Carmo–Wallach (Theorem 2.3.4), whose implications will be applied later in this article.

Suppose $\mathbf{V} \to M$ is a complex (resp. real/real oriented) vector bundle of rank q and consider an N-dimensional space of sections $W \subset \Gamma(\mathbf{V})$. By definition of $\underline{W} \to M$, there is a bundle homomorphism $ev : \underline{W} \to \mathbf{V}$, called *evaluation*, defined by $(x, t) \mapsto t(x)$ for all $t \in W, x \in M$. The vector bundle $\mathbf{V} \to M$ is said to be *globally generated by* W if the evaluation is surjective. Under this hypothesis, there is a map $f : M \to Gr_p(W)$, where $Gr_p(W)$ is a complex (resp. real/real oriented) Grassmannian and $p = N - q$, defined by

$$f(x) = \mathrm{Ker}\, ev_x = \{t \in W \,|\, t(x) = 0\}\,,$$

where $ev_x \equiv ev(x, \cdot)$. The map f is said to be *induced by* the couple $(\mathbf{V} \to M,\ W)$, or simply by W if the vector bundle $\mathbf{V} \to M$ is specified (cf. [24]).

Notice that, by the definition of induced map, $\mathbf{V} \to M$ can be *naturally identified* with $f^*Q \to M$. Therefore, given a smooth map $f : M \to Gr_p(W)$, it can be regarded as the induced map determined the by the couple $(f^*Q \to M,\ W)$. If the linear map of $W \subset \Gamma(\mathbf{V})$ into $\Gamma(f^*Q)$ is injective, we say that the map f is *full* [24, Definition 5.2]. This definition of fullness coincides with the ones used in [5] when the target space is the sphere or complex projective space.

Moreover, assume M to be Riemannian and $\mathbf{V} \to M$ to be equipped with a fiber-metric and a connection. From these data a Laplace operator acting on sections can be defined.

The model special case is that in which M is a compact reductive homogeneous space G/K (where G is a compact Lie group and K is a closed subgroup of G), and $\mathbf{V} \to M$ is a homogeneous complex (resp. real) vector bundle of rank q, i.e. $\mathbf{V} \cong G \times_K V_0$ where V_0 is a q-dimensional complex (resp. real) K-module (cf. [24]). If additionally V_0 admits a K-invariant Hermitian (resp. symmetric) inner product, $\mathbf{V} \to M$ inherits a G-invariant Hermitian (resp. symmetric) fiber-metric.

Because of reductivity, $\mathbf{V} \to M$ is equipped with a canonical connection too, the one for which the horizontal subspace on the principal K-bundle $G \to M$ is given by the complement $\mathfrak{m}$ to $\mathfrak{k} = L(K)$ in $\mathfrak{g} = L(G)$.

Using the Levi–Civita connection and the canonical connection, $\Gamma(\mathbf{V})$ can be decomposed into eigenspaces of the Laplacian each being a finite-dimensional not necessarily irreducible G-module and equipped with a G-invariant L^2-inner product. Then, we say that the induced map by $(\mathbf{V} \to M,\ W)$ is *standard* if a G-submodule $W \subseteq W_\mu$ globally generates the bundle, where W_μ is the eigenspace of the Laplacian with eigenvalue μ.

Evidently, the definition of standard map generalizes the special homogeneous case. However, the homogeneous setting will be enough for the purposes of the present work.

The spaces of sections inducing standard maps have the following interesting property which will be useful later:

Lemma 2.3.1 ([24, Lemma 5.17]). *Let W be a G-subspace of W_μ. If W globally generates $\mathbf{V} \to G/K$, then V_0 can be regarded as a subspace of W.*

Denote by U_0 the orthogonal complement of V_0 in W. Then, the induced standard map $f_0 : M \to Gr_p(W)$ is expressed as

$$f_0([g]) = gU_0 \subset W,$$

for all $[g] \in G/K$, and is G-equivariant.

Notice that, besides its assumed fiber-metric and connection, $\mathbf{V} \to M$ is endowed with a secondary couple of fiber-metric and connection inherited from the natural identification $\phi : \mathbf{V} \cong f^*Q$, i.e. the fiber-metric and canonical connection on $Q \to Gr_p(W)$ pulled-back to $f^*Q \to M$. In general, these structures do not need to be gauge equivalent unless the splitting $W = U_0 \oplus^\perp V_0$ satisfies extra conditions:

Lemma 2.3.2 ([24, Lemma 5.18]). *The pull-back connection is gauge equivalent to the canonical connection if and only if*

$$\mathfrak{m}V_0 \subset U_0.$$

Lemma 2.3.3 ([24, Lemma 5.19]). *If a G-representation $W \subseteq W_\mu$ globally generates $\mathbf{V} \to M$ and satisfies the condition $\mathfrak{m}V_0 \subset U_0$, then the standard map $f_0 : M \to Gr_p(W)$ is harmonic with constant energy density $e(f_0) = q\mu$ and the mean curvature operator is proportional to the identity $A = -\mu Id_{\mathbf{V}}$.*

Let us introduce the two increasingly stronger equivalence relations [24, Definitions 5.3 and 5.4], up to which we shall later define moduli spaces of maps. Let f_1 and f_2 : $M \to Gr_p(W)$. Then f_1 is called *image equivalent* to f_2 if there exists an isometry ψ of $Gr_p(W)$ such that $f_2 = \psi \circ f_1$. Next, we fix a vector bundle $\mathbf{V} \to M$, a fiber metric and a connection compatible with the metric. Furthermore, denote by $\tilde{\psi}$ the bundle isomorphism of $Q \to Gr_p(W)$ which covers the isometry ψ of $Gr_p(W)$. Then, the pair (f_1, ϕ_1) is said to be *gauge equivalent* to (f_2, ϕ_2), where $\phi_i : \mathbf{V} \to f_i^*Q (i = 1, 2)$ are bundle isomorphisms, if there exists an isometry ϕ of $Gr_p(W)$ such that $f_2 = \psi \circ f_1$ and $\phi_2 = \tilde{\psi} \circ \phi_1$.

Aside from the geometric background, some algebraic preliminaries regarding Hermitian/symmetric operators are needed.

Let G be a compact Lie group, W a complex G-module together with an invariant Hermitian product $(\ ,\ )_W$ and denote by $\mathrm{H}(W)$ the set of Hermitian endomorphisms of W. We equip $\mathrm{H}(W)$ with a G-invariant inner product $(A, B)_H = \operatorname{trace} AB$, for $A, B \in \mathrm{H}(W)$. Define a Hermitian operator $\mathrm{H}(u, v)$ for u, $v \in W$ as

$$\mathrm{H}(u, v) := \frac{1}{2} \left\{ u \otimes (\cdot, v)_W + v \otimes (\cdot, u)_W \right\}.$$

If U and V are subspaces of W, we define a real subspace $\mathrm{H}(U, V) \subset \mathrm{H}(W)$ spanned by $\mathrm{H}(u, v)$ where $u \in U$ and $v \in V$. In a similar fashion, $\mathrm{GH}(U, V)$ denotes the subspace of $\mathrm{H}(W)$ spanned by $g\mathrm{H}(u, v)$, where $g \in \mathrm{G}$.

If W is a real G-module together with an invariant inner product, then symmetric endomorphisms take the place of Hermitian ones and we get analogous definitions of $\mathrm{S}(W)$, $\mathrm{S}(u, v)$, $\mathrm{S}(U, V)$, $\mathrm{GS}(U, V)$.

Now we have all the needed ingredients to introduce a version of the generalization of the theorem of do Carmo–Wallach for holomorphic maps.

Theorem 2.3.4. *Let $M = G/K$ be a compact irreducible Hermitian symmetric space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and fix a complex homogeneous line bundle $\mathbf{V} = G \times_K V_0$ over M with invariant metric h and canonical connection ∇. Regard $\mathbf{V} \to M$ as a real vector bundle with complex structure J^c. Finally, let $f : M \to Gr_n(\mathbf{R}^{n+2})$ be a full holomorphic map satisfying the following two conditions:*

(G) *The pull-back $f^*Q \to M$ of the universal quotient bundle $Q \to Gr_n(\mathbf{R}^{n+2})$ with the pull-back metric, connection and complex structure is gauge equivalent to $\mathbf{V} \to M$ with h, ∇ and J^c.*

(EH) *The mean curvature operator $A \in \Gamma(\operatorname{End} \mathbf{V})$ of f is expressed as $-\mu\, Id_V$ with some positive real number μ, and so $e(f) = 2\mu$.*

Hence the space of holomorphic sections $W = H^0(\mathbf{V}) \subset \Gamma(\mathbf{V})$ is also an eigenspace of the Laplacian with eigenvalue μ. Regard W as a real vector space with L^2-inner product $(\cdot, \cdot)_W$ induced from the L^2-Hermitian product. Then, there exists a positive semi-definite symmetric endomorphism $T \in \mathrm{S}(W)$ such that the pair (W, T) satisfies the following three conditions:

(I) *The vector space* $\mathbf{R}^{n+2}$ *is a subspace of* W *with the inclusion* $\iota : \mathbf{R}^{n+2} \to W$ *preserving the orientation, and* $V \to M$ *is globally generated by* $\mathbf{R}^{n+2}$.

(II) *As a subspace,* $\mathbf{R}^{n+2} = \operatorname{Ker} T^{\perp}$ *and the restriction of* T *is a positive definite symmetric endomorphism of* $\mathbf{R}^{n+2}$.

(III) *The endomorphism* T *satisfies the orthogonality conditions*

$$\left(T^2 - Id_W, \mathrm{GH}(V_0, V_0)\right)_H = 0, \qquad \left(T^2, \mathrm{GH}(\mathfrak{m}V_0, V_0)\right)_H = 0. \tag{2.3.1}$$

By (I), (II) *and* (III), *the endomorphism* T *provides the following.*

(a) *A holomorphic totally geodesic embedding of* $Gr_n(\mathbf{R}^{n+2})$ *into* $Gr_{n'}(W)$ *by* $U \mapsto U \oplus \operatorname{Ker} T$ *where* $n' = n + \dim \operatorname{Ker} T$

(b) *A bundle isomorphism* $\phi = T \circ ev^* : \mathbf{V} \to f^*Q$ *which preserves the metric* h *and the connection* ∇ *where* ev^* *is the adjoint bundle map of* ev *with respect to* h *and* $(\cdot,\cdot)_W$. *We consider that* $T \circ ev^*$ *is a map of* $\mathbf{V} \to f^*Q$ *by the identification* $(f^*Q)_x = (\operatorname{Ker} ev_x)^{\perp}$ $(x \in M)$.

(c) *A expression of* $f : M \to Gr_n(\mathbf{R}^{n+2})$,

$$f\left([g]\right) = \left(\iota^* T \iota\right)^{-1} \left(f_0\left([g]\right) \cap \operatorname{Ker} T^{\perp}\right), \tag{2.3.2}$$

where ι^* *denotes the adjoint operator of* ι *under the induced inner product on* $\mathbf{R}^{n+2}$ *from* $(\cdot,\cdot)_W$ *on* W *and* f_0 *is the standard map by* W.

Moreover two such pairs (f_i, ϕ_i), $(i = 1, 2)$ *are gauge equivalent if and only if* $\operatorname{Ker} T_1 = \operatorname{Ker} T_2$ *and* $\iota_1^* T_1 \iota_1 = \iota_2^* T_2 \iota_2$, *where* (T_i, ι_i) *correspond to* f_i $(i = 1, 2)$ *under the expression in* (2.3.2), *respectively.*

Conversely, suppose that a vector space $\mathbf{R}^{n+2}$, *the space of holomorphic sections* $W \subset \Gamma(\mathbf{V})$ *regarded as real vector space and a positive semi-definite symmetric endomorphism* $T \in \mathrm{S}(W)$ *satisfying conditions* (I), (II) *and* (III) *are given. Then we have a unique holomorphic embedding of* $Gr_n(\mathbf{R}^{n+2})$ *into* $Gr_{n'}(W)$ *and the map* $f : M \to Gr_n(\mathbf{R}^{n+2})$ *defined by* (2.3.2) *is a full holomorphic map into* $Gr_n(\mathbf{R}^{n+2})$ *satisfying conditions* (G) *and* (EH) *with bundle isomorphism* $\mathbf{V} \cong f^*Q$.

Proof. This is obtained by a combination of Theorems 5.16 and 5.20 in [24], themselves refinements and following the same proof as that of Theorem 5.5. □

Remark 1. Conditions (G) and (EH) in the theorem are named respectively gauge and Einstein–Hermitian conditions.

Chapter 3

Holomorphic isometric embeddings of the projective line into quadrics

3.1 Holomorphic isometric embeddings

The aim of this section is to introduce holomorphic isometric embeddings from $\mathbf{C}P^1$ into $Gr_n(\mathbf{R}^{n+2})$ and to show that they satisfy the hypothesis of Theorem 2.3.4. Then the universal quotient bundle has a holomorphic bundle structure. Notice that the curvature two-form R of the canonical connection on the universal quotient bundle is the fundamental two-form ω_Q on $Gr_n(\mathbf{R}^{n+2})$ up to a constant multiple:

$$R = -2\pi\sqrt{-1}\omega_Q.$$

Denote by ω_0 the fundamental two-form on $\mathbf{C}P^1$. When R_1 denotes the curvature two-form of the canonical connection on the hyperplane bundle $\mathcal{O}(1) \to \mathbf{C}P^1$ (cf. [10, p. 145]), we also have $R_1 = -2\pi\sqrt{-1}\omega_0$. In what follows, we will denote by $\mathcal{O}(k) \to \mathbf{C}P^1$ the k-th tensor power of the hyperplane bundle.

Definition 1. Let $f : \mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_n(\mathbf{R}^{n+2})$ be a holomorphic embedding. Then f is called an isometric embedding of degree k if $f^*\omega_Q = k\omega_0$ (and so, k must be a positive integer).

In order to show that holomorphic isometric embeddings $\mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_n(\mathbf{R}^{n+2})$ satisfy the conditions of Theorem 2.3.4 we need the following two lemmas. Their proofs rely heavily on properties of the (unique) Einstein–Hermitian connection. For additional details we refer the interested reader to the excellent book by Kobayashi [16, Ch. IV].

Lemma 3.1.1. *Let $f : \mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_n(\mathbf{R}^{n+2})$ be a holomorphic embedding. Then f is an isometric embedding of degree k if and only if the pull-back bundle $f^*Q \to \mathbf{C}P^1$ with the pull-back connection is gauge equivalent to $\mathcal{O}(k) \to \mathbf{C}P^1$ with the canonical connection.*

Proof. If the degree of the isometric embedding f equals k, the pull-back of the universal quotient bundle is holomorphically isomorphic to the holomorphic line bundle of degree

k on $\mathbf{C}P^1$ (by uniqueness of the holomorphic bundle structure), which by homogeneity admits a unique Einstein–Hermitian structure up to homotheties of the fiber-metric (cf. [16, Proposition IV.6.1]). Uniqueness of the Einstein–Hermitian connection yields the result.

Conversely, if the pull-back of the universal quotient bundle is holomorphically isomorphic as Einstein–Hermitian bundle to the holomorphic line bundle, the pull-back fiber-metric and the Einstein–Hermitian connection coincide up to homothety, and the statement in the lemma follows. □

Lemma 3.1.2. *Let $f : \mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_n(\mathbf{R}^{n+2})$ be a holomorphic isometric embedding of degree k. Then, the mean curvature operator $A \in \Gamma(\mathbf{V})$ of f is the identity on $\mathbf{V}$ up to a negative real constant.*

Proof. It is well-known that every holomorphic section t of $\mathcal{O}(k) \to \mathbf{C}P^1$ satisfies $\Delta t - K_{EH}t = 0$ (cf. [24, Lemma 4.2]), where the Laplacian is defined through a compatible connection, and K_{EH} is the mean curvature arising from the Hermitian structure in the sense of Kobayashi [16, p. 99]. Since the canonical connection is the Einstein–Hermitian connection, $K_{EH} = \mu Id$.

On the other hand, a generalization of the theorem of Takahashi (Theorem 2.2.3) yields that $\Delta t + At = 0$ for $t \in \mathbf{R}^{n+2}$. Regarding $\mathbf{R}^{n+2}$ as a subspace of $H^0(\mathbf{C}P^1, \mathcal{O}(k))$, then $\mathbf{R}^{n+2}$ globally generates $\mathcal{O}(k) \to \mathbf{C}P^1$. Therefore $K_{EH} = -A$, and the lemma follows. □

These two lemmas amount to say that the holomorphic embedding f is isometric if and only if it satisfies the gauge condition (G), and then the (EH) condition is automatically satisfied. Hence we can apply Theorem 2.3.4 to obtain the moduli space $\mathcal{M}_k$ of holomorphic isometric embeddings of degree k by the gauge equivalence of maps.

3.2 Hermitian/Symmetric endomorphisms

In order to apply the generalized do Carmo–Wallach theory we need a deeper understanding of the space of symmetric endomorphisms of the space of holomorphic sections of the bundles of interest. Since in the present work the spaces of holomorphic sections are real SU(2)-modules, in this section we describe how the space of symmetric endomorphisms of a real irreducible SU(2)-module splits into irreducible components. To do so we need certain spectral formulae for decomposing tensor products of real SU(2)-modules. Being standard, proofs of the spectral formulae (Lemmas 3.2.4–3.2.6) are ommitted. The interested reader might consult [1].

Let W be a $\mathbf{C}$-vector space with a Hermitian inner product and write $W_{\mathbf{R}}$ for the underlying $\mathbf{R}$-vector space naturally equipped with the complex structure J^c. The Hermitian inner product induces a symmetric inner product on $W_{\mathbf{R}}$, simply by taking the real part.

If $\mathrm{H}(W)$ denotes the $\mathbf{R}$-vector space of Hermitian endomorphisms on W and $\mathrm{S}(W_{\mathbf{R}})$

the $\mathbf{R}$-vector space of all symmetric endomorphisms on $W_{\mathbf{R}}$, it follows from general considerations above that $\mathrm{H}(W) \subset \mathrm{S}(W_{\mathbf{R}})$, while $\mathbf{C}$-linearity of $A \in \mathrm{H}(W)$ is reflected in $\mathrm{S}(W_{\mathbf{R}})$ by commutation of A and J^c.

Suppose that W has a real (resp. quaternionic) structure denoted by σ compatible with the Hermitian inner product. Then $\mathrm{H}(W)$ has a regular action of σ such that $A \mapsto \sigma A \sigma$, where A is a Hermitian endomorphism. Hence, we can define the subspaces $\mathrm{H}_{\pm}(W)$ of $\mathrm{H}(W)$ as the set of invariant/anti-invariant Hermitian endomorphisms with respect to σ. The action of σ extends to $\mathrm{S}(W_{\mathbf{R}})$ in the obvious way.

Lemma 3.2.1. *If $A \in \mathrm{H}_+(W)$, then real endomorphisms σA and $J^c\sigma A$ are symmetric endomorphisms on $W_{\mathbf{R}}$.*

Proof. For simplicity, we assume that σ is a real structure. If σ is a quaternionic structure the proof goes along the same lines.

Let $A \in \mathrm{H}_+(W)$ so that $\sigma A = A\sigma$. Also, denote the Hermitian inner product on W by $(\ ,\ )$, with the convention in which it is $\mathbf{C}$-linear in the first argument, and let $\langle\ ,\ \rangle$ be the induced symmetric inner product on $W_{\mathbf{R}}$. Then, for $u, v \in W \cong W_{\mathbf{R}}$,

$$\begin{aligned} \langle \sigma A u, v\rangle &= \mathrm{Re}(\sigma A u, v) = \mathrm{Re}\overline{(Au, \sigma v)} = \mathrm{Re}(u, A\sigma v) \\ &= \mathrm{Re}(A\sigma v, u) = \mathrm{Re}(\sigma A v, u) = \langle \sigma A v, u\rangle. \end{aligned}$$

Therefore, $\sigma A \in \mathrm{S}(W_{\mathbf{R}})$. The proof for $J^c\sigma A$ is analogous. □

Notice that σA (resp. $J^c\sigma A$) above is not a Hermitian operator since σ is by definition conjugate-linear. We put

$$\sigma\mathrm{H}_+(W) := \{\sigma A \mid A \in \mathrm{H}_+(W)\} \subset \mathrm{S}(W_{\mathbf{R}}),$$

$$J^c\sigma\mathrm{H}_+(W) := \{J^c\sigma A \mid A \in \mathrm{H}_+(W)\} \subset \mathrm{S}(W_{\mathbf{R}}).$$

A characterization of these subspaces is given as follows:

Lemma 3.2.2. *Let B be a symmetric endomorphism of $W_{\mathbf{R}}$. Then,*

1. *B belongs to $\sigma\mathrm{H}_+(W)$ if and only if $J^cB = -BJ^c$ and $\sigma B\sigma = B$;*
2. *B belongs to $J^c\sigma\mathrm{H}_+(W)$ if and only if $J^cB = -BJ^c$ and $\sigma B\sigma = -B$.*

Proof. For B in $\sigma\mathrm{H}_+(W)$ (resp. in $J^c\sigma\mathrm{H}_+(W)$), there exists $A \in \mathrm{H}_+(W)$ such that $B = \sigma A$ (resp. $J^c\sigma A$). Writing BJ^c, $\sigma B\sigma$ in terms of A, then commutation relations for A, J^c, σ yield the implications.

Conversely, condition $J^cB = -BJ^c$ implies that B is not Hermitian. Hence, $A := \sigma B$ (resp. $A := J^c\sigma B$) is Hermitian, for commutation relations between J^c and σ lead to $AJ^c = J^cA$. Invariance under the regular action of σ on $\mathrm{H}(W)$ shows $A \in \mathrm{H}_+(W)$, therefore B belongs to $\sigma\mathrm{H}_+(W)$ (resp. $J^c\sigma\mathrm{H}_+(W)$). □

Subspaces $\sigma\mathrm{H}_+(W)$ and $J^c\sigma\mathrm{H}_+(W)$ are orthogonal with respect to the inherited inner product on $\mathrm{S}(W_{\mathbf{R}})$, Then, counting dimensions we have

Corollary 3.2.3. *We have a decomposition of* $\mathrm{S}(W_{\mathbf{R}})$*:*

$$\mathrm{S}(W_{\mathbf{R}}) = \mathrm{H}_+(W) \oplus \mathrm{H}_-(W) \oplus \sigma\mathrm{H}_+(W) \oplus J^c\sigma\mathrm{H}_+(W).$$

Remark 2. As a result, the orthogonal complement of $\mathrm{H}(W)$ in $\mathrm{S}(W_{\mathbf{R}})$ has the induced complex structure.

Let $S^k\mathbf{C}^2$ be the k-th symmetric power of the standard complex $\mathrm{SU}(2)$-module $\mathbf{C}^2$. Since $\mathbf{C}^2$ has an invariant quaternionic structure j, $S^{2k}\mathbf{C}^2$ inherits an invariant real structure $\sigma = j^{2k}$, while $S^{2k+1}\mathbf{C}^2$ is equipped with an induced invariant quaternionic structure j^{2k+1}. We shall denote the standard real $\mathrm{SO}(3)$-module by $\mathbf{R}^3$ and its l-th symmetric power by $S^l\mathbf{R}^3$.

The fundamental relation between real irreducible $\mathrm{SU}(2)$- and $\mathrm{SO}(3)$-modules is as follows.

Lemma 3.2.4. *For $k \geqq 2$, $S^k\mathbf{R}^3$ admits the following decomposition:*

$$S^k\mathbf{R}^3 = S_0^k\mathbf{R}^3 \oplus S^{k-2}\mathbf{R}^3$$

where

$$S_0^k\mathbf{R}^3 = (S^{2k}\mathbf{C}^2)^{\mathbf{R}}$$

is the real irreducible $\mathrm{SU}(2)$*-module defined as the σ-invariant real subspace of $S^{2k}\mathbf{C}^2$.*

Once we have identified the real irreducible $\mathrm{SU}(2)$-modules we would like to have a spectral formula for the tensor product. To that end, it is enough to restrict to the real stable subspace of the real structure.

Lemma 3.2.5. *For $k \geqq l$, we have*

$$S_0^k\mathbf{R}^3 \otimes S_0^l\mathbf{R}^3 = \bigoplus_{r=0}^{2l} S_0^{k+l-r}\mathbf{R}^3. \tag{3.2.1}$$

Any complex irreducible $\mathrm{SU}(2)$-module $S^n\mathbf{C}^2$ can be interpreted as a real module by considering its underlying $\mathbf{R}$-vector space $\mathbf{R}^{2n+2}$. For odd n, this is a real irreducible module. When n is even, this is reducible and we have further splittings into the stable subspaces for the action of the induced real structure.

It will be useful to have a spectral formula for the decomposition of tensor products of the underlying $\mathbf{R}$-vector spaces of a given complex $\mathrm{SU}(2)$-module into real irreducible ones.

Lemma 3.2.6. *When we regard $S^{2k}\mathbf{C}^2$ as a real* $\mathrm{SU}(2)$*-module* $\mathbf{R}^{4k+2}$*, the second symmetric power* $S^2\mathbf{R}^{4k+2}$ *has the following irreducible decomposition:*

$$S^2\mathbf{R}^{4k+2} = 3\left(\bigoplus_{r=0}^{k} S_0^{2k-2r}\mathbf{R}^3\right) \oplus \left(\bigoplus_{r=0}^{k-1} S_0^{(2k-1)-2r}\mathbf{R}^3\right). \tag{3.2.2}$$

When we regard $S^{2k+1}\mathbf{C}^2$ as a real $\mathrm{SU}(2)$*-module* $\mathbf{R}^{4k+4}$*, the second symmetric power* $S^2\mathbf{R}^{4k+4}$ *has the following irreducible decomposition:*

$$S^2\mathbf{R}^{4k+4} = 3\left(\bigoplus_{r=0}^{k} S_0^{(2k+1)-2r}\mathbf{R}^3\right) \oplus \left(\bigoplus_{r=0}^{k-1} S_0^{2k-2r}\mathbf{R}^3\right). \tag{3.2.3}$$

Applying Corollary 3.2.3 to the real $\mathrm{SU}(2)$-modules discussed in the previous three lemmas yields

Proposition 3.2.7.

$$\mathrm{H}_+(S^{2k}\mathbf{C}^2) = \bigoplus_{r=0}^{k} S_0^{2k-2r}\mathbf{R}^3, \qquad \mathrm{H}_-(S^{2k}\mathbf{C}^2) = \bigoplus_{r=0}^{k-1} S_0^{2k-1-2r}\mathbf{R}^3,$$
$$\mathrm{H}_+(S^{2k+1}\mathbf{C}^2) = \bigoplus_{r=0}^{k} S_0^{2k+1-2r}\mathbf{R}^3, \qquad \mathrm{H}_-(S^{2k+1}\mathbf{C}^2) = \bigoplus_{r=0}^{k} S_0^{2k-2r}\mathbf{R}^3.$$

3.3 Rigidity of the real standard map

Let G be a compact Lie group. An irreducible G-module is said to be a *class-one representation of* (G, K), for K a closed subgroup of G, if it contains non-zero K-invariant elements.

Essential at this stage is to prove Proposition 3.3.3 (and its real invariant counterpart Proposition 3.3.5). This is a technical result that states in short that if each factor in the normal decomposition of a G-module W is inequivalent as a K-module to any other factor, there is a certain G-orbit in $\mathrm{H}(W)$ which contains all class-one representations of (G, K). Since in our case $\mathrm{H}(W)$ itself is composed of class-one representations only, the G-orbit mentioned earlier fills $\mathrm{H}(W)$.

The proposition has a practical reading: the Hermitian/symmetric operators parametrising the moduli spaces belong to the orthogonal complement in $\mathrm{H}(W)$ to the aforesaid G-orbit, but in the present situation this space is null. Therefore the induced map will be rigid. We use this information to study the real standard map, the outcome naming the section (Theorem 3.3.4).

A detailed description of the normal decomposition can be found in [5]. Let us sketch the central ideas: Consider the situation described in §2, i.e. $W \subset \Gamma(\mathbf{V})$ is a space of sections of the vector bundle $\mathbf{V} \to M,\ M = G/K$ associated to the principal homogeneous bundle $G \to G/K$ with standard fiber the irreducible K-module $V_0 \subset W$. Furthermore, suppose $\mathbf{V} \to M$ to be equipped with its canonical connection. Let $f : G/K \to Gr_p(W)$ be the corresponding induced map by $(\mathbf{V} \to M,\ W)$. The space of sections W splits into V_0 and its orthogonal complement $N_0 = U_0$. Assume the condition of Lemma 2.1, i.e. $\mathfrak{m}V_0 \subset U_0$ such that the canonical connection and the pull-back connection coincide.

From now on, our considerations will be restricted at a point $o \in M$ for the sake of simplicity. The second fundamental form K at $o \in M$ is an element of $T_o^*M \otimes V_0^* \otimes U_0$ so that for all $X \in T_oM,\ v \in V_0,\ (\mathrm{K}_X(v))_o \in U_0$. The image of this mapping, also designated by B_1, is a well-defined subspace of N_0 and thus gives a further orthogonal decomposition of W as $V_0 \oplus \mathrm{Im}B_1 \oplus (V_0 \oplus \mathrm{Im}B_1)^\perp$. Call $N_1 = (V_0 \oplus \mathrm{Im}B_1)^\perp$ the *first normal subspace*. Applying the connection to the second fundamental form at the point $o \in M$ we have $\nabla \mathrm{K} \in S^2T_o^*M \otimes V_0^* \otimes U_0$ (where symmetrization follows from Gauss–Codazzi equations and flatness of the connection on $\underline{W}$). If π_1 denotes the orthogonal projection $\pi_1 : W \to N_1$, then B_2 is defined as $\pi_1 \circ \nabla \mathrm{K} \in S^2T_o^*M \otimes V_0^* \otimes N_1$, and we have $W = V_0 \oplus \mathrm{Im}B_1 \oplus \mathrm{Im}B_2 \oplus N_2$ where N_2 is the *second normal subspace*. Recursively, $B_p = \pi_{p-1} \circ \nabla^{p-1}\mathrm{K} \in S^pT_o^*M \otimes V_0^* \otimes N_{p-1}$. This reiterative process leads to

$$W = V_0 \oplus \mathrm{Im}\, B_1 \oplus \mathrm{Im}\, B_2 \oplus \cdots \oplus \mathrm{Im}\ B_n \oplus N_n.$$

If $N_n = 0$ this is called the *normal decomposition of* W *with respect to* V_0.

Let us enunciate without proof two results regarding the normal decomposition which are needed in the sequel to establish Proposition 3.3.3.

Proposition 3.3.1 ([24, Prop. 7.7])**.** *If W is an irreducible G-module, then for any K-module, $V_0 \subset W$ there exists a positive integer n such that $N_n = 0$, i.e.*

$$W = V_0 \oplus \mathrm{Im}\, B_1 \oplus \cdots \oplus \mathrm{Im}\, B_n \tag{3.3.1}$$

which is a normal decomposition of (W, V_0).

Proposition 3.3.2 ([24, Prop. 7.8])**.** *Let W be a G-module and $V_0 \subset W$ a K-module. Suppose that (W, V_0) has a normal decomposition. Assume that each term in the decomposition (3.3.1) shares no common K-irreducible factor with any other term in the decomposition. Let T be a non-negative Hermitian endomorphism of W which satisfies $(Tgv_1, Tgv_2) = (v_1, v_2)$ for all $g \in G,\ v_1, v_2 \in V_0$. Then, if T is K-equivariant, $T = Id_W$.*

Remark 3. See also Lemma 4.2 in [5].

Hereafter, we assume $G = \mathrm{SU}(2))$, during this chapter. Then, we can state the following

Proposition 3.3.3. *Let $W = H^0(\mathbf{C}P^1, \mathcal{O}(k))$ and V_0 the K-module regarded as the standard fiber for $\mathcal{O}(k) \to \mathbf{C}P^1$. Then,* $\mathrm{GH}(V_0, V_0) = \mathrm{H}(W)$.

Proof. By Borel–Weil theorem, W is identified with the SU(2)-module $S^k\mathbf{C}^2$ and, using Lemma 2.3.1, V_0 can be regarded as a subspace of W. The space W decomposes under the U(1)-action as

$$W = \mathbf{C}_{-k} \oplus \mathbf{C}_{-k+2} \oplus \cdots \oplus \mathbf{C}_k,$$

where $\mathbf{C}_l$ denotes the irreducible U(1)-module of weight l. Indeed, this is the normal decomposition by Proposition 3.3.1 where $V_0 = \mathbf{C}_{-k}$.

Let H be a class-one subrepresentation of (G, K) in $\mathrm{H}(W)$. Suppose that $H \not\subset \mathrm{GH}(V_0, V_0)$. Then, by a standard argument, we can assume that $H \perp \mathrm{GH}(V_0, V_0)$. Since H is a class-one representation, there exists a non-zero $C \in H$ such that $kCk^{-1} = C$ for all $k \in K$. It follows from the orthogonality assumption that

$$\begin{aligned} 0 =&(C, gH(v_1, v_2))_{\mathrm{H}(W)} = (C, H(gv_1, gv_2))_{\mathrm{H}(W)} \\ =&\frac{1}{2}\left\{(Cgv_1, gv_2)_W + (Cgv_2, gv_1)_W\right\}, \end{aligned}$$

for arbitrary $g \in G$ and v_1, $v_2 \in V_0 \subset W$. Polarization gives

$$0 = (Cgv_1, gv_2), \quad g \in G,\ v_1, v_2 \in V_0.$$

If C is sufficiently small, then $Id + C > 0$ and so, we can define a positive Hermitian operator T satisfying $T^2 = Id + C$. Then we have

$$(Tgv_1, Tgv_2) = (v_1, v_2) \quad g \in G,\ v_1, v_2 \in V_0.$$

Since T is also K-equivariant, Proposition 3.3.2 yields that $T = Id$ and so, $C = 0$, which is a contradiction. Hence, every class-one subrepresentation of (G, K) in $\mathrm{H}(W)$ is included in $\mathrm{GH}(V_0, V_0)$. However, it follows from the Clesbsch–Gordan formulae that $\mathrm{H}(W)$ is composed by class-one representation of (G, K) only, therefore $\mathrm{GH}(V_0, V_0) = \mathrm{H}(W)$. □

Remark 4. A more general version of our Proposition 3.3.3 can be found in [24, Proposition 7.9]. Our proof is essentially the same with the obvious particularizations.

We shall prove the following interesting result.

Theorem 3.3.4. *Let $W = S^{2k}\mathbf{C}^2$ such that $W^{\mathbf{R}} = S_0^k\mathbf{R}^3 \cong \mathbf{R}^{2k+1}$. If $f : \mathbf{C}P^1 \hookrightarrow Gr_{2k-1}(\mathbf{R}^{2k+1})$ is a holomorphic isometric embedding of degree $2k$, then f is the standard map by $W^{\mathbf{R}}$ up to gauge equivalence.*

Before proving Theorem 3.3.4, let us clarify the construction of the mapping $f : \mathbf{C}P^1 \hookrightarrow Gr_{2k-1}(\mathbf{R}^{2k+1})$ from the vector bundle viewpoint.

If we regard the complex projective line as the symmetric space G/K where $G = \mathrm{SU}(2)$ and $K = \mathrm{U}(1)$, then by Borel–Weil theorem the space of sections $\Gamma(\mathcal{O}(2k))$ becomes a

G-module such that $W = H^0(\mathbf{C}P^1; \mathcal{O}(2k)) \cong S^{2k}\mathbf{C}^2$. The decomposition of $S^{2k}\mathbf{C}^2$ into irreducible U(1)-modules is as follows:

$$S^{2k}\mathbf{C}^2 = \bigoplus_{r=0}^{2k} \mathbf{C}_{2k-2r}. \tag{3.3.2}$$

The typical fiber of $\mathcal{O}(2k) \to \mathbf{C}P^1$ is regarded as a subspace $\mathbf{C}_{-2k}$ in the decomposition by Lemma 2.3.1.

Since W has an invariant real structure, we have an invariant real subspace denoted by $W^{\mathbf{R}} = (S^{2k}\mathbf{C}^2)^{\mathbf{R}} \cong S_0^k\mathbf{R}^3$ of real dimension $2k+1$. The real structure descends to the splitting (3.3.2) but now each irreducible U(1)-module is not invariant under the real structure, but $\sigma(\mathbf{C}_{2k-2r}) = \mathbf{C}_{-2k+2r}$. Therefore for each $r = 0, \ldots, k$ the space $(\mathbf{C}_{2k-2r} \oplus \mathbf{C}_{-2k+2r})$ is stable under the real structure and decomposes in two real isomorphic irreducible U(1)-modules, denoted by $(\mathbf{C}_{2k-2r} \oplus \mathbf{C}_{-2k+2r})^{\mathbf{R}}$, such that (3.3.2) would be rewritten as

$$S_0^k\mathbf{R}^3 = \bigoplus_{r=0}^{2k} (\mathbf{C}_{2k-2r} \oplus \mathbf{C}_{-2k+2r})^{\mathbf{R}}. \tag{3.3.3}$$

This implies that $\mathcal{O}(2k) \to \mathbf{C}P^1$ is globally generated by $W^{\mathbf{R}}$. Thus, we can define a *real standard map* $f_0 : \mathbf{C}P^1 \to Gr_{2k-1}(\mathbf{R}^{2k+1})$ by $W^{\mathbf{R}}$, which turns out to be a holomorphic isometric embedding of degree $2k$ by Lemma 2.3.3. Using the inner product on $W^{\mathbf{R}}$ and the fiber-metric on $\mathcal{O}(2k) \to \mathbf{C}P^1$, it is possible to define the adjoint of the evaluation which at the identity of G/K determines a mapping $ev_{[e]}^* : \mathcal{O}(2k) \to \underline{W}^{\mathbf{R}}$ whose image is just $(\mathbf{C}_{2k} \oplus \mathbf{C}_{-2k})^{\mathbf{R}}$.

Within this framework we have a real version of Proposition 3.3.3, which is the core of the proof of Theorem 3.3.4:

Proposition 3.3.5. *Let* $W = H^0(\mathbf{C}P^1, \mathcal{O}(2k))$ *and* V_0 *the* K*-module regarded as the standard fiber for* $\mathcal{O}(2k) \to \mathbf{C}P^1$*. Then,* $\mathrm{GS}(V_0, V_0) = \mathrm{S}(W^{\mathbf{R}})$.

Proof. Equation (3.3.3) gives the normal decomposition of $W^{\mathbf{R}}$ where now $V_0 = (\mathbf{C}_{-2k} \oplus \mathbf{C}_{2k})^{\mathbf{R}}$. The space of symmetric endomorphisms of $W^{\mathbf{R}}$ can be identified by decomposing first the tensor product using Lemma 3.2.1, and identifying the symmetric components

$$\mathrm{S}(W^{\mathbf{R}}) = \bigoplus_{r=0}^{k} S_0^{4k-4r}\mathbf{R}^3 \subset \otimes^2 W^{\mathbf{R}} = \bigoplus_{r=0}^{2k} S_0^{4k-2r}\mathbf{R}^3.$$

Notice that all these modules are class-one representations. Then, a similar argument as the one in the proof of Proposition 3.3.3 yields the desired result. $\square$

We can now proceed to prove Theorem 3.3.4.

Proof. Consider the real standard map by the holomorphic line bundle $\mathcal{O}(2k) \to \mathbf{C}\mathrm{P}^1$ and $W^{\mathbf{R}}$ as depicted above. Therefore by Proposition 3.3.5, $\mathrm{S}(W^{\mathbf{R}}) = \mathrm{GS}(V_0, V_0)$ and replacing $\mathbf{R}^{n+2}$ by $W^{\mathbf{R}}$ in Theorem 2.3.4 the real standard map admits no deformations as holomorphic isometric embedding of degree $2k$ into $Gr_{2k-1}(\mathbf{R}^{2k+1})$. □

Remark 5. If the target space is replaced by a higher-dimensional Grassmannian including $Gr_{2k-1}(\mathbf{R}^{2k+1})$ as a totally geodesic submanifold the resulting moduli space could be non-trivial. This situation will be discussed in the next section.

3.4 Moduli space by gauge equivalence

We undertake now the task of giving an accurate description of the moduli space of holomorphic isometric embeddings $\mathbf{C}\mathrm{P}^1 \to Gr_p(W)$ up to gauge equivalence. Our strategy will be to capitalize on the representation-theoretic formulae of §4 to explicitly determine the subspaces of linear operators in $\mathrm{S}(W)$ which specify the moduli. Such subspaces are sharply characterized by condition (III) in Theorem 2.3.4. This is achieved after a sequence of stepping-stone results culminating in Lemma 3.4.2 and its Corollary, which allows to compute the moduli dimension.

As indicated Theorem 2.3.4, the gauge equivalence relation is to be taken into account to obtain the moduli space and to give a geometric meaning to its compactification in the natural L^2-topology. A qualitative description of these spaces is given in Theorem 3.4.4.

Let W be the space of holomorphic sections of $\mathcal{O}(k) \to \mathbf{C}\mathrm{P}^1$ which, by Borel–Weil theorem, is identified with the $\mathrm{SU}(2)$-module $S^k\mathbf{C}^2$. Equation (3.3.2) gives a weight decomposition of W with respect to $\mathrm{U}(1)$. When $\mathcal{O}(k) \to \mathbf{C}P^1$ is regarded as the homogeneous line bundle $\mathrm{SU}(2) \times_{\mathrm{U}(1)} V_0 \to \mathbf{C}\mathrm{P}^1$, then V_0 is identified with the $\mathrm{U}(1)$-irreducible subspace $\mathbf{C}_{-k}$ of W by Lemma 2.3.1.

In order to apply Theorem 2.3.4 we shall regard the universal quotient bundle as a real vector bundle of rank 2. Following the generalization of do Carmo–Wallach theory, we must determine the subspaces $\mathrm{GS}(V_0, V_0)$ and $\mathrm{GS}(\mathfrak{m}V_0, V_0)$ of $\mathrm{S}(W)$.

From now on V_0 and W shall stand either for the complex modules or for their underlying $\mathbf{R}$-vector spaces whenever the meaning is clear, avoiding the heavier notation $(V_0)_{\mathbf{R}}$ or $W_{\mathbf{R}}$. In the remaining sections, we will adopt this convention.

Since $\mathrm{GH}(V_0, V_0)$ is a proper subspace of $\mathrm{GS}(V_0, V_0)$, we have that $\mathrm{H}(W) \subset \mathrm{GS}(V_0, V_0)$. We must determine the intersection between $\mathrm{GS}(V_0, V_0)$ and subspaces $\sigma\mathrm{H}_+(W) \oplus J\sigma\mathrm{H}_+(W)$ appearing in Corollary 3.2.3. The same is true for the intersection $\mathrm{GS}(\mathfrak{m}V_0, V_0)$ with $\sigma\mathrm{H}_+(W) \oplus J\sigma\mathrm{H}_+(W)$ as we shall consider immediately.

Lemma 3.4.1. $\mathfrak{m}V_0 = \mathbf{C}_{-k-2}$.

Proof. By the decomposition of $S^2\mathbf{C}^2$ into irreducible $\mathrm{U}(1)$-modules $S^2\mathbf{C}^2 = \mathbf{C}_2 \oplus \mathbf{C}_0 \oplus \mathbf{C}_{-2}$ and using the real structure we have $(S^2\mathbf{C}^2)^{\mathbf{R}} \cong \mathfrak{su}(2)$, $(\mathbf{C}_0)^{\mathbf{R}} \cong \mathfrak{u}(1)$ therefore

$(\mathbf{C}_2 \oplus \mathbf{C}_{-2})^{\mathbf{R}} \cong \mathfrak{m}$. Then,

$$\mathfrak{m} \otimes V_0 = (\mathbf{C}_2 \oplus \mathbf{C}_{-2}) \otimes \mathbf{C}_{-k} = \mathbf{C}_{-k+2} \oplus \mathbf{C}_{-k-2}.$$

The action of $\mathfrak{m}$ on V_0 is then obtained by projecting $\mathfrak{m} \otimes V_0$ back to $S^k\mathbf{C}^2$. Therefore

$$\mathfrak{m}V_0 = (\mathfrak{m} \otimes V_0) \cap S^k\mathbf{C}^2 = \mathbf{C}_{-k+2}.$$

$\square$

Lemma 3.4.2. $\mathrm{GS}(\mathfrak{m}V_0, V_0) \cap \sigma\mathrm{H}_+(W) \oplus J\sigma\mathrm{H}_+(W)$ *is the highest weight representations of* $SU(2)$ *appeared in Proposition 3.2.7.*

Proof. Let u_{-k} and u_{-k+2} be unitary bases for the complex one-dimensional $\mathrm{U}(1)$-modules $V_0 = \mathbf{C}_{-k}$ and $\mathfrak{m}V_0 = \mathbf{C}_{-k+2}$, respectively. Then, the space $\mathrm{H}(\mathfrak{m}V_0, V_0) \equiv \mathrm{H}(\mathbf{C}_{-k+2}, \mathbf{C}_{-k})$ is the real span of

$$2\mathrm{H}(u_{-k+2}, u_{-k}) = u_{-k+2} \otimes (\cdot, u_{-k})_W + u_{-k} \otimes (\cdot, u_{-k+2})_W$$

where $(,)_W$ denotes the Hermitian inner product on $S^k\mathbf{C}^2$. When $\mathbf{C}_{-k}$ and $\mathbf{C}_{-k+2}$ are regarded as their underlying two-dimensional $\mathbf{R}$-vector spaces $\mathbf{R}^2_k$ and $\mathbf{R}^2_{k-2}$, real bases are given respectively by $\{u_{-k}, J^c u_{-k}\}$ and $\{u_{-k+2}, J^c u_{-k+2}\}$ where J^c is the almost complex structure induced by the multiplication by the imaginary unit. Using these real bases the complex form $2\mathrm{H}(u_{-k+2}, u_{-k})$ can be rewritten as a real operator

$$\begin{aligned} 2\mathrm{H}(u_{-k+2}, u_{-k})|\mathbf{R} &= u_{-k+2} \otimes \langle \cdot, u_{-k}\rangle_W + J^c u_{-k+2} \otimes \langle \cdot, J^c u_{-k}\rangle_W \\ &\quad + u_{-k} \otimes \langle \cdot, u_{-k+2}\rangle_W + J^c u_{-k} \otimes \langle \cdot, J^c u_{-k+2}\rangle_W \end{aligned}$$

where $\langle,\rangle_W$ is the inner product on $W_{\mathbf{R}}$ induced from the Hermitian inner product on W. Write the basis for $\mathrm{S}(\mathfrak{m}V_0, V_0) \equiv \mathrm{S}(\mathbf{R}^2_{-k+2}, \mathbf{R}^2_{-k})$ as $\{\mathrm{S}(u_{-k+2}, u_{-k})$, $\mathrm{S}(J^c u_{-k+2}, u_{-k})$, $\mathrm{S}(u_{-k+2}, J^c u_{-k})$, $\mathrm{S}(J u_{-k+2}, J^c u_{-k})\}$, e.g. ,

$$2\mathrm{S}(u_{-k+2}, u_{-k}) = u_{-k+2} \otimes \langle \cdot, u_{-k}\rangle_W + u_{-k} \otimes \langle \cdot, u_{-k+2}\rangle_W, \quad \text{etc.}$$

Comparing both equations we have that

$$\mathrm{H}(u_{-k+2}, u_{-k})|\mathbf{R} = \mathrm{S}(u_{-k+2}, u_{-k}) + \mathrm{S}(J^c u_{-k+2}, J^c u_{-k}).$$

Analogously,

$$\mathrm{H}(u_{-k+2}, iu_{-k})|\mathbf{R} = \mathrm{S}(u_{-k+2}, J^c u_{-k}) - \mathrm{S}(J^c u_{-k+2}, u_{-k}).$$

Let us define a new elements $\{X, Y\}$

$$\begin{aligned} X &= \mathrm{S}(u_{-k+2}, u_{-k}) - \mathrm{S}(J^c u_{-k+2}, J^c u_{-k}), \\ Y &= \mathrm{S}(u_{-k+2}, J^c u_{-k}) + \mathrm{S}(J^c u_{-k+2}, u_{-k}). \end{aligned}$$

$X, Y \in \mathrm{S}(W_{\mathbf{R}})$ are orthogonal to the subspace of Hermitian matrices $\mathrm{H}(W) \subset \mathrm{S}(W_{\mathbf{R}})$, therefore they belong to $\sigma H_+(W) \oplus J\sigma H_+(W)$ according to Corollary 3.2.3.

Let us consider the contragredient action of the structure map σ on X, the case of Y being analogous. Firstly,

$$\sigma \left(u \otimes \langle \cdot, v\rangle_W\right) \sigma = \sigma u \otimes \langle \sigma \cdot, v\rangle_W = \sigma u \otimes \langle \cdot, \sigma v\rangle_W$$

and as such $\sigma \mathrm{S}(u, v)\sigma = \mathrm{S}(\sigma u, \sigma v)$.

Secondly, the U(1)-modules $\mathbf{C}_i$ are not σ-invariant but $\sigma(\mathbf{C}_{\pm i}) = \mathbf{C}_{\mp i}$, for all i that is, $\sigma u_{\pm i} = u_{\mp i}$. which, together with conjugate-linearity of the structure map yields $\sigma(\mathbf{R}^2_{\pm i}) = \mathbf{R}^2_{\mp i} : \{u_{\pm i}, J^c u_{\pm i}\} \mapsto \{u_{\mp i}, -J^c u_{\mp i}\}$. Hence we have

$$\begin{aligned} X^\sigma = \sigma X \sigma &= \mathrm{S}(\sigma u_{-k+2}, \sigma u_{-k}) - \mathrm{S}(\sigma J^c u_{-k+2}, \sigma J^c u_{-k}) \\ &= \mathrm{S}(u_{k-2}, u_k) - \mathrm{S}(J^c u_{k-2}, J^c u_k). \end{aligned}$$

This is not an element of $\mathrm{S}(\mathfrak{m}V_0, V_0) \equiv \mathrm{S}(\mathbf{R}^2_{-k+2}, \mathbf{R}^2_{-k})$ but $X^\sigma \in \mathrm{S}(\mathbf{R}^2_{k-2}, \mathbf{R}^2_k)$. Note that we can find $g \in \mathrm{SU}(2)$ such that $\mathrm{S}(u_{k-2}, u_k) = \mathrm{S}(gu_{-k+2}, gu_{-k}) = g \cdot \mathrm{S}(u_{-k+2}, u_{-k}) \in \mathrm{GS}(\mathfrak{m}V_0, V_0)$ up to a sign. Let us add $Y^\sigma = \mathrm{S}(u_{k-2}, J^c u_k) + \mathrm{S}(J^c u_{k-2}, u_k)$ for the sake of completeness.

The preceding argument also shows that a subspace of $\mathrm{GS}(\mathfrak{m}V_0, V_0)$ is spanned by $\{S(u_{k-2}, u_k),\ S(u_{k-2}, J^c u_k),\ S(J^c u_{k-2}, u_k),\ S(J^c u_{k-2}, J^c u_k)\}$.

Moreover, using the characterization given in Lemma 3.2.2 we have

$$X + X^\sigma \in \sigma \mathrm{H}_+(W), \qquad X - X^\sigma \in J^c \sigma \mathrm{H}_+(W).$$

The same inclusions are also true for $Y \pm Y^\sigma$.

From the expression of the action of σ on $\mathrm{H}(u, v)$

$$\begin{aligned} \sigma \cdot \mathrm{H}(u, v) &= \sigma \left(u \otimes (\cdot, v)_W + v \otimes (\cdot, u)_W\right) \sigma = \sigma u \otimes (\sigma \cdot, v)_W + \sigma v \otimes (\sigma \cdot, u)_W \\ &= \sigma u \otimes \overline{(\cdot, \sigma v)_W} + \sigma v \otimes \overline{(\cdot, \sigma u)_W}, \end{aligned}$$

it is easy to write $X \pm X^\sigma$ back in terms of Hermitian operators as

$$X \pm X^\sigma = \sigma \cdot \left(\mathrm{H}(u_{k-2}, u_{-k}) \pm \mathrm{H}(u_{-k+2}, u_k)\right) |\mathbf{R}.$$

The toral action of a U(1)-element of SU(2) on $u_{\pm k},\ u_{\pm(k-2)}$ yields

$$\exp(i\theta) \cdot u_{\pm k} = \exp(\pm ik\theta) u_{\pm k}, \qquad \exp(i\theta) \cdot u_{\pm(k-2)} = \exp(\pm i(k-2)\theta) u_{\pm(k-2)}$$

and as such, $X \pm X^\sigma$ (considered as the Hermitian operators above) have weight $\pm(2k-2)$. However, from Corollary 3.2.7 we know that the only component in the real decomposition of $\sigma \mathrm{H}_+(W)$ and $J^c \sigma \mathrm{H}_+(W)$ (both isomorphic to $\mathrm{H}_+(W)$) which can host such a vector is the top term $S^k_0 \mathbf{R}^3$ on each space. Therefore

$$\mathrm{GS}(\mathfrak{m}V_0, V_0) \cap \sigma \mathrm{H}_+(W) = S^k_0 \mathbf{R}^3 \qquad (\text{ resp. for } J^c \sigma \mathrm{H}_+(W)\ ).$$

And as a result

$$\mathrm{GS}(\mathfrak{m}V_0, V_0) = \mathrm{H}(W) \oplus S_0^k\mathbf{R}^3 \oplus S_0^k\mathbf{R}^3.$$

□

In other words, we obtain

Corollary 3.4.3. *The orthogonal complement to* $\mathrm{GS}(\mathfrak{m}V_0, V_0) \oplus \mathbf{R}\,Id$ *in* $\mathrm{S}(W)$ *is*

$$2\bigoplus_{r=1}^{k\geq 2r} S_0^{k-2r}\mathbf{R}^3.$$

This follows from applying the previous lemma to the explicit expressions for the components of $\mathrm{S}(W)$ as described in Proposition 3.2.7, and accounts for the space of symmetric operators T described by the second relation in (2.3.1), i.e. condition (III) in Theorem 2.3.4.

Remark 6. The first condition in (2.3.1) is for all our purposes inessential. Let $\mathrm{GS}_0(V_0, V_0)$ be the orthogonal complement of the G-invariant, irreducible subrepresentation generated by the identity in $\mathrm{GS}(V_0, V_0)$. We denote by $\mathrm{S}_0(W)$ the set of tracefree symmetric operators on W with the induced inner product from $\mathrm{S}(W)$. Then,

$$\mathrm{GS}_0(V_0, V_0) \subset \mathrm{GS}(\mathfrak{m}V_0, V_0),$$

which stems from an analogous result to Lemma 3.4.2 applied to $\mathrm{GS}_0(V_0, V_0)$. The proof is equivalent, changing the weight $\pm(2k-2)$ by $\pm 2k$ in the crucial final step.

Condition (III) in Theorem 2.3.4 is fulfilled by the family of operators in Corollary 3.4.3 (see remark above) thus accounting for all holomorphic embeddings $f : \mathbf{C}P^1 \hookrightarrow Gr_p(\mathbf{R}^{p+2})$ up to possible degeneracies. Quantitative information about the moduli (i.e. its dimension) can therefore be derived from the Corollary:

$$\dim_{\mathbf{R}} \mathcal{M}_k = k(k-1). \tag{3.4.1}$$

The following theorem summarizes the qualitative information about the moduli space and gives a neat geometric interpretation to its compactification.

Theorem 3.4.4. *If* $f : \mathrm{CP}^1 \hookrightarrow Gr_n(\mathbf{R}^{n+2})$ *is a full holomorphic isometric embedding of degree* k*, then* $n \leq 2k$*.*

Let $\mathcal{M}_k$ *be the moduli space of pairs* (f, ϕ) *by the gauge equivalence, where* f *is a full holomorphic isometric embeddings of degree* k *of the complex projective line into* $Gr_{2k}(\mathbf{R}^{2k+2})$ *and* ϕ *is the bundle isomorphism* $\mathcal{O}(k) \to f^*Q$ *in Theorem 2.3.4. Then,* $\mathcal{M}_k$ *can be regarded as an open bounded convex body in* $2\bigoplus_{r=1}^{k\geq 2r} S_0^{k-2r}\mathbf{R}^3$*.*

Let $\overline{\mathcal{M}_k}$ *be the closure of the moduli* $\mathcal{M}_k$ *by the inner product. Boundary points of* $\overline{\mathcal{M}_k}$ *describe those maps whose images are included in some totally geodesic submanifold* $Gr_p(\mathbf{R}^{p+2})$ *of* $Gr_{2k}(\mathbf{R}^{2k+2})$*, where* $p < 2k$*.*

The totally geodesic submanifold $Gr_p(\mathbf{R}^{p+2})$ *can be regarded as the common zero set of some sections of* $Q \to Gr_{2k}(\mathbf{R}^{2k+2})$*, which belongs to* $\mathbf{R}^{2k+2}$*.*

Proof. The restriction $n \leq 2k$ follows from (I) in Theorem 2.3.4 and Borel–Weil theorem.

It is evident from (III) in Theorem 2.3.4 that $\mathrm{GS}(\mathfrak{m}V_0, V_0)^\perp$ is a parametrization of the space of full holomorphic isometric embeddings $f : \mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_{2k}(\mathbf{R}^{2k+2})$ of degree k. Positivity of T being guaranteed by fullness, we can apply the original do Carmo–Wallach argument [5, §5.1], to conclude that $\mathcal{M}_k$ is a bounded connected *open* convex set in $\mathrm{H}(W)$ with the topology induced by the L^2 scalar product.

Under the natural compactification in the L^2-topology, the boundary points correspond to operators T which are not positive definite, but positive semi-definite. It follows from Theorem 2.3.4 that each of these operators defines in turn a full holomorphic isometric embedding $\mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_p(\mathbf{R}^{p+2})$, of degree k with $p = 2k - \dim \operatorname{Ker} T$, whose target embeds in $Gr_{2k}(\mathbf{R}^{2k+2})$ as a totally geodesic submanifold. The image Z of the embedding $Gr_p(\mathbf{R}^{p+2}) \hookrightarrow Gr_{2k}(\mathbf{R}^{2k+2})$ is determined by the common zero-set of sections in Ker T. □

3.5 Moduli space by image equivalence

The moduli space $\mathcal{M}_k$ has a natural complex structure induced by that on $Q \to Gr_{2k}(\mathbf{R}^{2k+2})$ which coincides with the one in Remark 2. Hence, $\mathcal{M}_k$ can be regarded as holomorphically included in the $\mathbf{C}$-vector space $\oplus_{r=1}^{k \geqq 2r} S^{2k-4r}\mathbf{C}^2$. We can show that the centralizer of the holonomy group acts on $\mathcal{M}_k$ with weight $-k$. Hence we have

Theorem 3.5.1. *Let $\mathbf{M}_k$ be the moduli space of holomorphic isometric embeddings of the complex projective line into $Gr_{2k}(\mathbf{R}^{2k+2})$ of degree k by the image equivalence of maps. Then we have $\mathbf{M}_k = \mathcal{M}_k/S^1$.*

Proof. Assume two full holomorphic isometric embeddings $\mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_{2k}(\mathbf{R}^{2k+2})$ of degree k to be image equivalent. They may represent distinct points in $\mathcal{M}_k$. By definition of image equivalence, there is an isometry ψ of $Gr_{2k}(\mathbf{R}^{2k+2})$ such that $f_2 = \psi \circ f_1$, then $f_2^*Q = f_1^*\tilde{\psi}Q$ as sets. Using the natural identifications ϕ_1, ϕ_2 of Theorem 2.3.4 we introduce new bundle isomorphisms $\mathcal{O}(k) \to f_2^*Q$ defined by $\tilde{\psi} \circ \phi_1$ and ϕ_2. Hence, we have a gauge transformation $\phi_2^{-1}\tilde{\psi}\phi_1$ on the line bundle $\mathcal{O}(k) \to M$ preserving the metric and the connection. By connectedness of $\mathbf{C}\mathrm{P}^1$ such a gauge transformation is regarded as an element of the centralizer of the holonomy group of the connection in the structure group of V, i.e. $\mathrm{U}(1) \equiv S^1$ acting with weight $-k$ on the standard fiber $V_0 \cong \mathbf{C}_{-k}$. Modding out the S^1-action yields the true moduli space by image equivalence $\mathbf{M}_k$. □

Remark 7. The moduli space $\mathcal{M}_k$ has a complex structure (see remark in §4) and a metric induced by the inner product both preserved by the S^1-action. Hence, it is a Kähler manifold together with an S^1-action preserving the Kähler structure. Therefore, $\mathcal{M}_k$ is naturally equipped with a moment map $\mu : \mathcal{M}_k \to \mathbf{R}$ expressed as $\mu = |T|^2$.

Corollary 3.5.2. *There exists a one-parameter family* $\{f_t\}$, $t \in [0,1]$, *of* $\mathrm{SU}(2)$*-equivariant image-inequivalent holomorphic isometric embeddings of even degree of* $\mathbf{C}\mathrm{P}^1$ *into complex quadrics, where* f_0 *corresponds to the standard map and* f_1 *is the real standard map.*

Proof. The moduli space by gauge equivalence $\mathcal{M}_k$ sits in $\oplus_{r=1}^{k\geq 2r} S^{2k-4r}\mathbf{C}^2$. For even k this last expression includes the trivial representation $\mathbf{C}$, which using the real structure can be described as $\mathbf{C} = \mathbf{R}\sigma \oplus \mathbf{R}J^c\sigma$. Let $C \in \mathbf{C} \subset \oplus_{r=1}^{k\geq 2r} S^{2k-4r}\mathbf{C}^2$. If it is small enough, then by Theorem 2.4, $Id + C$ determines a holomorphic isometric embedding into $Gr_{2k}(\mathbf{R}^{2k+2})$. The group $\mathrm{SU}(2)$ acts on each component of $Id + C$ trivially, so the associated holomorphic isometric embedding is $\mathrm{SU}(2)$-equivariant. The S^1 action of the centralizer of the holonomy group acts on $\mathbf{C}$ with weight $-k$ (see proof of Theorem 3.5.1) therefore, taking quotient by the S^1-action, we obtain a half-open segment parametrising the described maps, which becomes a closed segment under the natural compactification in the L^2-topology. Let $C = t\sigma + sJ^c\sigma$. Then we can show that $Id + C$ is positive if and only if $t^2 + s^2 < 1$. Suppose that $t^2 + s^2 = 1$. Then $(t + sJ^c)\sigma$ is also an invariant real structure on $S^{2k}\mathbf{C}^2$. Hence we may consider only the case that $t = 1$ and $s = 0$. Since the kernel of $Id + \sigma$ is $J^cW^{\mathbf{R}}$, Theorem 2.3.4 implies that $Id + \sigma$ determines a totally geodesic submanifold $Gr_{2k-1}(\mathbf{R}^{2k+1})$ of $Gr_{4k}(\mathbf{R}^{4k+2})$ and a holomorphic isometric embedding into the submanifold $Gr_{2k-1}(\mathbf{R}^{2k+1})$ represented by $2\, Id_{W^{\mathbf{R}}}$. This map is nothing but the real standard map by $W^{\mathbf{R}}$, because constant multiples of the identity give the same subspace of $W^{\mathbf{R}}$. $\square$

Chapter 4

Killing vector fields on complex hypersurfaces in the complex projective space

4.1 Main theorem

In this chapter, we identify $\mathbf{C}P^{n+1}$ with $Gr_{n+1}(\mathbf{C}^{n+2^*})$. In this case, $\mathbf{C}P^{n+1}$ is regarded as the Hermitian symmetric space $\mathrm{SU}(n+2)/\mathrm{U}(n+1)$. Hence the Lie algebra of Killing vector fields is $\mathfrak{su}(n+2)$.

We have the short exact sequence of holomorphic vector bundles:

$$0 \longrightarrow S \longrightarrow \underline{\mathbf{C}}^{n+2^*} \longrightarrow Q \longrightarrow 0, \tag{4.1.1}$$

where $Q \to \mathbf{C}P^{n+1}$ is the universal quotient bundle $\mathcal{O}(1) \to \mathbf{C}P^{n+1}$. $S \to \mathbf{C}P^{n+1}$ and $Q \to \mathbf{C}P^{n+1}$ can be recognized as homogeneous vector bundles:

$$S = \mathrm{SU}(n+2) \times_{\mathrm{U}(n+1)} E_0^*, \qquad Q = \mathrm{SU}(n+2) \times_{\mathrm{U}(n+1)} L_0^*,$$

Though we are mainly interested in various induced metrics on the complex quadric hypersurface in $\mathbf{C}P^{n+1}$, the complex quadric is regarded as an oriented real 2-plane Grassmannian $Gr_n(\mathbf{R}^{n+2}) \cong \mathrm{SO}(n+2)/\mathrm{SO}(n)\times\mathrm{SO}(2)$. Thus the complex quadric has $\mathrm{SO}(n+2)$ invariant Kähler metric, which is unique up to a constant multiple. In this case, the Lie algebra of Killing vector fields on the complex quadric is regarded as the Lie algebra $\mathfrak{so}(n+2)$ of $\mathrm{SO}(n+2)$.

In order to state the main theorem of this chapter, we denote by P^{n+2} the set of diagonal matrices with positive entries. A subset D of P^{n+2} and the interior $\mathring{D}$ of D are defined by

$$D = \left\{ \mathrm{diag}(\lambda_1, \cdots, \lambda_{n+2}) \in P^{n+2} \;\middle|\; 0 < \lambda_1 \leqq \cdots \leqq \lambda_{n+2},\ \sum_i \lambda_i = n+2 \right\}, \tag{4.1.2}$$

$$\mathring{D} = \left\{ \mathrm{diag}(\lambda_1, \cdots, \lambda_{n+2}) \in D \;\middle|\; 0 < \lambda_1 < \cdots < \lambda_{n+2} \right\}.$$

Notice that D is a subset of the space of real symmetric matrices which is a representation space of $\mathrm{SO}(n+2)$.

Theorem 4.1.1. *Let S be a compact connected complex hypersurface in $\mathbf{C}P^{n+1}$ with the Fubini-Study metric. If the induced metric on S admits a non-trivial Killing vector field, then S is a hyperplane $\mathbf{C}P^n$ or a complex quadric hypersurface $Gr_n(\mathbf{R}^{n+2})$ as a complex manifold.*

When $S = \mathbf{C}P^n$, the induced metric is the Fubini-Study metric and the Lie algebra of Killing vector fields is $\mathfrak{su}(n+1)$.

When $S = Gr_n(\mathbf{R}^{n+2})$, $D \setminus \mathring{D}$ can be regarded as the moduli space of the induced metrics with non-trivial Killing vector fields. The space of Killing vector fields of the metric corresponding to T in $D \setminus \mathring{D}$ is the Lie algebra of Lie subgroup of $\mathrm{SO}(n+2)$*, which is the stabilizer of T.*

In each case, the Lie algebra of Killing vector fields is a subalgebra of $\mathfrak{su}(n+2)$, which is the Lie algebra of Killing vector fields on $\mathbf{C}P^{n+1}$.

In this chapter, we prove this theorem.

Remark 8. The set of induced metrics on $Gr_n(\mathbf{R}^{n+2})$ is in one-to-one correspondence with D. The metric in $\mathring{D}$ has no non-trivial Killing vector field. The metric corresponding to $\mathrm{diag}(1, \cdots, 1)$ has the Lie algebra of $\mathrm{SO}(n+2)$ as the space of Killing vector fields.

4.2 Proof of Theorem 4.1.1

A compact connected complex hypersurface in $\mathbf{C}P^{n+1}$ of degree d is obtained by zero locus of a holomorphic section which is transverse to the zero section of the holomorphic line bundle $\mathcal{O}(d) \to \mathbf{C}P^{n+1}$ of degree d. Such a section will be called a *generic* section.

First of all, we refer to the following result of K. Yano (see also [15, Theorem 4.3]).

Theorem 4.2.1. [35] *Let M be a compact Kähler manifold and $Z = X - \sqrt{-1}JX$ a complex vector field of type (1,0) with real part X. Then X is a Killing vector field if and only if Z is holomorphic and* $\mathrm{div}X = 0$*.*

Due to Theorem 4.2.1, we focus on the complex Lie algebra of holomorphic vector fields on a hypersurface instead of the Lie algebra of Killing vector fields. For the space of holomorphic vector fields on a hypersurface in $\mathbf{C}P^{n+1}$, K. Kodaira and D. C. Spencer have proved the following result. This is also obtained by a theorem of H. Matsumura and P. Monsky in [21].

Theorem 4.2.2. [19] *Let S be a non-singular hypersurface in $\mathbf{C}P^{n+1}$ of degree d. If $n \geqq 2$ and $d \geqq 3$, then S admits no non-trivial holomorphic vector field.*

We are therefore concerned with the induced metrics on hypersurfaces of degree one or two.

Since a complex hypersurface of degree one is a totally geodesic submanifold $\mathbf{C}P^n$, the induced metric on $\mathbf{C}P^n$ is the Fubini-Study metric. Hence the Lie algebra of Killing vector fields on $\mathbf{C}P^n$ is a subalgebra of $\mathfrak{su}(n+2)$ and so every Killing vector field on $\mathbf{C}P^n$ can be obtained by restriction.

To consider the case of degree two, let $H^0(\mathbf{C}P^{n+1};\mathcal{O}(2))$ be the space of holomorphic sections of $\mathcal{O}(2) \to \mathbf{C}P^{n+1}$. It follows from Borel-Weil theory that $H^0(\mathbf{C}P^{n+1};\mathcal{O}(2))$ is identified with the space $S^2\mathbf{C}^{n+2^*}$ of symmetric quadratic forms on $\mathbf{C}^{n+2}$ as $\mathrm{SU}(n+2)$-modules. Thus every generic holomorphic section t of $\mathcal{O}(2) \to \mathbf{C}P^{n+1}$ corresponds to a non-degenerate quadratic form on $\mathbf{C}^{n+2}$, which is also denoted by the same symbol. The zero locus of t is denoted by Z_t, which is expressed as

$$Z_t = \left\{ x \in \mathbf{C}P^{n+1} \;\middle|\; t(v,v) = 0,\ v \in x \right\}.$$

Recall that $(\cdot,\cdot)_{n+2}$ is the Hermitian inner product on $\mathbf{C}^{n+2}$. We denote by $e_1, \cdots, e_{n+2}$ a unitary basis on $\mathbf{C}^{n+2}$ and by $e^1, \cdots, e^{n+2}$ its dual. An element t in $S^2\mathbf{C}^{n+2^*}$ is written as

$$t = \sum_{i,j=1}^{n+2} a_{ij} e^i \otimes e^j, \qquad a_{ij} = a_{ji} \in \mathbf{C}.$$

We denote by $t_0 \in H^0(\mathbf{C}P^{n+1};\mathcal{O}(2))$ the holomorphic section corresponding to the normal form $\sum_{i=1}^{n+2} e^i \otimes e^i$. An involutive anti-linear endomorphism σ is induced from t_0 and $\mathrm{SU}(n+2)$-structure, which satisfies

$$t_0(v,w) = (v, \sigma(w))_{n+2}, \qquad \text{for any } v, w \in \mathbf{C}^{n+2}.$$

We call σ a *real structure* on $\mathbf{C}^{n+2}$. We have the decomposition of $\mathbf{C}^{n+2}$:

$$\mathbf{C}^{n+2} = \mathbf{R}^{n+2} \oplus \sqrt{-1}\mathbf{R}^{n+2},$$

where $\mathbf{R}^{n+2}$ and $\sqrt{-1}\mathbf{R}^{n+2}$ are eigenspaces of σ with eigenvalues 1 and -1, respectively. The restriction of $(\cdot,\cdot)_{n+2}$ on $\mathbf{C}^{n+2}$ to $\mathbf{R}^{n+2}$ induces an inner product, which is denoted by the same symbol. Thus we have a subgroup $\mathrm{SO}(n+2)$ of $\mathrm{SU}(n+2)$.

The special unitary group $\mathrm{SU}(n+2)$ acts on $S^2\mathbf{C}^{n+2^*}$ by

$$(g \cdot t)(u,v) = t(g^{-1}u, g^{-1}v),$$

where $g \in \mathrm{SU}(n+2),\ t \in S^2\mathbf{C}^{n+2^*}$ and $u, v \in \mathbf{C}^{n+2}$. An element $g \in \mathrm{SU}(n+2)$ preserves t_0 if and only if $g \in \mathrm{SO}(n+2)$. Thus $\mathrm{SO}(n+2)$ acts on Z_{t_0}. Therefore the induced metric on Z_{t_0} is $\mathrm{SO}(n+2)$ invariant. Consequently, Z_{t_0} is the complex quadric equipped with the standard Kähler metric ω_0. The Lie algebra of Killing vector fields on Z_{t_0} is the corresponding Lie algebra $\mathfrak{so}(n+2)$.

Let t be a generic element in $H^0(\mathbf{C}P^{n+1}; \mathcal{O}(2)) = S^2\mathbf{C}^{n+2^*}$. Then we have an $A_t \in GL(n+2; \mathbf{C})$ such that

$$t(u,v) = t_0(A_t u, A_t v), \qquad \text{for any } u, v \in \mathbf{C}^{n+2}.$$

Since t_0 is symmetric, we have

$$t_0(A_t u, A_t v) = t_0({}^t A_t A_t u, v).$$

The matrix ${}^t A_t A_t$ is a symmetric matrix with complex entries. Then a result of T. Takagi [28] yields that there exist a unitary matrix U_t and uniquely determined positive real numbers $\lambda_1, \cdots, \lambda_{n+2}$ such that

$${}^t A_t A_t = {}^t U_t T_t^2 U_t, \quad T_t = \mathrm{diag}\Big(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_{n+2}}\,\Big), \quad 0 < \lambda_1 \leqq \cdots \leqq \lambda_{n+2}$$

Therefore we obtain

$$t(u,v) = t_0({}^t U_t T_t^2 U_t u, v) = t_0(T_t U_t u, T_t U_t v),$$

and so $Z_t = U_t^{-1} T_t^{-1} Z_{t_0}$. Since a unitary matrix U_t^{-1} induces a holomorphic isometry on $\mathbf{C}P^{n+1}$, the zero locus Z_t is congruent to $T_t^{-1} Z_{t_0}$. Moreover, since $Z_t = Z_{ct}$ for any constant c, we may suppose that Trace $T_t^2 = n+2$.

To identify the induced metric on Z_t, we consider $t \in S^2\mathbf{C}^{n+2^*}$ corresponding to $T_t = \mathrm{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_{n+2}})$, $0 < \lambda_1 \leqq \cdots \leqq \lambda_{n+2}$, $\sum_{i=1}^{n+2} \lambda_i = n+2$. We denote by $f_0 : Gr_n(\mathbf{R}^{n+2}) \hookrightarrow \mathbf{C}P^{n+1}$ the natural inclusion of $Gr_n(\mathbf{R}^{n+2}) = Z_{t_0}$. Then f_0 is an $\mathrm{SO}(n+2)$-equivariant holomorphic isometric embedding. Since Z_t is $T_t^{-1} Z_{t_0}$ and T_t^{-1} is holomorphic transformation on $\mathbf{C}P^{n+1}$, Z_t is the image of the holomorphic embedding $T_t^{-1} \circ f_0 : Gr_n(\mathbf{R}^{n+2}) \to \mathbf{C}P^{n+1}$. It follows that the Kähler manifold Z_t with the induced metric is isometric to $Gr_n(\mathbf{R}^{n+2})$ with the induced metric by $T_t^{-1} \circ f_0$. Since T_t is the positive real diagonal matrix, the rigidity theorem of Calabi [2] implies that the moduli space of induced metrics is identified with D in (4.1.2).

To compute the Kähler form, we may consider the fiber metric on the pull-back vector bundle of $Q \to \mathbf{C}P^{n+1}$ by $T_t^{-1} \circ f_0$.

At first we pull back (4.1.1) by f_0 to obtain the short exact sequence of holomorphic vector bundles:

$$0 \longrightarrow f_0^* S \xrightarrow{i_0} Gr_n(\mathbf{R}^{n+2}) \times \mathbf{C}^{n+2^*} \xrightarrow{ev_0} f_0^* Q \longrightarrow 0.$$

Since the Chern class of $f_0^* Q \to Gr_n(\mathbf{R}^{n+2})$ is the positive generator of $H^2(Gr_n(\mathbf{R}^{n+2}))$, $f_0^* Q$ is denoted by $\mathcal{O}(1) \to Gr_n(\mathbf{R}^{n+2})$. For simplicity, $f_0^* S \to Gr_n(\mathbf{R}^{n+2})$ is denoted by $F \to Gr_n(\mathbf{R}^{n+2})$ when regarded as C^∞ complex vector bundle. Since f_0 is $\mathrm{SO}(n+2)$-equivariant, $F \to Gr_n(\mathbf{R}^{n+2})$ and $\mathcal{O}(1) \to Gr_n(\mathbf{R}^{n+2})$ can also be recognized as homogeneous vector bundles:

$$F = \mathrm{SO}(n+2) \times_{\mathrm{SO}(n)\times\mathrm{SO}(2)} E_0^*, \qquad \mathcal{O}(1) = \mathrm{SO}(n+2) \times_{\mathrm{SO}(n)\times\mathrm{SO}(2)} L_0^*.$$

An element in $F \to Gr_n(\mathbf{R}^{n+2})$ (resp. $\mathcal{O}(1) \to Gr_n(\mathbf{R}^{n+2})$) can be expressed as $[g, u]$ (resp. $[g, v]$) for $g \in \mathrm{SO}(n+2)$ and $u \in E_0^*$, (resp. $v \in L_0^*$). Then the homomorphisms $i_0 : F \to Gr_n(\mathbf{R}^{n+2}) \times \mathbf{C}^{n+2^*}$ and $ev_0 : Gr_n(\mathbf{R}^{n+2}) \times \mathbf{C}^{n+2^*} \to \mathcal{O}(1)$ are written in the form:

$$i_0\big([g, u]\big) = ([g], gu), \qquad ev_0\big(([g], w)\big) = [g, \pi_0(g^{-1}w)],$$

where $g \in \mathrm{SO}(n+2)$, $u \in E_0^*$, $w \in \mathbf{C}^{n+2^*}$ and π_0 is the orthogonal projection of $\mathbf{C}^{n+2^*}$ onto L_0^*.

Let t be the holomorphic section of $\mathcal{O}(2) \to \mathbf{C}P^{n+1}$ corresponding to a positive diagonal matrix T_t such that $T_t^2 \in D$. We use a monomorphism $T_t^{-1} \circ i_0$ and an epimorphism $ev_0 \circ T_t$ to obtain a short exact sequence of complex vector bundles

$$0 \longrightarrow F \xrightarrow{T_t^{-1} \circ i_0} Gr_n(\mathbf{R}^{n+2}) \times \mathbf{C}^{n+2^*} \xrightarrow{ev_0 \circ T_t} \mathcal{O}(1) \longrightarrow 0, \tag{4.2.1}$$

where T_t and T_t^{-1} are regarded as automorphisms of a vector bundle $Gr_n(\mathbf{R}^{n+2}) \times \mathbf{C}^{n+2^*} \to Gr_n(\mathbf{R}^{n+2})$. Since $f_0(x) = \mathrm{Ker} ev_{0_x}$, (4.2.1) can also be considered as the pull-back of (4.1.1) by $T_t^{-1} \circ f_0$.

The induced Kähler form ω_{T_t} on $Gr_n(\mathbf{R}^{n+2})$ by $T_t^{-1} \circ f_0$ is the curvature form of the Hermitian metric g_{T_t} of $\mathcal{O}(1) \to Gr_n(\mathbf{R}^{n+2})$ induced by $ev_0 \circ T_t$ up to a constant multiple. Since $\mathrm{Ker}(ev_0 \circ T_{[g]}) = \mathrm{Im}(T^{-1} \circ i_{0[g]}) = T^{-1} g E_0^*$ at a point $[g] \in Gr_n(\mathbf{R}^{n+2})$, where $g \in \mathrm{SO}(n+2)$, we see

$$\big(\mathrm{Ker}(ev_0 \circ T_{t\,[g]})\big)^{\perp} = T_t g L_0^*.$$

Therefore for a unit vector $v \in L_0^*$ we have

$$g_{T_t}((ev \circ T_t)([g], T_t g v), (ev \circ T_t)([g], T_t g v)) = (T_t g v, T_t g v)_{n+2}.$$

Since L_0^* is of complex dimension one, we obtain

$$ev \circ T_t([g], T_t g v) = [g, \pi_0(g^{-1} T_t^2 g v)] = [g, (g^{-1} T_t^2 g v, v)_{n+2} v],$$

and so

$$g_{T_t}([g, v], [g, v]) = \frac{1}{(T_t g v, T_t g v)_{n+2}} = \frac{1}{(g^{-1} T_t^2 g v, v)_{n+2}}. \tag{4.2.2}$$

Since we have a decomposition $\mathbf{C}^{n+2} = \mathbf{R}^{n+2} \oplus \sqrt{-1}\mathbf{R}^{n+2}$ as a real $\mathrm{SO}(n+2)$-module, the space $\mathrm{Herm}(\mathbf{C}^{n+2})$ of Hermitian endomorphisms of $\mathbf{C}^{n+2}$ has the decomposition of irreducible modules:

$$\mathrm{Herm}(\mathbf{C}^{n+2}) = \mathbf{R} \oplus S_0^2\mathbf{R}^{n+2} \oplus \wedge^2\mathbf{R}^{n+2}. \tag{4.2.3}$$

Notice that the second symmetric power $S^2\mathbf{R}^{n+2}$ has the one-dimensional irreducible component $\mathbf{R}$ generated by the identity endomorphism. Its orthogonal complement is denoted by $S_0^2\mathbf{R}^{n+2}$, which is a class one representation space of $(\mathrm{SO}(n+2), \mathrm{SO}(n) \times \mathrm{SO}(2))$. An $\mathrm{SO}(n) \times \mathrm{SO}(2)$-invariant vector $C_0 \in S_0^2\mathbf{R}^{n+2}$ gives a function $\varphi_C = (C, gC_0)_{S_0^2\mathbf{R}^{n+2}}$ on

$Gr_n(\mathbf{R}^{n+2})$, where $g \in \mathrm{SO}(n+2)$ and $(\cdot,\cdot)_{S_0^2\mathbf{R}^{n+2}}$ is the inner product on $S_0^2\mathbf{R}^{n+2}$ inherited from $(\cdot,\cdot)_{n+2}$.

According to (4.2.3), T_t^2 is decomposed into $T_t^2 = \mathrm{Id} + C$, where $C \in S_0^2\mathbf{R}^{n+2}$. Substituting $T_t^2 = \mathrm{Id} + C$ into (4.2.2), we obtain

$$g_{T_t}([g,v],[g,v]) = \frac{1}{1 + (g^{-1}Cgv, v)_{n+2}},$$

for a unit vector v in L_0^*. It is easily seen that $\varphi_C([g])$ coincides with $(g^{-1}Cgv, v)_{n+2}$ up to a constant multiple. Therefore, with an appropriate choice of C_0, we have

$$g_{T_t}([g,v],[g,v]) = \frac{1}{1 + \varphi_C([g])}.$$

It follows that

$$\omega_{T_t} = \omega_0 - \sqrt{-1}\partial\bar{\partial}(1 + \varphi_C). \tag{4.2.4}$$

Remark 9. Since $\sqrt{-1}C$ is an element in $\mathfrak{su}(n+2)$, $\sqrt{-1}\varphi_C([g])$ can be regarded as the restriction to $Gr_n(\mathbf{R}^{n+2})$ of a moment map on $\mathbf{C}P^{n+1}$.

Finally, we specify the Lie algebra $\mathfrak{g}$ of Killing vector fields on the hypersurface Z_t. By Theorem 4.2.1, $\mathfrak{g}$ coincides with the Lie algebra of the group G of holomorphic isometric transformations on Z_t. Since Z_t is holomorphically isomorphic to $Gr_n(\mathbf{R}^{n+2})$, the group of holomorphic transformations of Z_t is $\mathrm{SO}(n+2,\mathbf{C})$ for any generic holomorphic section t of $\mathcal{O}(2) \to \mathbf{C}P^{n+1}$. Thus G is a subgroup of $\mathrm{SO}(n+2,\mathbf{C})$, which preserves ω_{T_t}. Notice that ω_{T_t} belongs to the complexification of $\mathbf{R} \oplus S_0^2\mathbf{R}^{n+2}$, which is an $\mathrm{SO}(n+2,\mathbf{C})$-module. It follows from (4.2.4) that $\psi \in \mathrm{SO}(n+2,\mathbf{C})$ preserves ω_T if and only if both of ω_0 and φ_C are preserved by ψ. This means that ψ preserves ω_{T_t} if and only if $\psi \in \mathrm{SO}(n+2)$ and $\psi T\,{}^t\psi = T$. Hence G is the stabilizer of T_t in $\mathrm{SO}(n+2)$. Thus $\mathfrak{g}$ is a non-trivial subalgebra of $\mathfrak{so}(n+2)$ if and only if $T_t \in D \setminus \mathring{D}$.

Since $\mathrm{SO}(n+2)$ is a subgroup of $\mathrm{SU}(n+2)$, G can also be regarded as a subgroup of $\mathrm{SU}(n+2)$. Therefore $\mathfrak{g}$ is a subalgebra of $\mathfrak{su}(n+2)$.

Chapter 5

Isoparametric functions and Radon transforms on symmetric spaces

5.1 Definition of isoparametric functions

First of all, we give a definition of an isoparametic function on a Riemannian manifold in this paper.

Definition 5.1.1. Let $f : M \to \mathbf{R}$ be a function on a Riemannian manifold (M, g_M). The function f is called an *isoparametric function* if there exist functions $F, G : \mathbf{R} \to \mathbf{R}$ such that

$$(1)\ g_M(df, df) = F(f), \quad (2)\ \Delta f = G(f).$$

The regular level set of an isoparametric function is called an *isoparametric hypersurface*. We recommend [32] for a review of isoparametric hypersurfaces.

Amongst isoparametric hypersurfaces, an isoparametric hypersurface of a sphere is well-known and has been researched for a long time. An isoparametric hypersurface of a sphere has g distinct constant principal curvatures, where $g = 1, 2, 3, 4, 6$ [22]. We give examples of isoparametric functions on a sphere.

Example. ($g = 2$) Let $S^{N-1} \subset \mathbf{R}^N$ be a unit sphere. If we denote a standard coordinate functions on $\mathbf{R}^N$ by $(x_1, \cdots, x_N)$, then

$$\frac{1}{N}\left\{ q\sum_{i=1}^{p} x_i^2 - p\sum_{\alpha=1}^{q} x_\alpha^2 \right\},$$

where $2 \leqq p \leqq N-2$ and $p + q = N$, is an isoparametric function. The regular level set is identified with $S^{p-1} \times S^{q-1}$.

Each isoparametic hypersurface with $g = 1, 2, 3$ is homogeneous in the sense that it is one of orbits of an isometry group of a sphere. Such homogeneous isoparametric hypersurfaces of a sphere are completely classified in Takagi-Takahashi [27] using a result in Hsiang-Lawson [14]. However, there exist a lot of examples of non-homogeneous isoparametric hypersurfaces of a sphere with $g = 4$.

First of all, Nomizu [25] found an isoparametric function with $g = 4$.

Example. ($g = 4$) Let $S^{2N-1} \subset \mathbf{C}^N$ ($N \geqq 3$) be a unit sphere. If a standard coordinate functions on $\mathbf{C}^N$ are denoted by $(x_1 + iy_1, \cdots, x_N + iy_N)$, then

$$\left(\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2\right)^2 + 4\left(\sum_{i=1}^{n} x_i y_i\right)^2$$

is an isoparametric function.

The regular level set is homogeneous in this example.

Ozeki and Takeuchi [26] gave first examples of non-homogeneous isoparametric hypersurfaces with $g = 4$ and Ferus, Karcher and Münzner systematically constructed such hypersurfaces [9], which are nowadays called of OT-FKM type.

5.2 Critical Submanifolds

Let (G, K) be an irreducible symmetric pair of compact type, where G is a simply-connected compact Lie group and K is a closed subgroup of G. The standard involution gives a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the corresponding Lie algebras of G and K, respectively.

We denote by W an irreducible G-module with a G-invariant scalar product, which has a hypersphere as a principal orbit. Such a representation W is called a representation of *spherical type*. Those are classified in Hsiang-Hsiang [13].

G	$SU(n)$	$Spin(n)$	$Spin(7)$	$Spin(9)$	$Sp(n)$
W	$\mathbb{C}^n$, $\mathbb{C}^{n^*}$	$\mathbb{R}^n$	S_7	S_9	$\mathbb{C}^{2n} \cong \mathbb{C}^{2n^*}$

G	$Spin(8)$	G_2
W	S_8^+, S_8^-	$\mathbf{R}^7$

• Table 5.1

In this table, S_n denotes a spin representation of $Spin(n)$ and $S_n^{\pm}$ denote half-spin representations of $Spin(n)$.

Then, it is easily checked that the following happens: either W is decomposed into two irreducible components as K-module $W = U \oplus V$, or W itself is an irreducible K-module. We consider only the former cases. Then on a case-by-case basis, we can show

Lemma 5.2.1. *The decomposition $W = U \oplus V$ is a generalized Cartan decomposition.*

We define two irreducible vector bundles $G \times_K U$ and $G \times_K V$, which are denoted by the same symbols $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$, with canonical connections ∇^U and ∇^V, respectively.

Fix an element $w \in W$ such that $|w| = 1$ and consider the corresponding section $s \in \Gamma(U)$. Denote by H the isotropy subgroup of G at $w \in W$. Our assumption yields that the homogeneous space G/H is a unit sphere in W.

The square of a pointwise norm $f([g]) = |s|^2([g])$ $(g \in G)$ of the section s is a function on G/K. Here, we can take $w \in U \subset W$ without loss of generality, since W is of spherical type.

First of all, we can show

Lemma 5.2.2. *Only the zero set S_0 and the set S_M where the function f attains the maximum value (, which is called the maximum set) are critical submanifolds of $f : G/K \to \mathbf{R}$.*

Lemma 5.2.3. *If neither U nor V is a trivial representation of K, then both sets S_0 and S_M are connected and H-orbits.*

Lemma 5.2.4. *The function is a Morse-Bott function.*

For proofs, see [23] Lemmas 7.3, 7.8 and 7.10. The assumption that W is a G-representation of spherical type is exploited in proofs and we have that K-modules U and V are K-representations of spherical type, if they are not trivial representations of dimension 1. Indeed, we obtain

$$S_0 = \left\{ [g] \in G/K \mid \pi_U([g^{-1}w]) = 0 \right\}, \tag{5.2.1}$$

$$S_M = \left\{ [g] \in G/K \mid \pi_V([g^{-1}w]) = 0 \right\}. \tag{5.2.2}$$

If we denote by T_0 the zero set and by T_M the maximum set of $|t|^2$, then $T_0 = S_M$ and $T_M = S_0$. For this duality, we do not distinguish module U from V. In the case that neither U nor V is a trivial module of K, S_0 and S_M are assumed to be expressed as H/H_0 and H/H_M, respectively, as homogeneous spaces.

Lemma 5.2.5. *If U is not a trivial module of H_0, then S_0 is a singular H-orbit.*

Proof. Since W globally generates a bundle $\mathbf{U} \to G/K$, generic sections in W are transverse to the zero section. The hypothesis that W is a representation of spherical type implies that every section in W except zero is transverse to the zero section. From the transversality of the section, the normal spaces of S_0 can be identified with U. Then the assumption yields the result by so-called slice theorem. □

If we replace U, H_0 and S_0 by V, H_M and S_M, respectively, then the same conclusion holds. In this case, by Hsiang-Lawson [14], S_0 and S_M are minimal submanifolds. However, we can say more.

Theorem 5.2.6. *The critical submanifolds S_0 and S_M are totally geodesic submanifolds of G/K.*

Proof. First of all, we can consider a map into a Grassmannian $i : G/K \to Gr_p(W)$ as the induced map by $(\mathbf{V} \to G/K, W)$ [24, Definition 3.2] (and so, p denotes the dimension of U). Then i is a totally geodesic immersion from Lemma 5.2.1.

On a Grassmannian $Gr_p(W)$, the module U gives the tautological vector bundle $S \to Gr_p(W)$ in a similar fashion, whose pull-back bundle by i is naturally identified with $\mathbf{U} \to G/K$. Then the element $w \in W$ also gives a section $\tilde{s}$ of $S \to Gr_p(W)$ and the pull-back of $\tilde{s}$ is nothing but the section s. Let $\tilde{S}_0$ and $\tilde{S}_M$ be the zero set and the maximum set of $|\tilde{s}|^2$. We take the orthogonal complement space $W^\perp$ of w in W. Then (5.2.1) and (5.2.2) imply that

$$\tilde{S}_0 = Gr_p(W^\perp), \quad \tilde{S}_M = Gr_{p-1}(W^\perp),$$

which are totally geodesic submanifolds of $Gr_p(W)$.

Then S_0 and S_M are the intersections of two totally geodesic submanifolds of $Gr_p(W)$ respectively ($S_0 = G/K \cap \tilde{S}_0$ and $S_M = G/K \cap \tilde{S}_M$), which yields the desired result. □

We give a table which includes symmetric spaces G/K, representation spaces W, stabilizers H, decompositions as K-modules $W = U \oplus V$ and pairs S_0 and S_M. We give a complete list in the table. To do so, we use the coincidences that happen in low dimensions between the various classical Lie groups, which are listed in the Remark after the Table 3.2.

G/K	W	H	$U \oplus V$	S_0, S_M
$\mathrm{SU}(n)/\mathrm{SO}(n)$	$\mathbf{C}^n$	$\mathrm{SU}(n-1)$	$\mathbf{R}^n \oplus \mathbf{R}^n$	$\mathrm{SU}(n-1)/\mathrm{SO}(n-1)$
$Gr_p(\mathbf{C}^n)$	$\mathbf{C}^n$	$\mathrm{SU}(n-1)$	$\mathbf{C}^p \oplus \mathbf{C}^q$	$Gr_p(\mathbf{C}^{n-1})$, $Gr_{p-1}(\mathbf{C}^{n-1})$
$Gr_p(\mathbf{R}^n)$	$\mathbf{R}^n$	$\mathrm{Spin}(n-1)$	$\mathbf{R}^p \oplus \mathbf{R}^q$	$Gr_p(\mathbf{R}^{n-1})$, $Gr_{p-1}(\mathbf{R}^{n-1})$
S^{n-1}	$\mathbf{R}^n$	$\mathrm{Spin}(n-1)$	$\mathbf{R} \oplus \mathbf{R}^{n-1}$	S^{n-1}, 2points
$Gr_4(\mathbf{R}^7)$	S_7	G_2	$\mathbf{R}^4 \oplus \mathbf{R}^4$	$G_2/\mathrm{SO}(4)$, $G_2/\mathrm{SO}(4)$
$Gr_4(\mathbf{R}^8)$	$S_8^\pm$	$\mathrm{Spin}(7)$	$\mathbf{R}^4 \oplus \mathbf{R}^4$	$Gr_4(\mathbf{R}^7)$, $Gr_3(\mathbf{R}^7)$
$Gr_4(\mathbf{R}^9)$	S_9	$\mathrm{Spin}(7)$	$\mathbf{R}^8 \oplus \mathbf{R}^8$	$Gr_4(\mathbf{R}^7)$, $Gr_3(\mathbf{R}^7)$
$\mathrm{Sp}(n)/\mathrm{U}(n)$	$\mathbf{C}^{2n}$	$\mathrm{Sp}(n-1)$	$\mathbf{C}^n \oplus \mathbf{C}^{n^*}$	$\mathrm{Sp}(n-1)/\mathrm{U}(n-1)$
$Gr_p(\mathbf{H}^n)$	$\mathbf{H}^n$	$\mathrm{Sp}(n-1)$	$\mathbf{H}^p \oplus \mathbf{H}^q$	$Gr_p(\mathbf{H}^{n-1})$, $Gr_{p-1}(\mathbf{H}^{n-1})$
$G_2/\mathrm{SO}(4)$	$\mathbf{R}^7$	$\mathrm{SU}(3)$	$\mathbf{R}^4 \oplus \mathbf{R}^3$	$\mathrm{SU}(3)/\mathrm{SO}(3)$, $\mathbf{C}P^2$

• Table 5.2

Remark 10. We now list the coincidences of a pair of symmetric spaces and representations

W omitted in the table.

$$\begin{aligned}
&(\mathrm{SU}(2)/\mathrm{SO}(2), \mathfrak{su}(2)) = (S^2, \mathbf{R}^3),\\
&\left(\mathrm{SU}(4)/\mathrm{SO}(4), \mathbf{R}^6 = \wedge^2 \mathbf{C}^{4^{\mathbf{R}}}\right) = \left(Gr_3(\mathbf{R}^6), \mathbf{R}^6\right),\\
&\left(\mathrm{SU}(4)/\mathrm{Sp}(2), \mathbf{R}^6\right) = \left(S^5, \mathbf{R}^6\right),\\
&\left(Gr_4(\mathbf{R}^6), \mathbf{C}^4\right) = \left(Gr_2(\mathbf{C}^4), \mathbf{C}^4\right),\\
&\left(Gr_2(\mathbf{R}^5), \mathbf{C}^4\right) = \left(\mathrm{Sp}(2)/\mathrm{U}(2), \mathbf{C}^4\right),\\
&\left(\mathrm{SO}(6)/\mathrm{U}(3), \mathbf{C}^4\right) = (\mathbf{C}P^3, \mathbf{C}^4),\\
&(\mathrm{Sp}(1)/\mathrm{U}(1), \mathfrak{sp}(1)) = (S^2, \mathbf{R}^3).
\end{aligned}$$

5.3 Isoparametric functions

Let G/K, W, H and f be as in the previous section. In this section, the level set of the function $f : G/K \to \mathbb{R}$ is our main concern. Since $H \subset G$ is an isotropy subgroup at $w \in W$, f is invariant under the action of H. Hence, H acts on the level set of f.

We can easily show

Lemma 5.3.1. *If the action of H on G/K is of cohomogeneity one, then f is an isoparametric function.*

Because $|\mathrm{grad}\, f|^2$ and Δf are also invariant under the action of H, and so they are constant functions on the level set of f.

The actions of H are of cohomogeneity one except the following cases:

$$(\mathrm{SU}(n)/\mathrm{SO}(n), \mathbf{C}^n), \quad \left(\mathrm{Sp}(n)/\mathrm{U}(n), \mathbf{C}^{2n}\right), \quad \left(Gr_4(\mathbf{R}^9), S_9\right).$$

In the above cases, the cohomogeneity of the actions are 2, 3 and 2, respectively.

In the case of cohomogeneity one, we can easily describe the level set of f as a unit sphere bundle of S_0 or S_M, and show that all level sets are H-orbits, which are left to the reader.

From now on, we would like to compute geometric invariants of submanifolds, more precisely, mean curvatures and principal curvatures. These invariants are related to invariants of vector bundles.

Theorem 5.3.2. *We have*

$$\Delta s = \frac{n}{p} s, \quad \Delta t = \frac{n}{q} t, \quad n := \dim G/K,$$

for arbitrary $s \in W \subset \Gamma(\mathbf{U})$ and $t \in W \subset \Gamma(\mathbf{V})$, when W is an orthogonal representation.

We also have

$$\Delta s = \frac{n}{2p} s, \quad \Delta t = \frac{n}{2q} t, \quad n := \dim G/K,$$

for arbitrary $s \in W \subset \Gamma(\mathbf{U})$ *and* $t \in W \subset \Gamma(\mathbf{V})$*, when* W *is a unitary representation.*

Proof. From Theorem 2.2.2, we see that the mean curvature operators A and B are parallel. Since $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$ are irreducible, we have

$$B = -\mu Id_U, \quad A = -\nu Id_V$$

for some constant μ and ν. Since $i : G/K \to Gr_p(W)$ is totally geodesic (hence harmonic), Theorem 2.2.3 yields that

$$\Delta s = \mu s, \quad \Delta t = \nu t.$$

Since $i : G/K \to Gr_p(W)$ is an isometric immersion, the definition of the Riemannian metric g_{Gr} yields that

$$n = \sum g_{Gr}(e_i, e_i) = -\sum \operatorname{trace} J_{e_i} I_{e_i} = -\operatorname{trace} A = -\operatorname{trace} B,$$

when W is a real representation, and

$$n = -2\operatorname{trace} A = -2\operatorname{trace} B,$$

when W is a complex representation. Hence we have our desired results. □

We fix $w \in W$ ($|w| = 1$) again and consider the function $f = |s|^2$.

Theorem 5.3.3. *We have that*

$$\Delta f = \frac{2nN}{pq} \left(f - \frac{p}{N} \right),$$

when W *is an orthogonal representation and*

$$\Delta f = \frac{nN}{pq} \left(f - \frac{p}{N} \right),$$

when W *is a unitary representation.*

Proof. Notice that $w \in W$ also induces a section of $S \to Gr_p(W)$ denoted by $\tilde{s}$. It follows that the pull-back section of $\tilde{s}$ is nothing but $s \in \Gamma(\mathbf{U})$. From Proposition 2.1.2, we see that $\nabla^S \tilde{s} = -J\tilde{t}$ on Grassmannian, where $\tilde{t}$ is the corresponding section. Since $i : G/K \to Gr_p(W)$ is a totally geodesic immersion and ∇^U is regarded as the pull-back connection of ∇^S, we also have $\nabla^U s = -Jt$. Then we obtain

$$|Jt|^2 = \sum g_U \left(J_{e_i} t, J_{e_i} t \right) = -g_V \left(At, t \right) = g_V \left(\Delta t, t \right).$$

The well-known formula

$$\Delta|s|^2 = g_U\left(\Delta s, s\right) + g_U\left(s, \Delta s\right) - 2|\nabla^U s|^2$$

yields that

$$\Delta|s|^2 = 2g_U\left(\Delta s, s\right) - 2g_V\left(\Delta t, t\right).$$

Theorem 5.3.2 yields the result. □

Hence, the function f always satisfies the condition (2) of the definition of an isoparametric function.

However, $|\mathrm{grad}\, f|^2 = |df|^2$ does not satisfy the condition (1) in general. We distinguish the case that the action of H is of the cohomogeneity one from others.

5.3.1 The case of cohomogeneity one

In this subsection, we omit the case that G/K is a sphere. Hence, in the decomposition $W = U \oplus V$, U and V are K-representations of spherical type. Moreover, S_0 and S_M are singular H-orbits, which are expressed as H/H_0 and H/H_M, respectively. Since the action of H is of cohomogeneity one, U is a representation of H_0 of spherical type and V is a representation of H_M of spherical type.

Let $\mathbf{n}$ be a unit normal vector field defined by

$$\mathbf{n} = \frac{\mathrm{grad}\, f}{|\mathrm{grad}\, f|},$$

on the regular point of f. We denote by $A_{\mathbf{n}}$ the shape operator of $f^{-1}(c)$, where c is a regular value. By definition, we have that

$$A_{\mathbf{n}}X = -D_X\mathbf{n} = -X\left(\frac{1}{|df|}\right)\mathrm{grad}\, f - \frac{1}{|df|}D_X\mathrm{grad}\, f,$$

where X is a tangent vector to $f^{-1}(c)$ and D is the Levi-Civita connection on G/K. Since f is an isoparametric function, the first term of the right-hand-side vanishes. Consequently, we have that

$$g\left(A_{\mathbf{n}}X, Y\right) = -\frac{1}{|df|}\left(D_X df\right)(Y),$$

where X and Y are tangent vectors to $f^{-1}(c)$ and g is the Riemannian metric on G/K. The definition of f yields that

$$\begin{aligned}\left(D_X df\right)(Y) = g_U&\left(\nabla^U_X\left(\nabla^U s\right)(Y), s\right) + g_U\left(s, \nabla^U_X\left(\nabla^U s\right)(Y)\right)\\ &+ g_U\left(\nabla^U_X s, \nabla^U_Y s\right) + g_U\left(\nabla^U_Y s, \nabla^U_X s\right).\end{aligned}$$

Since $W = U \oplus V$ is a generalized Cartan decomposition, Proposition 2.1.2 yields that

$$\nabla^U_X s = -Jt, \quad \nabla^U_X \left(\nabla^U s\right)(Y) = J_Y I_X s,$$

where t is the corresponding section. It follows that

$$\begin{aligned}(D_X df)(Y) = &-g_V\left(I_X s, I_Y s\right) - g_V\left(I_Y s, I_X s\right) \\ &+ g_U\left(J_X t, J_Y t\right) + g_U\left(J_Y t, J_X t\right).\end{aligned}$$

We define endomorphisms $\tilde{I}$ and $\tilde{J}$ of the tangent bundle of G/K by

$$g(\tilde{I}X, Y) = \frac{1}{2}\left\{g_V\left(I_X s, I_Y s\right) + g_V\left(I_Y s, I_X s\right)\right\}$$

and

$$g(\tilde{J}X, Y) = \frac{1}{2}\left\{g_U\left(J_X t, J_Y t\right) + g_U\left(J_Y t, J_X t\right)\right\}.$$

By definition, we obtain

$$A_{\mathbf{n}} = \frac{2}{|df|}\left(\tilde{I} - \tilde{J}\right). \tag{5.3.1}$$

We can immediately see

Lemma 5.3.4. *The endomorphisms $\tilde{I}$ and $\tilde{J}$ are H-invariant symmetric operators.*

To see properties of $\tilde{I}$ and $\tilde{J}$, we give a key algebraic theorem.

We denote by $\mathfrak{h}$ the corresponding Lie subalgebra to H and a natural projection by $\pi : G \to G/K$.

Theorem 5.3.5. *In the case that the H-action on G/K is of cohomogeneity one, for an arbitrary $\xi \in \mathfrak{m}$ such that $\xi \perp \mathfrak{h}$ and $|\xi w| = 1$, we have that*

$$\xi^2 w = -w.$$

Proof. Let N be the normal space of S_M at $\pi(e)$, where e is a unit element of G. The subgroup $L \subset G$ defined as $L := K \cap H$ is isomorphic to H_M and acts on N as a representation of spherical type, since the action of H is of cohomogeneity one.

Since W globally generates $\mathbf{V} \to G/K$ and is a representation of spherical type, t is transverse to the zero section. Hence we have that

$$T_x S_M = \operatorname{Ker} \nabla^V t = \operatorname{Ker} Is = \left\{X \in T_x G/K \mid I_X s = 0\right\}.$$

It follows that $T_{\pi(e)} G/K = \operatorname{Ker} Is \oplus_\perp N$.

We may regard $Is : T_{\pi(e)} G/K \to V_{\pi(e)}$ as an homomorphism $Is : \mathfrak{m} \to V$ and consider $N \subset \mathfrak{m}$. Since $s\left(\pi(e)\right) = [e, w]$, Is is an L-equivariant homomorphism. Hence V is also an L-representation of spherical type which is isomorphic to N.

Let $\xi \in N$ such that $|I_\xi s| = |\xi w| = 1$. Then we obtain $L_\xi \subset L$ as an isotropy subgroup at ξ and $I_\xi : \mathbf{U} \to V$ is an L_ξ-equivariant homomorphism. The endomorphism $J_\xi I_\xi : U \to U$ can be now regarded as $\xi^2 : U \to U$, which is a restriction of $\xi^2 : W \to W$ to $U \subset W$. Note that the eigenvalues except zero of $\xi^2|_U$ are the same as ones of $\xi^2|_V$ with multiplicities, since $W = U \oplus V$ is a generalized Cartan decomposition. Then L_ξ irreducible decompositions of U and V, which are given after the proof, yield that $\xi^2 w = cw$ with some constant $c \in \mathbf{R}$ by Schur's lemma. It follows that $c = \langle \xi^2 w, w\rangle = -\langle \xi w, \xi w\rangle = -1$. □

We shall exploit L_ξ-decomposition in the sequel. We denote by $\mathfrak{l}$ and $\mathfrak{l}_\xi$ the corresponding Lie subalgebras to L and L_ξ, respectively.

• L_ξ-decomposition of $(G/K, W)$.

(1) $\big(Gr_p(\mathbf{R}^N), \mathbf{R}^N\big)$,

Let $e_1, \cdots, e_N$ be an orthonormal basis of $\mathbf{R}^N$ such that $e_1, \cdots, e_p$ spans $\mathbf{R}^p$. We take $w = e_1$ and so, $\mathfrak{l} = \mathfrak{so}(p-1) \oplus \mathfrak{so}(q)$, where $q := N - p$. Let ξ be a skew endomorphism of $\mathbf{R}^N$ such that

$$\xi e_1 = e_{p+1},\ \xi e_{p+1} = -e_1, \text{and } \xi e_A = 0,\ A \neq 1, p+1.$$

Notice that $\xi \in \mathfrak{m} \cap \mathfrak{h}^\perp$ with $|\xi w| = 1$. It follows that $\mathfrak{l}_\xi$ is isomorphic to $\mathfrak{so}(p-1) \oplus \mathfrak{so}(q-1)$. Then we have

$$U = \mathbf{R}^p = \mathbf{R}w \oplus \mathbf{R}^{p-1}, \quad V = \mathbf{R}^q = \mathbf{R}\xi w \oplus \mathbf{R}^{q-1}.$$

(2) $\big(Gr_p(\mathbf{C}^N), \mathbf{C}^N\big)$,

Let $e_1, \cdots, e_N$ be a unitary basis of $\mathbf{C}^N$ such that $e_1, \cdots, e_p$ spans $\mathbf{C}^p$. We take $w = e_1$ and so, $\mathfrak{l} = \mathfrak{u}(1) \oplus \mathfrak{su}(p-1) \oplus \mathfrak{su}(q)$. Let ξ be a skew Hermitian endomorphism of $\mathbf{C}^N$ such that

$$\xi e_1 = e_{p+1},\ \xi e_{p+1} = -e_1, \text{and } \xi e_A = 0,\ A \neq 1, p+1.$$

Notice that $\xi \in \mathfrak{m} \cap \mathfrak{h}^\perp$ with $|\xi w| = 1$. It follows that $\mathfrak{l}_\xi$ is isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{su}(p-1) \oplus \mathfrak{su}(q-1)$. Then we have

$$U = \mathbf{C}^p = \mathbf{C}w \oplus \mathbf{C}^{p-1}, \quad V = \mathbf{C}^q = \mathbf{C}\xi w \oplus \mathbf{C}^{q-1}.$$

(3) $\big(Gr_p(\mathbf{H}^N), \mathbf{H}^N\big)$,

Let $e_1, \cdots, e_N$ be a quaternion-unitary basis of $\mathbf{H}^N$ such that $e_1, \cdots, e_p$ spans $\mathbf{H}^p$. We take $w = e_1$ and so, $\mathfrak{l} = \mathfrak{sp}(p-1) \oplus \mathfrak{sp}(q)$. Let ξ be a quaternion-skew Hermitian endomorphism of $\mathbf{H}^N$ such that

$$\xi e_1 = e_{p+1},\ \xi e_{p+1} = -e_1, \text{and } \xi e_A = 0,\ A \neq 1, p+1.$$

Notice that $\xi \in \mathfrak{m} \cap \mathfrak{h}^\perp$ with $|\xi w| = 1$. It follows that $\mathfrak{l}_\xi$ is isomorphic to $\mathfrak{sp}(p-1) \oplus \mathfrak{sq}(q-1)$. Then we have

$$U = \mathbf{H}^p = \mathbf{H}w \oplus \mathbf{H}^{p-1}, \quad V = \mathbf{H}^q = \mathbf{H}\xi w \oplus \mathbf{H}^{q-1}.$$

(4) $(Gr_4(\mathbf{R}^7), S_7)$,

The isotropy subalgebra is isomorphic to $\mathfrak{so}(4)\oplus\mathfrak{sp}(1)$. The Lie algebra $\mathfrak{so}(4)$ is a direct sum of two copies of $\mathfrak{sp}(1)$. To distinguish these copies of $\mathfrak{sp}(1)$, the isotropy subalgebra is denoted by $\mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1) \oplus \mathfrak{sp}(1)$.

Under the action of the isotropy subalgebra on S_7, we have an irreducible decomposition:

$$S_7 = \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2\right)^{\mathbf{R}} \oplus \left(\mathbf{C}^2_- \otimes \mathbf{C}^2\right)^{\mathbf{R}},$$

where $\mathbf{C}^2_{(\pm)}$ denote the standard representations of $\mathfrak{sp}_{(\pm)}(1)$, respectively and $\left(\mathbf{C}^2_\pm \otimes \mathbf{C}^2\right)^{\mathbf{R}}$ denote real invariant spaces of $\mathbf{C}^2{}_\pm \otimes \mathbf{C}^2$, respectively.

We pick up a unit vector $w \in \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2\right)^{\mathbf{R}}$ and so, $\mathfrak{l}$ is regarded as the diagonal subalgebra of $\mathfrak{sp}_+(1)\oplus\mathfrak{sp}(1)$. Let $v \in \left(\mathbf{C}^2_- \otimes \mathbf{C}^2\right)^{\mathbf{R}}$ be a unit vector. Since L_ξ ($\xi \in \mathfrak{m}\cap\mathfrak{h}^\perp$ with $|\xi w| = 1$) can be identified with an isotropy subgroup of the L-action on S_7 at v, it follows that $\mathfrak{l}_\xi$ is isomorphic to the subalgebra $\{(X, X, X)\}$ of $\mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1) \oplus \mathfrak{sp}(1)$. Then we have

$$U = \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2\right)^{\mathbf{R}} = \mathbf{R}w \oplus \mathbf{R}^3, \quad V = \left(\mathbf{C}^2_- \otimes \mathbf{C}^2\right)^{\mathbf{R}} = \mathbf{R}v \oplus \mathbf{R}^3,$$

where $\mathbf{R}^3$ denotes the adjoint representation of $\mathfrak{l}_\xi$.

(5) $(G_2/\mathrm{SO}(4), \mathbf{R}^7)$,

To distinguish two copies of $\mathfrak{sp}(1)$, the isotropy subalgebra is denoted by $\mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1)$.

Under the action of the isotropy subalgebra on $\mathbf{R}^7$, we have an irreducible decomposition:

$$\mathbf{R}^7 = \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2_-\right)^{\mathbf{R}} \oplus \mathfrak{sp}_-(1),$$

where $\mathbf{C}^2_\pm$ denote the standard representations of $\mathfrak{sp}_\pm(1)$, respectively and $\left(\mathbf{C}^2_+ \otimes \mathbf{C}^2_-\right)^{\mathbf{R}}$ denotes a real invariant space of $\mathbf{C}^2{}_+ \otimes \mathbf{C}^2_-$.

We pick up a unit vector $w \in \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2_-\right)^{\mathbf{R}}$ and so, $\mathfrak{l}$ is regarded as the diagonal subalgebra Δ of $\mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1)$. Let $v \in \mathfrak{sp}_-(1)$ be a unit vector. Since L_ξ ($\xi \in \mathfrak{m}\cap\mathfrak{h}^\perp$ with $|\xi w| = 1$) can be identified with an isotropy subgroup of L-action on $\mathbf{R}^7$ at v, we have that $\mathfrak{l}_\xi$ is isomorphic to $\mathfrak{u}(1)$ which is the standard subalgebra of Δ. Then we have

$$U = \left(\mathbf{C}^2_+ \otimes \mathbf{C}^2_-\right)^{\mathbf{R}} = \mathbf{R}w \oplus \mathbf{R} \oplus \mathbf{C}_2, \quad V = \mathfrak{sp}_-(1) = \mathbf{R}v \oplus \mathbf{C}_2,$$

where $\mathbf{C}_\alpha$ denotes an irreducible representation of $\mathfrak{u}(1)$ with weight α.

Remark 11. We should consider the case of $\left(Gr_4(\mathbf{R}^8), S_8^\pm\right)$. However, the triality gives the same picture as in the case of $(Gr_4(\mathbf{R}^8), \mathbf{R}^8)$, and so we omit it.

Corollary 5.3.6. *We can find a geodesic on G/K which intersects all H-orbits orthogonally.*

Proof. For any $\xi \in \mathfrak{m}$ such that $\xi \perp \mathfrak{h}$ and $|\xi w| = 1$, Theorem 5.3.5 yields that

$$e^{t\xi} w = \sum \frac{1}{n!} \xi^n w = \cos t w + \sin t v,$$

where we put $v := \xi w \in V$. Then, $\pi(e^{t\xi})$ is a geodesic through $\pi(e)$.

Moreover, we get

$$s\left(\pi(e^{t\xi})\right) = \left[e^{t\xi}, \pi_U\left(e^{-t\xi} w\right)\right] = \cos t\left[e^{t\xi}, w\right] = \cos t e^{t\xi} s\left(\pi(e)\right).$$

Hence,

$$f\left(\pi(e^{t\xi})\right) = \cos^2 t,$$

and so, the geodesic $\pi(e^{t\xi})$ meets all H-orbits.

Since $\xi \perp \mathfrak{h}$, ξ can be regarded as a normal vector of S_M in G/K. We can identify the normal bundle of an H-orbit with a neighbourhood of the H-orbit G-equivariantly via an exponential map restricted to the normal space. Hence the geodesic $\pi(e^{t\xi})$ intersects all H-orbits orthogonally by Gauss's lemma. □

Remark 12. The existence of a geodesic which intersects all orbits orthogonally is well-known in the case that the action is of cohomogeneity one. However, we exploit our geodesic $\pi(e^{t\xi})$ to compute submanifold-geometric invariants including principal curvatures of the regular level set explicitly. To do so, we fix the notation $\pi(e^{t\xi})$ to express the specified geodesic.

For simplicity, we put $o := \pi(e) \in G/K$.

Theorem 5.3.7. *The endomorphism $\tilde{I}$ has only two eigenspaces, which is expressed as $T_x G/K = E_1 \oplus E_2$, where $s(x) \neq 0$. The eigenspace E_1 with zero eigenvalue is indeed* $\mathrm{Ker}\, Is$*, where we regard Is as a homomorphism $Is : TG/K \to \mathbf{V}$. Both E_1 and E_2 can be identified with $T_o S_M$ and V, respectively, via a parallel transport along the geodesic $\pi\left(e^{t\xi}\right)$ and an action of H, where $\xi \in \mathfrak{m} \cap \mathfrak{h}^\perp$.*

Proof. As we already show,

$$T_o S_M = \mathrm{Ker}\, \nabla^V t = \left\{X \in T_o G/K \mid \nabla^V_X t = 0\right\}.$$

It follows from $\nabla^V t = -Is$ that $T_o S_M$ is included in the eigenspace of $\tilde{I}$ with zero eigenvalue.

Let L be an isotropy subgroup of H at $o \in S_M$. Then we already see that $N(\cong V)$ is an irreducible representation of L. From Lemma 5.3.4, V must be an eigenspace of $\tilde{I}$, because L acts on each eigenspace.

Let $x \in G/K$ be a point outside S_M and suppose that $s(x) \neq 0$. It follows from H-invariance of $\tilde{I}$ that x can be assumed to be joined to o by $\pi(e^{t\xi})$ and $x = \pi(e^{t_0\xi})$ for some $\xi \in \mathfrak{m} \cap \mathfrak{h}^\perp$. If we put $g(x) = e^{t_0\xi}$, then $x = g(x)o$ and $s(x) = \cos t_0 g(x) s(o)$.

Since I is G-invariant, if X and $Y \in T_x G/K$, then we obtain

$$
\begin{aligned}
g_x\left(\tilde{I}X, Y\right) &= \frac{1}{2}\left\{g_{V_x}\left(I_X s(x), I_Y s(x)\right) + g_{V_x}\left(I_Y s(x), I_X s(x)\right)\right\} \\
&= \frac{1}{2}\Big\{g_{V_x}\left(I_{g(x)g(x)^{-1}X}\cos t_0 g(x)s(o), I_{g(x)g(x)^{-1}Y}\cos t_0 g(x)s(o)\right) \\
&\quad + g_{V_x}\left(I_{g(x)g(x)^{-1}Y}\cos t_0 g(x)s(o), I_{g(x)g(x)^{-1}X}\cos t_0 g(x)s(o)\right)\Big\} \\
&= \frac{f(y)}{2}\Big\{g_{V_y}\left(g(x)I_{g(x)^{-1}X}s(o), g(x)I_{g(x)^{-1}Y}s(o)\right) \\
&\quad + g_{V_y}\left(g(x)I_{g(x)^{-1}Y}s(o), g(x)I_{g(x)^{-1}X}s(o)\right)\Big\} \\
&= \frac{f(y)}{2}\Big\{g_{V_o}\left(I_{g(x)^{-1}X}s(o), I_{g(x)^{-1}Y}s(o)\right) + g_{V_o}\left(I_{g(x)^{-1}Y}s(o), I_{g(x)^{-1}X}s(o)\right)\Big\} \\
&= f(y)g_o\left(\tilde{I}g(x)^{-1}X, g(x)^{-1}Y\right).
\end{aligned}
$$

It follows that $T_x G/K = g(x)T_o S_M \oplus g(x)V_o$ is the eigenspace decomposition of the endomorphism $\tilde{I}_x$. It also follows that $g(x)T_o S_M = \operatorname{Ker} Is$. □

Lemma 5.3.8. *The normal vector field* $\mathbf{n}$ *belongs to* E_2*, where* $df \neq 0$.

Proof. From Corollary 5.3.6, the velocity vector of the geodesic $\pi(e^{t\xi})$ is a constant multiple of the unit normal vector field $\mathbf{n}$.

By Theorem 5.3.7, the eigenspace E_1 corresponding to zero eigenvalue is the image of a parallel transport of TS_M along $\pi(e^{t\xi})$. Then, we have that $\mathbf{n} \perp E_1$. The H-invariance gives our desired result. □

Remark 13. It is well-known (and easily shown) that the unit normal vector field $\mathbf{n}$ generates a geodesic if the function satisfies the condition (1) of Definition 5.1.1.

We denote by λ an eigenvalue of $\tilde{I}$ whose eigenspace is $E_2 \cong V$.

Theorem 5.3.9. *The eigenvalue* λ *is equal to* $\frac{n}{pq}|s|^2$ *when* W *is real,* $\frac{n}{4pq}|s|^2$ *when* W *is complex.*

Proof. In both cases, we have

$$\sum g(\tilde{I}e_i, e_i) = g_V\left(I_{e_i}s, I_{e_i}s\right) = -g_U(Bs, s) = g_U(\Delta s, s). \tag{5.3.2}$$

• W:**real**. By definition, we have $\sum g(\tilde{I}e_i, e_i) = q\lambda$. From (5.3.2) and Theorem 5.3.2, we get

$$q\lambda = \frac{n}{p}|s|^2.$$

• W:**complex**. By definition, we have $\sum g(\tilde{I}e_i, e_i) = 2q\lambda$. From (5.3.2) and Theorem 5.3.2, we get

$$2q\lambda = \frac{n}{2p}|s|^2.$$

□

In a similar way, we have

Theorem 5.3.10. *The eigenspaces of $\tilde{J}$ can be identified with U and TS_0 via a parallel transport along the geodesic $\pi(e^{t\xi})$ and an H-action. The eigenvalue corresponding to the eigenspace U is $\frac{n}{pq}|t|^2$ when W is real, $\frac{n}{4pq}|t|^2$ when W is complex. The eigenspace TS_0 is the kernel of $\tilde{J}$.*

For simplicity, it is said that the eigenspaces of $\tilde{I}$ are V and TS_M and the eigenspaces of $\tilde{J}$ are U and TS_0, when no confusion can arise.

Lemma 5.3.11. *The unit normal vector field* $\mathbf{n}$ *is the eigenvector of $\tilde{J}$ which belongs to U.*

We can compute the norm of the velocity vector of the geodesic $\pi(e^{t\xi})$.

Lemma 5.3.12. *Let $\xi \in \mathfrak{m} \cap \mathfrak{h}^{\perp}$ with $|\xi w| = 1$. The square of the norm $|\xi|^2$ is equal to $\frac{pq}{n}$, when W is real, $\frac{4pq}{n}$, when W is complex.*

Proof. On the one hand, since ξ is a constant multiple of $\mathbf{n}$, we get $\tilde{I}\xi = \lambda\xi$, from Lemma 5.3.8, where λ is the eigenvalue different from zero. It follows that

$$g(\tilde{I}\xi, \xi) = \lambda|\xi|^2$$

On the other hand, the definition gives $g(\tilde{I}\xi, \xi) = g_V(I_\xi s, I_\xi s)$. Since G/K is a totally geodesic submanifold of $Gr_p(W)$, we can compute

$$I_\xi s\left(\pi(e^{t\xi})\right) = \left[e^{t\xi}, \xi \cos tw\right] = \cos t\left[e^{t\xi}, \xi w\right].$$

Since $|\xi w| = 1$, we obtain $|I_\xi s|^2 = \cos^2 t|\xi w|^2 = |s|^2$.

We immediately get $\lambda|\xi|^2 = |s|^2$, which provides us with the result by Theorem 5.3.9.

□

Theorem 5.3.13. *The norm of the gradient vector* $\operatorname{grad} f$ *is given by*

$$|df| = \begin{cases} 2|s||t|\sqrt{\frac{n}{pq}}, & \text{when } W \text{ is real,} \\ |s||t|\sqrt{\frac{n}{pq}}, & \text{when } W \text{ is complex.} \end{cases}$$

Proof. Let $\xi \in \mathfrak{m} \cap \mathfrak{h}^{\perp}$ with $|\xi w| = 1$. It is enough to compute the norm on the geodesic $\pi(e^{t\xi})$ due to H-invariance. Note that the corresponding section t is expressed as

$$t\left(\pi(e^{t\xi})\right) = \left[e^{t\xi}, -\sin tv\right] = -\sin t\left[e^{t\xi}, v\right] = -\sin t e^{t\xi} t\,(o)\,.$$

Moreover, we get from $\xi v = -w$ that

$$J_{\xi} t\left(\pi(e^{t\xi})\right) = -\sin t\left[e^{t\xi}, \xi v\right] = \tan t\left[e^{t\xi}, w\right] = \frac{|t|}{|s|} s\left(\pi(e^{t\xi})\right).$$

Then, we have

$$\begin{aligned}
|df|^2 =& \sum \left(g_U\left(\nabla^U_{e_i} s, s\right) + g_U\left(s, \nabla^U_{e_i} s\right)\right)^2 \\
=& \sum \left(g_U\left(J_{e_i} t, s\right) + g_U\left(s, J_{e_i} t\right)\right)^2 \\
=& \sum \frac{|s|^2}{|t|^2}\left(g_U\left(J_{e_i} t, J_{\xi} t\right) + g_U\left(J_{\xi} t, J_{e_i} t\right)\right)^2 = \sum 4\frac{|s|^2}{|t|^2}\left(g(\tilde{J} e_i, \xi)\right)^2.
\end{aligned}$$

We can take $e_n = \mathbf{n}$ and already see that $\xi = |\xi|\mathbf{n}$ (up to a sign). Theorem 5.3.10 and Lemma 5.3.12 yield that

$$|df|^2 = 4\frac{|t|^2}{|s|^2}|\xi|^2\mu^2,$$

where μ is the eigenvalue of $\tilde{J}$ different from zero. $\square$

Remark 14. From Theorems 5.3.3 and 5.3.13, it follows that

$$\Delta f = \frac{2nN}{pq}\left(f - \frac{p}{N}\right), \quad |df|^2 = \frac{4n}{pq} f(1-f),$$

when W is real, and

$$\Delta f = \frac{nN}{pq}\left(f - \frac{p}{N}\right), \quad |df|^2 = \frac{n}{pq} f(1-f),$$

when W is complex. If we define a new function $\tilde{f}$ by

$$\tilde{f} = f - \frac{p}{N},$$

then we have

$$\Delta \tilde{f} = \frac{2nN}{pq}\tilde{f}, \quad |d\tilde{f}|^2 = \frac{4n}{pq}\left(\tilde{f} + \frac{p}{N}\right)\left(\frac{q}{N} - \tilde{f}\right),$$

when W is real, and

$$\Delta \tilde{f} = \frac{nN}{pq}\tilde{f}, \quad |d\tilde{f}|^2 = \frac{n}{pq}\left(\tilde{f} + \frac{p}{N}\right)\left(\frac{q}{N} - \tilde{f}\right),$$

when W is complex.

Let c be a regular value of the function $f : G/K \to \mathbf{R}$. We can compute the mean curvature m of the hypersurface $f^{-1}(c)$. Notice that $|s|^2 = c$ and $|t|^2 = \sqrt{1-c}$, by definition. Hence, instead of using c, we employ $|s|$ and $|t|$ to express invariants on $f^{-1}(c)$.

Theorem 5.3.14. *Let m be the mean curvature of the regular level set $f^{-1}(c)$. Then m is expressed:*

$$m = \begin{cases} \frac{1}{|s||t|}\sqrt{\frac{n}{pq}}\left\{|s|^2(q-1) - |t|^2(p-1)\right\}, & \text{when } W \text{ is real,} \\ \frac{1}{2|s||t|}\sqrt{\frac{n}{pq}}\left\{|s|^2(2q-1) - |t|^2(2p-1)\right\}, & \text{when } W \text{ is complex.} \end{cases}$$

Proof. From (5.3.1), Theorems 5.3.7, 5.3.9 and 5.3.10 and Lemmas 5.3.8 and 5.3.12, it follows that

$$m = \sum_{i=1}^{n-1} g(A_{\mathbf{n}}e_i, e_i) = \frac{2}{|df|}\left\{\operatorname{trace}\tilde{I} - g(\tilde{I}\mathbf{n}, \mathbf{n}) - \operatorname{trace}\tilde{J} + g(\tilde{J}\mathbf{n}, \mathbf{n})\right\},$$

where $e_1, \cdots, e_n = \mathbf{n}$ is an orthonormal basis of TG/K. Using again Lemmas 5.3.8 and 5.3.12, Theorem 5.3.13 yield the result. $\square$

Remark 15. Using only the function f, m is described as

$$m = \begin{cases} \sqrt{\frac{n}{pqf(1-f)}}\left\{(N-2)f - (p-1)\right\}, & \text{when } W \text{ is real,} \\ \sqrt{\frac{n}{4pqf(1-f)}}\left\{2(N-1)f - (2p-1)\right\}, & \text{when } W \text{ is complex.} \end{cases}$$

Corollary 5.3.15. *There exists one and only one minimal regular level set of the function f. More precisely, $f^{-1}(c)$ is a minimal hypersurface, where*

$$c = \begin{cases} \frac{p-1}{N-2}, & \text{when } W \text{ is real,} \\ \frac{2p-1}{2(N-1)}, & \text{when } W \text{ is complex.} \end{cases}$$

Next, we compute principal curvatures, in other words, the eigenvalues of $A_{\mathbf{n}}$. From (5.3.1), we should see how the eigenspaces of $\tilde{I}$ and $\tilde{J}$ intersect with each other.

As we have already seen, the eigendecomposition of $\tilde{I}$ is expressed as

$$T_oG/K \cong \mathfrak{m} = \{X \in \mathfrak{m} \,|\, Xw = 0\} \oplus_\perp V = TS_M \oplus_\perp V.$$

We put $g_0 := e^{\frac{\pi}{2}\xi}$. In a similar way, we have

$$g_0^{-1}\left(T_{\pi(g_0)}G/K\right) \cong \mathfrak{m} = \{X \in \mathfrak{m} \,|\, Xv = 0\} \oplus_\perp U = g_0^{-1}\left(T_{\pi(g_0)}S_0\right) \oplus_\perp U.$$

We use the same notation as in L_ξ-decomposition of $(G/K, W)$ after Theorem 5.3.5.

• Principal curvatures.

By (5.3.1), Theorems 5.3.7, 5.3.9, 5.3.10 and 5.3.13 imply

Lemma 5.3.16. *The shape operator $A_{\mathbf{n}}$ satisfies*

$$A_{\mathbf{n}} = \begin{cases} 0 & (\text{on } T_oS_M \cap g_0^{-1}(T_{\pi(g_0)}S_0)) \\ \dfrac{|s|}{a|t|}\sqrt{\dfrac{n}{pq}} & (\text{on } g_0^{-1}(T_{\pi(g_0)}S_0) \cap V) \\ -\dfrac{|t|}{a|s|}\sqrt{\dfrac{n}{pq}} & (\text{on } T_oS_M \cap U) \\ \dfrac{|s|^2 - |t|^2}{a|s||t|}\sqrt{\dfrac{n}{pq}} & (\text{on } U \cap V), \end{cases}$$

where $a = 1$ (resp. 2) if W is real (resp. complex).

(1) $\big(Gr_p(\mathbf{R}^N), \mathbf{R}^N\big)$,

The tangent space $\mathfrak{m}$ is regarded as $\mathbf{R}^p \otimes \mathbf{R}^q$. We get the $\mathfrak{l}_\xi$-decomposition of $\mathfrak{m}$:

$$\begin{aligned} \mathfrak{m} =& \mathbf{R}w \otimes \mathbf{R}\xi w \oplus_\perp \mathbf{R}^{p-1} \otimes \mathbf{R}\xi w \oplus_\perp \mathbf{R}w \otimes \mathbf{R}^{q-1} \oplus_\perp \mathbf{R}^{p-1} \otimes \mathbf{R}^{q-1} \\ =& \mathbf{R} \oplus_\perp \mathbf{R}^{p-1} \oplus_\perp \mathbf{R}^{q-1} \oplus_\perp \mathbf{R}^{p-1} \otimes \mathbf{R}^{q-1}. \end{aligned}$$

In this decomposition, we can identify:

$$\begin{aligned} TS_0 =& \mathbf{R}^{q-1} \oplus_\perp \mathbf{R}^{p-1} \otimes \mathbf{R}^{q-1}, \quad TS_M = \mathbf{R}^{p-1} \oplus_\perp \mathbf{R}^{p-1} \otimes \mathbf{R}^{q-1}, \\ U =& \mathbf{R} \oplus_\perp \mathbf{R}^{p-1}, \quad V = \mathbf{R} \oplus_\perp \mathbf{R}^{q-1}. \end{aligned}$$

Since the both $\tilde{I}$ and $\tilde{J}$ are $\mathfrak{l}_\xi$-invariant, Schur's lemma yields the eigendecomposition of $A_{\mathbf{n}}$. Then Lemma 5.3.16 implies

Theorem 5.3.17. *The principal curvatures of the regular level set $f^{-1}(c)$ of the function f are*

$$\frac{|s|}{|t|}, \quad -\frac{|t|}{|s|}, \quad 0,$$

with multiplicities $q-1$, $p-1$, $(p-1)(q-1)$, respectively.

(2) $\big(Gr_p(\mathbf{C}^N), \mathbf{C}^N\big)$,

The holomorphic tangent space at o is regarded as $\mathbf{C}^{p^*} \otimes \mathbf{C}^q$. We identify $\mathfrak{m}$ with the holomorphic tangent space at o. We get the $\mathfrak{l}_\xi$-decomposition of $\mathfrak{m}$:

$$\begin{aligned} \mathfrak{m} =& \mathbf{C}w^* \otimes \mathbf{C}\xi w \oplus_\perp \mathbf{C}^{p-1^*} \otimes \mathbf{C}\xi w \oplus_\perp \mathbf{C}w^* \otimes \mathbf{C}^{q-1} \oplus_\perp \mathbf{C}^{p-1^*} \otimes \mathbf{C}^{q-1} \\ =& \mathbf{C} \oplus_\perp \mathbf{C}^{p-1^*} \oplus_\perp \mathbf{C}^{q-1} \oplus_\perp \mathbf{C}^{p-1^*} \otimes \mathbf{C}^{q-1}. \end{aligned}$$

In this decomposition, we can identify:

$$\begin{aligned} TS_0 =& \mathbf{C}^{q-1} \oplus_\perp \mathbf{C}^{p-1^*} \otimes \mathbf{C}^{q-1}, \quad TS_M = \mathbf{C}^{p-1^*} \oplus_\perp \mathbf{C}^{p-1^*} \otimes \mathbf{C}^{q-1}, \\ U^* =& \mathbf{C} \oplus_\perp \mathbf{C}^{p-1^*}, \quad V = \mathbf{C} \oplus_\perp \mathbf{C}^{q-1}. \end{aligned}$$

Since the both $\tilde{I}$ and $\tilde{J}$ are $\mathfrak{l}_\xi$-invariant, Schur's lemma yields the eigendecomposition of $A_{\mathbf{n}}$.

Then Lemma 5.3.16 implies

Theorem 5.3.18. *The principal curvatures of the regular level set $f^{-1}(c)$ of the function f are*

$$\frac{1}{\sqrt{2}|s||t|}(|s|^2-|t|^2), \quad \frac{|s|}{\sqrt{2}|t|}, \quad -\frac{|t|}{\sqrt{2}|s|}, \quad 0,$$

with multiplicities 1, $2(q-1)$, $2(p-1)$, $2(p-1)(q-1)$, *respectively.*

(3) $\big(Gr_p(\mathbf{H}^N), \mathbf{H}^N\big)$,

The tangent space $\mathfrak{m}$ is regarded as $\mathbf{H}^p \otimes \mathbf{H}^q$, in an appropriate sense. We get the $\mathfrak{l}_\xi$-decomposition of $\mathfrak{m}$:

$$\begin{aligned}\mathfrak{m} =& \mathbf{H}w \otimes \mathbf{H}\xi w \oplus_\perp \mathbf{H}^{p-1} \otimes \mathbf{H}\xi w \oplus_\perp \mathbf{H}w \otimes \mathbf{H}^{q-1} \oplus_\perp \mathbf{H}^{p-1} \otimes \mathbf{H}^{q-1}\\ =& \mathbf{H} \oplus_\perp \mathbf{H}^{p-1} \oplus_\perp \mathbf{H}^{q-1} \oplus_\perp \mathbf{H}^{p-1} \otimes \mathbf{H}^{q-1}.\end{aligned}$$

In this decomposition, we can identify:

$$\begin{aligned}TS_0 =& \mathbf{H}^{q-1} \oplus_\perp \mathbf{H}^{p-1} \otimes \mathbf{H}^{q-1}, \quad TS_M = \mathbf{H}^{p-1} \oplus_\perp \mathbf{H}^{p-1} \otimes \mathbf{H}^{q-1},\\ U =& \mathbf{H} \oplus_\perp \mathbf{H}^{p-1}, \quad V = \mathbf{H} \oplus_\perp \mathbf{H}^{q-1}.\end{aligned}$$

Since the both $\tilde{I}$ and $\tilde{J}$ are $\mathfrak{l}_\xi$-invariant, Schur's lemma yields the eigendecomposition of $A_{\mathbf{n}}$.

Then Lemma 5.3.16 implies

Theorem 5.3.19. *The principal curvatures of the regular level set $f^{-1}(c)$ of the function f are*

$$\frac{1}{2|s||t|}(|s|^2-|t|^2), \quad \frac{|s|}{2|t|}, \quad -\frac{|t|}{2|s|}, \quad 0,$$

with multiplicities 3, $4(q-1)$, $4(p-1)$, $4(p-1)(q-1)$, *respectively.*

(4) $(Gr_4(\mathbf{R}^7), S_7)$,

The tangent space $\mathfrak{m}$ is isomorphic to $\mathbf{R}^4 \otimes \mathbf{R}^3$ as $\mathfrak{so}(4) \oplus \mathfrak{sp}(1)$-module. Note that $\mathfrak{l}_\xi$ is isomorphic to the subalgebra $\{(X, X, X)\}$ of $\mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1) \oplus \mathfrak{sp}(1)$, and so we have a decomposition of $\mathfrak{m}$ as $\mathfrak{l}_\xi$-module:

$$\mathfrak{m} = \mathbf{R}^5 \oplus_\perp 2\mathbf{R}^3 \oplus_\perp \mathbf{R}.$$

Since

$$TS_0 = \mathbf{R}^5 \oplus_\perp \mathbf{R}^3, \quad TS_M = \mathbf{R}^5 \oplus_\perp \mathbf{R}^3, \quad U = \mathbf{R} \oplus \mathbf{R}^3, \quad V = \mathbf{R} \oplus \mathbf{R}^3,$$

we can not obtain the same conclusion as before.

We consider a $\mathfrak{l}_\xi$-irreducible decomposition of S_7:

$$S_7 = \mathbf{R}w \otimes \mathbf{R}^3 \, \oplus \, \mathbf{R}v \otimes \mathbf{R}^3,$$

We already see that $\xi w = v$ and $\xi v = -w$. Let u_1, u_2, u_3 be an orthonormal basis of $\mathbf{R}^3 \subset U$. We can put $\xi^2 u_i = x u_i$ for $i = 1, 2, 3$. Using the relation that $|\xi|^2 = g_V(I_\xi w, I_\xi w) + \sum_i g_V(I_\xi u_i, I_\xi u_i)$, Lemma 5.3.12 yields that

$$\frac{pq}{n} = 1 - 3x.$$

If we substitute p, q and n by 4, 4 and 12, then we obtain

$$x = -\frac{1}{9}.$$

Hence we can take an orthonormal basis v_1, v_2 and v_3 of $\mathbf{R}^3 \subset V$ such that

$$\xi u_i = \frac{1}{3} v_i, \quad \xi v_i = \frac{-1}{3} u_i, \quad i = 1, 2, 3.$$

Let η be a normal vector of $T_o S_M$ which is orthogonal to ξ, (which yields that $\eta \in \mathbf{R}^3 \subset V$), and satisfies that $|\eta w| = 1$. Theorem 5.3.12 gives

$$|\eta|^2 = \frac{pq}{n} = \frac{4}{3}.$$

The relation $\xi \perp \eta$ yields that $\xi w \perp \eta w$. Hence we may suppose that

$$\eta w = v_1, \quad \eta v = \frac{-1}{3} u_1.$$

We put $\eta = \eta_0 + \eta_1$ according to the decomposition $\mathfrak{m} = TS_0 \oplus U$. Note that $\eta_0 \in \mathbf{R}^3 \subset TS_0$ and $\eta_1 \in \mathbf{R}^3 \subset U$. Then we have

$$\eta v = \eta_0 v + \eta_1 v = \eta_1 v,$$

and so, $|\eta_1 v|^2 = \frac{1}{9}$. Since $\eta_1 \in U$, we get

$$g(\tilde{J}\eta_1, \eta_1) = \frac{n}{pq} |t|^2 |\eta_1|^2 = \frac{3}{4} |\eta_1|^2.$$

On the other hand, we have

$$g_o(\tilde{J}\eta_1, \eta_1) = g_V(J_{\eta_1} t, J_{\eta_1} t) = |t|^2 |\eta_1 v|^2 = \frac{1}{9}.$$

Consequently, we have $|\eta_1|^2 = \frac{4}{27}, \quad |\eta_0|^2 = \frac{32}{27}$, and so,

$$|\eta_0| : |\eta_1| = 2\sqrt{2} : 1.$$

Hence, if X is a vector of $\mathbf{R}^3 \subset T_oS_M$ and $X = X_0 + X_1$, where $X_0 \in \mathbf{R}^3 \subset T_oS_0$ and $X_1 \in \mathbf{R}^3 \subset U$, then we have

$$|X_0| : |X_1| = 1 : 2\sqrt{2}.$$

Let X_1, X_2 and X_3 be an orthonormal basis of $\mathbf{R}^3 \subset T_oS_M$ and v_1, v_2 and v_3 an orthonormal basis of $\mathbf{R}^3 \subset V$. Then we can take an orthonormal basis Y_1, Y_2 and Y_3 of $(\mathbf{R}^3 \oplus \mathbf{R}^3) \cap T_oS_0$ and an orthonormal basis u_1, u_2 and u_3 of $(\mathbf{R}^3 \oplus \mathbf{R}^3) \cap U$ such that

$$X_i = \frac{1}{3}\left(Y_i - 2\sqrt{2}u_i\right), \quad v_i = \frac{1}{3}\left(2\sqrt{2}Y_i + u_i\right), \quad i = 1, 2, 3.$$

It follows from Theorem 5.3.10 that

$$\tilde{J}X_i = \frac{2\sqrt{2}}{9}\frac{n}{pq}|t|^2\left(2\sqrt{2}X_i - v_i\right), \quad \tilde{J}v_i = \frac{1}{9}\frac{n}{pq}|t|^2\left(-2\sqrt{2}X_i + v_i\right)$$

From (5.3.1) and Theorems 5.3.7, 5.3.9, 5.3.10 and 5.3.13, we need to compute the eigenvalues of

$$\frac{1}{9}\sqrt{\frac{n}{pq}}\frac{1}{|s||t|}\begin{pmatrix} -8|t|^2 & 2\sqrt{2}|t|^2 \\ 2\sqrt{2}|t|^2 & 9|s|^2 - |t|^2 \end{pmatrix}$$

to obtain the principal curvatures. Then we have

Theorem 5.3.20. *The principal curvatures of the regular level set $f^{-1}(c)$ of the function f are*

$$\frac{\sqrt{3}}{12}\frac{1}{|s||t|}\left\{3(|s|^2 - |t|^2) \pm \sqrt{9 - 4|s|^2|t|^2}\right\}, \quad 0,$$

with multiplicities 3, 3, 5, *respectively.*

(5) $(G_2/\mathrm{SO}(4), \mathbf{R}^7)$,

We can proceed in the almost same way as in case of $(Gr_4(\mathbf{R}^7), S_7)$. So we shall sketch a proof.

The tangent space $\mathfrak{m}$ is regarded as $\left(\mathbf{C}^2_+ \otimes S^3\mathbf{C}^2_-\right)^{\mathbf{R}}$. Since $\mathfrak{l}_\xi$ is isomorphic to $\mathfrak{u}(1) \subset \Delta \subset \mathfrak{sp}_+(1) \oplus \mathfrak{sp}_-(1)$, we get a decomposition of $\mathfrak{m}$ as $\mathfrak{l}_\xi$-module:

$$\mathfrak{m}^{\mathbf{C}} = \mathbf{C}_4 \oplus_\perp 2\mathbf{C}_2 \oplus_\perp 2\mathbf{C} \oplus_\perp 2\mathbf{C}_{-2} \oplus_\perp \mathbf{C}_{-4}.$$

Considering real representations, we can take

$$\mathfrak{m} = \mathbf{C}_4 \oplus_\perp 2\mathbf{C}_2 \oplus_\perp 2\mathbf{R}.$$

Since

$$TS_0 = \mathbf{C}_4 \oplus \mathbf{C}_2, \quad TS_M = \mathbf{C}_4 \oplus \mathbf{C}_2 \oplus \mathbf{R}, \quad U = \mathbf{R} \oplus \mathbf{C}_2 \oplus \mathbf{R}, \quad V = \mathbf{R} \oplus \mathbf{C}_2,$$

we can not obtain the same conclusion as before.

We consider a $\mathfrak{u}(1)$-irreducible decomposition of $\mathbf{R}^7$:

$$\mathbf{R}^7 = \mathbf{R}w \oplus \mathbf{R} \oplus \mathbf{C}_2 \oplus \mathbf{R}v \oplus \mathbf{C}_2.$$

Let u_1, u_2 be an orthonormal basis of $\mathbf{C}_2 \subset U$ and $u_3 \in \mathbf{R} \subset U$ be an unit vector. We can put $\xi^2 u_i = x u_i$ for $i = 1, 2$ and $\xi u_3 = 0$. If follows from Lemma 5.3.12 that

$$\frac{pq}{n} = 1 - 2x,$$

and so

$$x = -\frac{1}{4}.$$

Let η be a normal vector of $T_o S_M$ which is orthogonal to ξ, (which yields that $\eta \in \mathbf{C}_2 \subset V$), and satisfies that $|\eta w| = 1$. We put $\eta = \eta_0 + \eta_1$ according to the decomposition $\mathfrak{m} = TS_0 \oplus U$. Note that $\eta_0 \in \mathbf{C}_2 \subset TS_0$ and $\eta_1 \in \mathbf{C}_2 \subset U$. Then we have $|\eta_1|^2 = \frac{3}{8}$, $\quad |\eta_0|^2 = \frac{9}{8}$, and so,

$$|\eta_0| : |\eta_1| = 3 : \sqrt{3}.$$

Hence, if X is a vector of $\mathbf{C}_2 \subset T_o S_M$ and $X = X_0 + X_1$, where $X_0 \in \mathbf{C}_2 \subset T_o S_0$ and $X_1 \in \mathbf{C}_2 \subset U$, then we obtain

$$|X_0| : |X_1| = 1 : \sqrt{3}.$$

From (5.3.1) and Theorems 5.3.7, 5.3.9, 5.3.10 and 5.3.13, we need to compute the eigenvalues of

$$\frac{1}{4}\sqrt{\frac{n}{pq}}\frac{1}{|s||t|}\begin{pmatrix} -3|t|^2 & \sqrt{3}|t|^2 \\ \sqrt{3}|t|^2 & 4|s|^2 - |t|^2 \end{pmatrix}$$

to obtain the principal curvatures. Then we have

Theorem 5.3.21. *The principal curvatures of the regular level set $f^{-1}(c)$ of the function f are*

$$\frac{1}{\sqrt{6}}\frac{1}{|s||t|}\left\{(|s|^2 - |t|^2) \pm \sqrt{1 - |s|^2|t|^2}\right\}, \quad -\sqrt{\frac{2}{3}}\frac{|t|}{|s|}, \quad 0,$$

with multiplicities 2, 2, 1 *and* 2, *respectively.*

5.3.2 The case of cohomogeneity greater than one

In this subsection, we see that $f = |s|^2$ is not an isoparametic function in each case. However, if we adopt Wang's definition of isoparametric functions ([33] or see also [6, p.55]), it will be shown that we can find a vector valued isoparametirc function $F : G/K \to \mathbf{R}^k$ which has f as a component, where k is the cohomogeneity of the H-action. Every H-orbit is included in a level set of F.

Moreover, we shall show that there exists a hidden symmetry in each case, in other words, $w \in W$ determines another subgroup of G. We obtain a subgroup $\tilde{H} \subset G$ such that $H \subset \tilde{H}$. The action of $\tilde{H}$ on G/K is of cohomogeneity one. Finally, the corresponding isoparametric functions are specified and we shall detect the relation between $w \in W$ and the new function.

Remark 16. For completeness, we give a definition of an isoparametric function by Wang. Let $f = (f_1, \cdots, f_k) : M \to \mathbf{R}^k$ be a function on a Riemannian manifold (M, g_M) with values in $\mathbf{R}^k$. The function f is called an *isoparametric function* if there exist functions $F_{ij}, G_i : \mathbf{R}^k \to \mathbf{R}$ $(1 \leqq i, j \leqq k)$ such that

$$(1)\ g_M(df_i, df_j) = F_{ij}(f_1, \cdots, f_k) \quad (2)\ \Delta f_i = G_i(f_1, \cdots, f_k).$$

This definition is different from a definition of Terng [30]. Though Terng's definition is stronger than one of Wang, Terng get a deep and beautiful structural theory. See also Heintze, Liu and Olmos [11] for isoparametric submanifolds. In both, the principal orbit of an hyperpolar action is a typical example.

• $(\mathrm{SU}(n)/\mathrm{SO}(n), \mathbf{C}^n)$

The tangent space $\mathfrak{m}$ is identified with a representation $S^2_0\mathbf{R}^n$ of $\mathrm{SO}(n)$, where $S^2_0\mathbf{R}^n$ denotes the set of tracefree symmetric transformations on $\mathbf{R}^n$. We denote by $\pi_0 : \mathbf{R}^n \otimes \mathbf{R}^n \to S^2_0\mathbf{R}^n$ the indicated orthogonal projection.

According to a generalized Cartan decomposition of $\mathbf{C}^n$, we obtain $z = x + iy \in \mathbf{R}^n \oplus i\mathbf{R}^n \cong \mathbf{R}^n \oplus \mathbf{R}^n$. Hence the vector bundle $\mathbf{V} \to G/K$ is naturally identified with $\mathbf{U} \to G/K$, and we do not distinguish one from the other.

Let Y be an element of $S^2_0\mathbf{R}^n$. Since $iY \in \mathfrak{m} \subset \mathfrak{su}(n)$, we have

$$(iY)(x + iy) = -Yy + iYx, \quad \text{and so,} \quad Y(x, y) = (-Yy, Yx).$$

When $\mathbf{C}^n$ is regarded as a *real* representation of $\mathrm{SU}(n)$ and the orthogonal projections are defined as $\pi_U(x + iy) = x$ and $\pi_V(x + iy) = y$, we have

$$\begin{aligned}\nabla_{\pi(L_g(iY))}s &= \left[g, -(iY)\pi_V(g^{-1}w)\right] = \left[g, Y\pi_V(g^{-1}w)\right], \\ \nabla_{\pi(L_g(iY))}t &= \left[g, -(iY)\pi_U(g^{-1}w)\right] = \left[g, -Y\pi_U(g^{-1}w)\right],\end{aligned}$$

where $g \in G$. For simplicity, we identify $Y \in \mathfrak{m}$ with the tangent vector $\pi(L_gY)$ to G/K and $\nabla_Y s$ and $\nabla_Y t$ are abbreviated to Yt and $-Ys$, respectively.

Then we get

$$df = 2g_U\left(\nabla s, s\right) = 2g_U\left(Yt, s\right) = 2g_{Gr}(Y, s \otimes t),$$

where g_{Gr} is the Riemannian metric on $Gr_n(\mathbf{R}^{2n})$, which is the target of the totally geodesic immersion of $G/K \to Gr_n(\mathbf{R}^{2n})$. Hence we obtain

$$df = 2\pi_0(s \otimes t) = 2\left(s \cdot t - \frac{g_U(s,t)}{n} I_n\right),$$

where

$$s \cdot t = \frac{1}{2}\left(s \otimes t + t \otimes s\right),$$

and I_n denotes the identity transformation of $U \to G/K$. Consequently, we have

$$|df|^2 = 2\left(|s|^2|t|^2 + \frac{n-2}{n} g_U(s,t)^2\right),$$

which shows that f is *not* an isoparametric function. Note that f is an isoparametric function in the case that $n = 2$, since we have

$$\left(\mathrm{SU}(2)/\mathrm{SO}(2), \mathbf{C}^2\right) = \left(\mathbf{C}P^1, \mathbf{C}^2\right),$$

which was already seen in the previous subsection.

We compute

$$dg_U(s,t)(Y) = g_U(Yt,t) - g_U(s,Ys) = g_{Gr}(Y, t \otimes t - s \otimes s)$$

and so, we get

$$dg_U(s,t) = \pi_0(t \cdot t - s \cdot s) = t \cdot t - s \cdot s + \frac{|s|^2 - |t|^2}{n} I_n.$$

It follows that

$$|dg_U(s,t)|^2 = |s|^4 - 2g_U(s,t)^2 + |t|^4 - \frac{1}{n}\left(|s|^2 - |t|^2\right)^2,$$

$$g\left(df, dg_U(s,t)\right) = -\frac{2(n-1)}{n} g_U(s,t)\left(|s|^2 - |t|^2\right),$$

Moreover, we have

$$\sum g_U(\nabla_{e_i} s, \nabla_{e_i} t) = -\sum g_U(e_i t, e_i s) = \sum g_U(e_i e_i t, s) = -g_U(\Delta t, s).$$

It follows from Theorem 5.3.2 that

$$\begin{aligned}\Delta g_U(s,t) =& g_U(\Delta s, t) - 2\sum g_U(\nabla s, \nabla t) + g_U(s, \Delta t)\\ =& \frac{2(n-1)(n+2)}{n} g_U(s,t).\end{aligned}$$

Consequently, we obtain an isoparametric function F with values in $\mathbf{R}^2$:

$$F := \left(|s|^2 - |t|^2, 2g_U(s,t)\right).$$

Since $g_U(s,t)$ is also H-invariant, the level sets of F consist of H-orbits.

We put $\tilde{f} = |F|^2 = \left(|s|^2 - |t|^2\right)^2 + 4g_U(s,t)^2$.

Theorem 5.3.22. *The function $\tilde{f}$ is an isoparametric function on the symmetric space $SU(n)/SO(n)$.*

Proof. Combined our direct computations with a well-known formula $\Delta f^2 = 2\{f\Delta f - |df|^2\}$, we have

$$\Delta\tilde{f} = 4(n+1)\tilde{f} - 8.$$

Moreover, it follows that

$$|d\tilde{f}|^2 = 8\tilde{f}(1-\tilde{f}).$$

□

We explain how $w \in W$ relates to $\tilde{f}$. Let h be an invariant Hermitian product on $W \cong \mathbf{C}^n$. Then $iw \otimes h(\cdot, w) - \frac{i}{n}I_n \in W \otimes W^*$ can be considered as an element of $\mathfrak{su}(n)$. We have a generalized Cartan decomposition of $\mathfrak{su}(n)$, which is a standard decomposition $\mathfrak{su}(n) = \mathfrak{so}(n) \oplus \mathfrak{m}$. Hence $iw \otimes h(\cdot, w) - \frac{i}{n}I_n$ determines a section $\tilde{s}$ of the holonomy bundle $\mathrm{SU}(n) \times_{\mathrm{SO}(n)} \mathfrak{so}(n)$. Since

$$\tilde{s} = \left[g, pr\left(g^{-1}\left(iw \otimes h(\cdot, w) - \frac{i}{n}I_n\right)\right)\right], \quad g \in \mathrm{SU}(n),$$

where $pr : \mathfrak{su}(n) \to \mathfrak{so}(n)$ is the orthogonal projection and $w = s + it$, we have

$$\tilde{s} = s \otimes g_U(\cdot, t) - t \otimes g_U(\cdot, s).$$

Consequently, we obtain

$$2|\tilde{s}|^2 = 4\left(|s|^2|t|^2 - g_U(s,t)^2\right) = 1 - \left\{\left(|s|^2 - |t|^2\right)^2 + 4g_U(s,t)^2\right\} = 1 - \tilde{f}. \qquad (5.3.3)$$

Since $iw \otimes h(\cdot, w) - \frac{i}{n}I_n$ is invariant under the action of $S(U(1) \times U(n-1))$ which is denoted by $\tilde{H}$, we have

Lemma 5.3.23. *The function $\tilde{f}$ is invariant under the action of $\tilde{H}$.*

If we check the action of $\tilde{H}$ on $\mathrm{SU}(n)/\mathrm{SO}(n)$ at o infinitesimally, it follows that the action of $\tilde{H}$ on $\mathrm{SU}(n)/\mathrm{SO}(n)$ is of cohomogeneity one.

Next, we determine critical points of $\tilde{f}$. We begin with a simple algebraic lemma, whose proof is left to the reader.

Lemma 5.3.24. *Let u and v be vectors in $\mathbf{R}^n$. Then $\pi_0(u \cdot v)$ and $\pi_0(u^2 - v^2)$ are linearly independent if and only if u and v are linearly independent.*

We have

$$d\tilde{f} = 8\left(|s|^2 - |t|^2\right)\pi_0(s \cdot t) + 8g_U(s,t)\pi_0(t^2 - s^2). \qquad (5.3.4)$$

Lemma 5.3.25. *The set of critical points of $\tilde{f}$ consists of those points in $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(1)$.*

Proof. From (5.3.4), $x \in \mathrm{SU}(n)/\mathrm{SO}(n)$ is a critical point of $\tilde{f}$ if and only if $F(x) = 0$ or $\pi_0(s \cdot t)(x)$ and $\pi_0(t^2 - s^2)(x)$ are linearly dependent and

$$\left(|s|^2 - |t|^2\right) \pi_0(s \cdot t) + g_U(s,t)\pi_0(t^2 - s^2) = 0 \tag{5.3.5}$$

at x.

Of course, $F(x) = 0$ is equivalent to $\tilde{f}(x) = 0$.

The latter condition yields that $s(x)$ and $t(x)$ are linearly dependent by lemma 5.3.24. Then the equation (5.3.5) is automatically satisfied. The Cauchy-Schwarz inequality implies that

$$\tilde{f} \leqq (|s|^2 - |t|^2)^2 + 4|s|^2|t|^2 = \left(|s|^2 + |t|^2\right)^2 = 1,$$

where the equality holds if and only if $s(x)$ and $t(x)$ are linearly dependent. □

We can describe $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(1)$ as $\tilde{H} \cong \mathrm{U}(n-1)$-orbits, respectively. In fact, we have

$$\begin{aligned}\tilde{f}^{-1}(0) &= \mathrm{U}(n-1)/\mathrm{U}(1) \times \mathrm{SO}(n-2) \cong \mathrm{SU}(n-1)/\mathrm{SO}(n-2),\\ \tilde{f}^{-1}(1) &= \mathrm{U}(n-1)/\mathrm{SO}(n-1) \supset S_0,\, S_M.\end{aligned}$$

We already see that

$$dF(x) = \left(4\pi_0(s \cdot t), 2\pi_0(t^2 - s^2)\right).$$

If $F(x) = 0$, then $\tilde{f}(x) = 0$ and so, $s(x)$ and $t(x)$ are linearly independent. Lemma 5.3.24 yields that $\pi_0(s \cdot t)$ and $\pi_0(t^2 - s^2)$ is also linearly independent. Hence x is a regular point of F. Indeed, from the above description, though $\tilde{f}^{-1}(0)$ is a singular orbit of $\tilde{H}$, $\tilde{F}^{-1}(0)$ is not a singular orbit of H.

Lemma 5.3.26. *One orbit $F^{-1}(0)$ of the action of H on $SU(n)/SO(n)$, which is not a singular orbit, is a minimal submanifold of $SU(n)/SO(n)$.*

Proof. The orbit $F^{-1}(0)$ is equal to $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(0)$ is a singular orbit of $\tilde{H}$. The theorem of Hsiang-Lawson [14] yields the result. □

Lemma 5.3.27. *The action of H on $SU(n)/SO(n)$ is not a hyperpolar action.*

Proof. We can apply [12, Theorem 3.13, p.231] to get the result. (We also refer to [12] to see the definition of the hyperpolar action.) □

Corollary 5.3.28. *The submanifold $F^{-1}(c)$, where c is a regular value of F, is not an equifocal submanifold of $SU(n)/SO(n)$.*

See [31] for the definition of the equifocal submanifold, which is considered as a generalization of isoparametric hypersurfaces.

Next we focus our attention on $\tilde{f}^{-1}(1)$. From (5.3.3), $\tilde{f}^{-1}(1)$ is nothing but a zero locus of the section $\tilde{s}$. In addition, since $\mathfrak{su}(n) = \mathfrak{so}(n) \oplus \mathfrak{m}$ is a generalized Cartan decomposition, we have a totally geodesic immersion from $\mathrm{SU}(n)/\mathrm{SO}(n)$ to a real Grassmannian $Gr_p(\mathfrak{su}(n))$, where $p = \dim \mathrm{SO}(n)$. Then the same method as in the proof of Theorem 5.2.6 yields

Theorem 5.3.29. *The level set $\tilde{f}^{-1}(1)$ is a totally geodesic submanifold of $SU(n)/SO(n)$.*

- $(\mathrm{Sp}(n)/\mathrm{U}(n), \mathbf{C}^n)$

The holomorphic tangent space is identified with $S^2\mathbf{C}^{n^*}$ as complex $\mathrm{U}(n)$-module, where $S^2\mathbf{C}^{n^*}$ denotes a symmetric power of $\mathbf{C}^{n^*}$. Hence, $S^2\mathbf{C}^{n^*} \oplus S^2\mathbf{C}^n$ is regarded as the complexification of $\mathfrak{m}$, which is denoted by $\mathfrak{m}^{\mathbf{C}}$. Let $\sigma : \mathfrak{m}^{\mathbf{C}} \to \mathfrak{m}^{\mathbf{C}}$ be the real structure. If $Y \in \mathfrak{m}$ is a real vector, then there exists a unique $Z \in S^2\mathbf{C}^{n^*}$ such that $Y = (Z, \sigma(Z)) \in \mathfrak{m}^{\mathbf{C}}$.

Let $j : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ be an invariant quaternion structure. We regard $\mathbf{C}^{2n}$ as a left $\mathbf{H}$-module with j. As $\mathrm{U}(n)$-module, we have $\mathbf{C}^{2n} = \mathbf{C}^n \oplus \mathbf{C}^{n^*}$. If $Z \in S^2\mathbf{C}^{n^*}$ is regarded as a homomorphism $Z : \mathbf{C}^n \to \mathbf{C}^{n^*}$, then we have $\sigma(Z) = jZj : \mathbf{C}^{n^*} \to \mathbf{C}^n$, where the quaternion structure j is restricted to $\mathbf{C}^{n^*}$. Consequently, $Y \in \mathfrak{m}$ acts on $(u, v) \in \mathbf{C}^n \oplus \mathbf{C}^{n^*}$ in the following way:

$$Y(u, v) = (\sigma(Z)v, Zu)\,,$$

where $Y = (Z, \sigma(Z)) \in \mathfrak{m}^{\mathbf{C}}$.

We put $U = G \times_K \mathbf{C}^n$ and $V = G \times_K \mathbf{C}^{n^*} \cong U^*$. With our convention, we have

$$\nabla_{\pi(L_g(Y))} s = \left[g, -\sigma(Z)\pi_V(g^{-1}w)\right], \quad \nabla_{\pi(L_g(Y))} t = \left[g, -Z\pi_U(g^{-1}w)\right],$$

where $g \in G$. For simplicity, we identify $Y \in \mathfrak{m}$ with the tangent vector $\pi(L_g Y)$ to G/K and $\nabla_Y s$ and $\nabla_Y t$ are abbreviated to $-\sigma(Z)t$ and $-Zs$, respectively.

Then we get

$$\begin{aligned} df(Y) =& g_U\left(\nabla_Y s, s\right) + g_U\left(s, \nabla_Y s\right) = -g_U\left(\sigma(Z)t, s\right) - g_U\left(s, \sigma(Z)t\right) \\ =& - h_{Gr}\left(\sigma(Z), g_V(\cdot, t) \otimes s\right) - h_{Gr}\left(g_V(\cdot, t) \otimes s, \sigma(Z)\right), \end{aligned}$$

where h_{Gr} is the Hermitian metric on $Gr_n(\mathbf{C}^{2n})$, which is the target of the totally geodesic immersion of $G/K \to Gr_n(\mathbf{C}^{2n})$.

Hence we obtain

$$df^{1,0} = s \cdot g_V(\cdot, t) = \frac{1}{2}\left(s \otimes g_V(\cdot, t) + g_V(\cdot, t) \otimes s\right).$$

Consequently, we have

$$|df|^2 = \left(|s|^2|t|^2 + |(s, t)|^2\right),$$

where $(\cdot,\cdot)$ denotes the pairing between $\mathbf{U} \to G/K$ and $\mathbf{V} \to G/K$. This shows that f is *not* an isoparametric function.

We compute

$$d(s,t)(Y) = -(\sigma(Z)t, t) - (s, Zs) = -(\sigma(Z), t \otimes t) - (s \otimes s, Z),$$

where $(\cdot,\cdot)$ in the right-hand-side denotes the obvious pairing. It follows that

$$d(s,t) = -t \cdot t - s \cdot s.$$

As a result, we have

$$\begin{aligned} &|d(s,t)|^2 = |s|^4 + |t|^4, \\ &h\,(d(s,t), df) = -\left(|s|^2 + |t|^2\right)(s,t) = -(s,t), \end{aligned}$$

Moreover, we have

$$\sum (\nabla_{e_i} s, \nabla_{e_i} t) = -(\Delta t, s).$$

It follows from Theorem 5.3.2 that

$$\Delta(s,t) = (\Delta s, t) - 2\sum(\nabla s, \nabla t) + (s, \Delta t) = 2(n+1)(s,t).$$

Consequently, we obtain an isoparametric function F with values in $\mathbf{R}^3$:

$$F := \left(|s|^2 - |t|^2, 2(s,t)\right).$$

Since (s,t) is also H-invariant, the level sets of F consists of H-orbits.

We put $\tilde{f} = |F|^2 = \left(|s|^2 - |t|^2\right)^2 + 4|(s,t)|^2$.

Theorem 5.3.30. *The function $\tilde{f}$ is an isoparametric function on the symmetric space $Sp(n)/U(n)$.*

Proof. In a similar computation to one in a proof of Theorem 5.3.22, we have

$$|d\tilde{f}|^2 = 4\tilde{f}(1 - \tilde{f}),$$

and

$$\Delta \tilde{f} = 2(2n+1)\tilde{f} - 6.$$

$\square$

We discuss a relation between $w \in W$ and $\tilde{f}$. Let ω be an invariant symplectic form on $W \cong \mathbf{C}^{2n}$ and we do not distinguish between $\mathbf{C}^{2n}$ and $\mathbf{C}^{2n^*}$. We can consider $w \wedge jw \in \wedge^2 \mathbf{C}^{2n}$. We have an irreducible decomposition $\wedge^2 \mathbf{C}^{2n} = \wedge^2_0 \mathbf{C}^{2n} \oplus \mathbf{C}\omega$ as $\mathrm{Sp}(n)$-module, and so we define the orthogonal projection $\pi_0 : \wedge^2 \mathbf{C}^{2n} \to \wedge^2_0 \mathbf{C}^{2n}$. As a $\mathrm{U}(n)$-module, we have $\wedge^2_0 \mathbf{C}^{2n} = \wedge^2 \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n^*} \oplus \mathfrak{su}(n)^{\mathbf{C}}$. Taking a real part, we get the orthogonal projection $pr : \left(\wedge_0^2 \mathbf{C}^{2n}\right)^{\mathbf{R}} \to \left(\wedge^2\mathbf{C}^n \oplus \wedge^2\mathbf{C}^{n^*}\right)^{\mathbf{R}}$. Hence $w \wedge jw$ determines a section $\tilde{s}$ of the bundle $\mathrm{Sp}(n) \times_{\mathrm{U}(n)} \left(\wedge^2\mathbf{C}^n \oplus \wedge^2\mathbf{C}^{n^*}\right)^{\mathbf{R}}$. Since

$$\tilde{s} = \left[g, pr\left(g^{-1}\pi_0\left(w \wedge jw\right)\right)\right], \quad g \in \mathrm{Sp}(n),$$

we have

$$\tilde{s} = -s \otimes g_V(\cdot, t) - t \otimes g_U(\cdot, s).$$

Consequently, we obtain

$$2|\tilde{s}|^2 = 4\left(|s|^2|t|^2 - |(s,t)|^2\right) = 1 - \left\{\left(|s|^2 - |t|^2\right)^2 + 4|(s,t)|^2\right\} = 1 - \tilde{f}.$$

Since $w \wedge jw$ is invariant under the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(n-1)$ which is denoted by $\tilde{H}$, we have

Lemma 5.3.31. *The function $\tilde{f}$ is invariant under the action of $\tilde{H}$.*

From the infinitesimal action of $\tilde{H}$ on $\mathrm{Sp}(n)/\mathrm{U}(n)$ at o, it follows that the action of $\tilde{H}$ on $\mathrm{Sp}(n)/\mathrm{U}(n)$ is of cohomogeneity one.

Remark 17. From the viewpoint of $\mathrm{Sp}(1)$, the function F is a moment map for the action of $\mathrm{Sp}(1)$ on $\mathrm{Sp}(n)/\mathrm{U}(n)$. Hence $\mathrm{Sp}(n-1)$ acts on the Kähler quotient. Indeed, the Kähler quotient is identified with a flag manifold $\mathrm{Sp}(n-1)/\mathrm{S}\left(\mathrm{U}(n-2) \times \mathrm{U}(1) \times \mathrm{U}(2)\right)$.

Next, we determine critical points of $\tilde{f}$. We have

$$d\tilde{f}^{1,0} = 4\left(|s|^2 - |t|^2\right) s \cdot g_V(\cdot, t) - 4\overline{(s,t)}s^2 + 4(s,t)g_V(\cdot,t)^2. \tag{5.3.6}$$

Lemma 5.3.32. *The set of critical points of $\tilde{f}$ consists of those points in $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(1)$.*

Proof. If s and $g_V(\cdot, t)$ are linearly dependent, then we have $d\tilde{f}^{1,0} = 0$ by (5.3.6).

Suppose that s and $g_V(\cdot, t)$ are linearly independent. Then, $s \cdot g_V(\cdot, t)$, s^2 and $g_V(\cdot, t)^2$ are linearly independent. It follows from (5.3.6) that $d\tilde{f}^{1,0} = 0$ if and only if $\tilde{f} = 0$.

Since $(s,t) = g_U\left(s, g_V(\cdot, t)\right)$, the Cauchy-Schwarz inequality implies that

$$\tilde{f} \leqq (|s|^2 - |t|^2)^2 + 4|s|^2|t|^2 = \left(|s|^2 + |t|^2\right)^2 = 1,$$

where the equality holds if and only if s and $g_V(\cdot, t)$ are linearly dependent. □

We can describe $\tilde{f}^{-1}(0)$ and $\tilde{f}^{-1}(1)$ as $\tilde{H}$-orbits, respectively. In fact, we have

$$\tilde{f}^{-1}(0) = \mathrm{Sp}(1) \times \mathrm{Sp}(n-1)/\mathrm{Sp}(1) \times \mathrm{U}(n-2) \cong \mathrm{Sp}(n-1)/\mathrm{U}(n-2),$$
$$\tilde{f}^{-1}(1) = S^2 \times \mathrm{Sp}(n-1)/\mathrm{U}(n-1) \supset S_0,\ S_M.$$

In similar ways in the case of $(\mathrm{SU}(n)/\mathrm{SO}(n), \mathbf{C}^n)$, we have

Lemma 5.3.33. *One orbit $F^{-1}(0)$ of the action of H on $Sp(n)/U(n)$, which is not a singular orbit, is a minimal submanifold of $Sp(n)/U(n)$.*

Lemma 5.3.34. *The action of H on $Sp(n)/U(n)$ is not a hyperpolar action.*

Corollary 5.3.35. *The submanifold $F^{-1}(c)$, where c is a regular value of F, is not an equifocal submanifold of $Sp(n)/U(n)$.*

Theorem 5.3.36. *The level set $\tilde{f}^{-1}(1)$ is a totally geodesic submanifold of $Sp(n)/U(n)$.*

- $(Gr_4(\mathbf{R}^9), S_9)$

Since $S_9 = S_4^+ \otimes S_5 \oplus S_4^- \otimes S_5$ as $\mathrm{Spin}(4) \times \mathrm{Spin}(5)$-module, we put $U = S_4^+ \otimes S_5$ and $V = S_4^- \otimes S_5$. More precisely, though we need to take a real part of each space, we omit the notation to indicate it. According to the decomposition

$$U \otimes V = \mathbf{R}^4 \otimes \left(\mathbf{R} \oplus \mathbf{R}^5 \oplus \mathfrak{so}(5)\right),$$

we define two orthogonal projections $\pi_0 : U \otimes V \to \mathbf{R}^4$ and $\pi_T : U \otimes V \to \mathbf{R}^4 \otimes \mathbf{R}^5$. Note that $\mathbf{R}^4$ and $\mathbf{R}^4 \otimes \mathbf{R}^5$ can also be considered as the tautological bundle and the cotangent bundle on $Gr_4(\mathbf{R}^9)$ with our convention.

We have

$$d|s|^2 = 2s \otimes t$$

on $Gr_8(S_9)$, where s and t are regarded as sections of the tautological bundle and the universal quotient bundle on $Gr_8(S_9)$, respectively. Since $S \otimes Q$ can be regarded as the cotangent bundle on $Gr_8(S_9)$, using a totally geodesic immersion $i : Gr_4(\mathbf{R}^9) \to Gr_8(S_9)$, we obtain

$$df = 2\pi_T(s \otimes t).$$

Lemma 5.3.37. *We have*

$$|df|^2 = 2\left(|s|^2|t|^2 - 6\,|\pi_0(s \otimes t)|^2\right).$$

Proof. First of all, we pay attention on $\mathrm{Spin}(5)$-modules. We identify $\mathrm{Spin}(5)$ with $\mathrm{Sp}(2)$. Then S_5 is recognized with the standard representation $\mathbf{C}^4$ with an invariant symplectic form ω of $\mathrm{Sp}(2)$ and we have $\mathbf{C}^4 \otimes \mathbf{C}^4 = \mathbf{C}\omega \oplus \wedge_0^2\mathbf{C}^4 \oplus \mathfrak{so}(5)^{\mathbf{C}}$. If $u, v \in \mathbf{C}^4$, then, under the decomposition

$$u \otimes v = u \wedge v + u \cdot v, \quad u \wedge v = \frac{1}{2}\left(u \otimes v - v \otimes u\right), \quad u \cdot v = \frac{1}{2}\left(u \otimes v + v \otimes u\right),$$

we have

$$u \wedge v \in \mathbf{C}\omega \oplus \wedge_0^2\mathbf{C}^4, \quad u \cdot v \in \mathfrak{so}(5)^{\mathbf{C}}.$$

It follows that

$$|u \wedge v|^2 = \frac{1}{2}\left(|u|^2|v|^2 - |h(u,v)|^2\right),$$

where $h(\cdot,\cdot)$ is an invariant Hermitian product on $\mathbf{C}^4$.

We denote two orthogonal projections by $p_0 : \wedge^2\mathbf{C}^4 \to \mathbf{C}\omega$ and $p_T : \wedge^2\mathbf{C}^4 \to \wedge^2_0\mathbf{C}^4$, respectively. It follows from $|u\wedge v|^2 = |p_0(u\wedge v)|^2 + |p_T(u\wedge v)|^2$ that

$$|p_0(u\wedge v)|^2 + |p_T(u\wedge v)|^2 = \frac{1}{2}\left(|u|^2|v|^2 - |h(u,v)|^2\right). \tag{5.3.7}$$

Since $|\omega|^2 = 4$, we get

$$p_0(u\wedge v) = \frac{1}{4}\omega(u,v)\omega, \quad |p_0(u\wedge v)|^2 = \frac{1}{4}\left|\omega(u,v)\right|^2.$$

It follows that

$$|p_T(u\wedge v)|^2 = \frac{1}{2}\left(|u|^2|v|^2 - |h(u,v)|^2\right) - \frac{1}{4}|\omega(u,v)|^2. \tag{5.3.8}$$

The subgroup Spin(4) is now identified with $\mathrm{Sp}_+(1)\times\mathrm{Sp}_-(1)$. Let $\mathbf{C}^2_\pm$ be standard representations with invariant quaternion structures $j_\pm$ of $\mathrm{Sp}_\pm(1)$, respectively. Note that $\mathbf{C}^2_\pm$ are equivalent to $S_4^\pm$, respectively. We denote by e_1, e_2 the standard basis of $\mathbf{C}^2_+$. This means that e_1, e_2 is a unitary basis with $e_2 = j_+e_1$. The standard basis of $\mathbf{C}^2_-$ is denoted by f_1, f_2. Let $a = e_1\otimes u_1 + e_2\otimes u_2$ be a real vector in $\mathbf{C}^2_+\otimes\mathbf{C}^4$. This yields that

$$ju_1 = u_2,$$

where j is an invariant quaternion structure on $\mathbf{C}^4$. We denote a real vector in $\mathbf{C}^2_-\otimes\mathbf{C}^4$ by $b = f_1\otimes v_1 + f_2\otimes v_2$ with $jv_1 = v_2$. We have

$$a\otimes b = \sum\left(e_i\otimes f_j\right)\otimes\left(u_i\otimes v_j\right).$$

By definition, we get

$$\pi_T(a\otimes b) = \sum\left(e_i\otimes f_j\right)\otimes p_T\left(u_i\wedge v_j\right),$$

and so,

$$|\pi_T(a\otimes b)|^2 = \sum\left|p_T\left(u_i\wedge v_j\right)\right|^2. \tag{5.3.9}$$

Since a and b are real vectors, we have, for instance,

$$h(u_1,v_1) = -h(u_1,jv_2) = \omega(u_1,v_2).$$

Consequently, it follows from (5.3.8) that

$$\left|p_T\left(u_1\wedge v_1\right)\right|^2 = \frac{1}{2}\left(|u_1|^2|v_1|^2 - |\omega(u_1,v_2)|^2\right) - \frac{1}{4}|\omega(u_1,v_1)|^2.$$

and so, (5.3.9) yields that

$$|\pi_T(a\otimes b)|^2 = \frac{1}{2}\left(|u_1|^2+|u_2|^2\right)\left(|v_1|^2+|v_2|^2\right) - \frac{3}{4}\sum|\omega(u_i,v_j)|^2. \tag{5.3.10}$$

The definition yields that

$$\pi_0(a\otimes b) = \sum (e_i\otimes f_j)\otimes p_0\left(u_i\wedge v_j\right), \tag{5.3.11}$$

and so,

$$|\pi_0(a\otimes b)|^2 = \sum |p_0\left(u_i\wedge v_j\right)|^2 = \frac{1}{4}\sum|\omega(u_i,v_j)|^2. \tag{5.3.12}$$

It follows from (5.3.10) and (5.3.12) that

$$|\pi_T(a\otimes b)|^2 = \frac{1}{2}|a|^2|b|^2 - 3\,|\pi_0(a\otimes b)|^2,$$

which yields the result. □

If $\pi_0(s\otimes t)\neq 0$, then it follows that f is *not* an isoparametric function on $Gr_4(\mathbf{R}^9)$. Since $\pi_0(s\otimes t)$ is a section of $\mathbf{R}^4$ determined by $w\in S^9$, we need to see how $\pi_0(s\otimes t)$ corresponds to w. Note that $w\otimes w$ is an element of S^2S_9 the symmetric power of S_9. As Spin(9)-module, we have a decomposition $S^2S_9 = \mathbf{R}\oplus\mathbf{R}^9\oplus\wedge^4\mathbf{R}^9$. Let $\Pi : S^2S_9\to\mathbf{R}^9$ be the orthogonal projection. We define a Spin(9)-equivariant map $\alpha : S_9\to\mathbf{R}^9$ as

$$\alpha(w) = \Pi(w\otimes w).$$

To describe $\alpha : S_9\to\mathbf{R}^9$ explicitly, we use a diagonal subgroup $\Delta\subset \mathrm{Sp}_+(1)\times\mathrm{Sp}_-(1)$ and regard S_9 and $\mathbf{R}^9$ as $\Delta\times\mathrm{Sp}(2)$-modules:

$$S_9 = \left(\mathbf{C}^2\otimes\mathbf{C}^4\right)^{\mathbf{R}}\oplus\left(\mathbf{C}^2\otimes\mathbf{C}^4\right)^{\mathbf{R}}, \quad \mathbf{R}^9 = \mathbf{R}\oplus\left(S^2\mathbf{C}^2\right)^{\mathbf{R}}\oplus\left(\wedge_0^2\mathbf{C}^4\right)^{\mathbf{R}},$$

where $\mathbf{C}^2$ denotes the standard representation of Δ. We use Δ to define a quaternion structure on $(\mathbf{C}^2\otimes\mathbf{C}^4)^{\mathbf{R}}$ and so, $\mathbf{R}\oplus(S^2\mathbf{C}^2)^{\mathbf{R}}\subset\mathbf{R}^9$ is identified with a scalar field $\mathbf{H}$. Then we have

$$S_9 = \mathbf{H}^2\oplus\mathbf{H}^2, \quad \mathbf{R}^9 = \mathbf{H}\oplus\left(\wedge_0^2\mathbf{C}^4\right)^{\mathbf{R}}.$$

Using a quaternion structure, we can also show

Lemma 5.3.38. *For an arbitrary $(u,v)\in S_9 = \mathbf{H}^2\oplus\mathbf{H}^2$, $\alpha : S_9\to\mathbf{R}^9$ can be expressed as:*

$$\alpha(u,v) = c\left(h_{\mathbf{H}}(u,v), p_T(u\wedge ju) - p_T(v\wedge jv)\right),$$

where c is a real non-zero constant and $h_{\mathbf{H}}$ denotes a quaternion hermitian inner product on $\mathbf{H}^2$.

The sections s and t are locally expressed as

$$s = e_1 \otimes s_1 + e_2 \otimes s_2, \quad t = f_1 \otimes t_1 + f_2 \otimes t_2,$$

where $\{e_1, e_2\}$ and $\{f_1, f_2\}$ are now regarded as local standard frames. Since s and t are real sections, we have

$$js_1 = s_2, \quad jt_1 = t_2.$$

Under the identification $S_9 = \mathbf{H}^2 \oplus \mathbf{H}^2$, this yields that

$$g^{-1}w = \sqrt{2}(s_1, t_1) \in \mathbf{H}^2 \oplus \mathbf{H}^2, \quad g \in \mathrm{Spin}(9).$$

It follows from (5.3.11) and our identification $\mathbf{R}^4 \cong \mathbf{H}$ that

$$\pi_0(s \otimes t) = \sqrt{2} h_{\mathbf{H}}(s_1, t_1), \tag{5.3.13}$$

which is nothing but the section of the tautological bundle corresponding to $\alpha(w)$ (up to constant) by Lemma 5.3.38. Consequently, f is *not* an isoparametric function on $Gr_4(\mathbf{R}^9)$, but a new function $\tilde{f} := |\pi_0(s \otimes t)|^2$ is an isoparametric function considered in the previous subsection. We have a subgroup $\mathrm{Spin}(8) \subset \mathrm{Spin}(9)$ as an isotropy subgroup at $\alpha(w)$, which is denoted by $\tilde{H}$. Of course, $\tilde{f}$ is invariant under the action of $\mathrm{Spin}(8)$. Since $|s|^2 = |s_1|^2 + |s_2|^2 = 2|s_1|^2$ and $|t|^2 = 2|t_1|^2$, the Cauchy-Schwarz inequality implies that

$$|\pi_0(s \otimes t)|^2 \leqq \frac{1}{2}|s|^2|t|^2 = \frac{1}{8}\left\{1 - \left(|s|^2 - |t|^2\right)^2\right\},$$

where the equality holds if and only if $|s|^2 = |t|^2 = \frac{1}{2}$. In particular, the maximum value of $\tilde{f}$ is $\frac{1}{8}$. This yields that $|\alpha(w)|^2 = \frac{1}{8}$. Hence we have

$$\alpha(u, v) = \sqrt{2}\left(h_{\mathbf{H}}(u, v), p_T(u \wedge ju) - p_T(v \wedge jv)\right). \tag{5.3.14}$$

It follows that

$$\tilde{f}^{-1}(0) = Gr_4(\mathbf{R}^8) \supset S_0, S_M, \quad \tilde{f}^{-1}\left(\frac{1}{8}\right) = Gr_3(\mathbf{R}^8).$$

We define a function $F : Gr_4(\mathbf{R}^9) \to \mathbf{R}^2$:

$$F := \left(|s|^2 - |t|^2, \tilde{f}\right).$$

Lemma 5.3.39. *The function F is an isoparametric function.*

Proof. From Lemma 5.3.37, we get

$$\left|d\left(|s|^2 - |t|^2\right)\right|^2 = \frac{1}{2}\left\{1 - \left(|s|^2 - |t|^2\right)^2 - 6\tilde{f}\right\}.$$

We need to compute $g\left(d\left(|s|^2-|t|^2\right), d\tilde{f}\right)$. Since $\pi_0(s\otimes t)$ is a section of the tautological bundle corresponding to $\alpha(w)$, it follows from (5.3.14) that

$$d\tilde{f} = 4h_{\mathbf{H}}(s_1,t_1)\otimes\{p_T(s_1\wedge s_2)-p_T(t_1\wedge t_2)\}.$$

On the other hand, we see that

$$d\left(|s|^2-|t|^2\right) = 2df = 4\pi_T(s\otimes t) = 4\sum (e_i\otimes f_j)\otimes p_T(s_i\wedge t_j).$$

It follows from $\mathbf{R}^4\cong\mathbf{H}$ that

$$\begin{aligned}
&\frac{1}{4}g\left(d\tilde{f},\pi_T(s\otimes t)\right)\\
=&\omega(s_1,t_1)g\left(p_T(s_1\wedge s_2)-p_T(t_1\wedge t_2),p_T(s_1\wedge t_1)\right)\\
+&\overline{\omega(s_1,t_1)}g\left(p_T(s_1\wedge s_2)-p_T(t_1\wedge t_2),p_T(s_2\wedge t_2)\right)\\
+&h(s_1,t_1)g\left(p_T(s_1\wedge s_2)-p_T(t_1\wedge t_2),p_T(s_1\wedge t_2)\right)\\
-&\overline{h(s_1,t_1)}g\left(p_T(s_1\wedge s_2)-p_T(t_1\wedge t_2),p_T(s_2\wedge t_1)\right)\\
=&\frac{1}{4}\left(|s|^2-|t|^2\right)^2\tilde{f}.
\end{aligned}$$

□

Since $F^{-1}\left(0,\frac{1}{8}\right)=\tilde{f}^{-1}\left(\frac{1}{8}\right)$, we obtain

Lemma 5.3.40. *One orbit $F^{-1}\left(0,\frac{1}{8}\right)$ of the action of H on $Gr_4(\mathbf{R}^9)$ is a totally geodesic submanifold of $Gr_4(\mathbf{R}^9)$.*

Remark 18. From $F^{-1}\left(0,\frac{1}{8}\right)=\tilde{f}^{-1}\left(\frac{1}{8}\right)$, we can get the well-known fact that $\mathrm{Spin}(7)/\mathrm{Sp}(1)\times\mathrm{Sp}(1)\cong Gr_3(\mathbf{R}^8)$.

Lemma 5.3.41. *The action of H on $Gr_4(\mathbf{R}^9)$ is not a hyperpolar action.*

Corollary 5.3.42. *The submanifold $F^{-1}(c)$, where c is a regular value of F, is not an equifocal submanifold of $Gr_4(\mathbf{R}^9)$.*

5.4 Radon transforms

We obtained isoparametric functions $\tilde{f}$ in the previous section. In the case that the action of H is of cohomogeneity one, $\tilde{f}$ is invariant under the action of H. Otherwise, $\tilde{f}$ is invariant under the action of $\tilde{H}$. In both cases, if we pull back $\tilde{f}$ to G under the natural fibration $\pi: G\to G/K$, then the pull-back function $\pi^*\tilde{f}$ is invariant under the action of $H\times K$ on G, where H acts on G on the left and K on the right. Hence, we can push down $\pi^*\tilde{f}$ to get a function on $H\backslash G$.

To be more precise, we introduce the Radon transform. Let $\psi : G \to H\backslash G$ be a natural fibration and $d\mu$ is the normalized Haar measure on H. We use the same notation to denote the measure on the fiber of $\psi : G \to H\backslash G$ induced by $d\mu$. We define a Radon transform $R : C^\infty(G/K) \to C^\infty(H\backslash G)$ for an arbitrary function f on G/K as

$$R(f)(x) = \int_{\psi^{-1}(x)} \pi^* f d\mu, \quad x \in H\backslash G.$$

By definition, the Radon transform is a G-equivariant linear map.

5.4.1 The case of cohomogeneity one

Let $\tilde{f} = |s|^2 - \frac{p}{N}$ be an isoparametric function defined in the Remark after Theorem 5.3.13. Let $\{e_1, \cdots, e_N\}$ be an orthogonal basis of a real representation W such that $\{w = e_1, \cdots, e_p\}$ is a basis of U and $\{e_{p+1}, \cdots, e_N\}$ is a basis of V. By definition, we have

$$\tilde{f}(\pi(g)) = \left|\pi_U\left(g^{-1}w\right)\right|^2 - \frac{p}{N}, \quad g \in G.$$

Let $\{x_1, \cdots, x_N\}$ be the standard coordinate functions with respect to $e_1, \cdots, e_N$ on W. We get

$$\begin{aligned}\left|\pi_U\left(g^{-1}w\right)\right|^2 - \frac{p}{N} =& \sum_{i=1}^{p} x_i(g^{-1}w)^2 - \frac{p}{N}\sum_{A=1}^{N} x_A(g^{-1}w)^2 \\ =& \frac{1}{N}\left\{q\sum_{i=1}^{p} x_i(g^{-1}w)^2 - p\sum_{\alpha=p+1}^{N} x_\alpha(g^{-1}w)^2\right\},\end{aligned}$$

and so,

$$R(\tilde{f}) = \frac{1}{N}\left\{q\sum_{i=1}^{p} x_i(g^{-1}w)^2 - p\sum_{\alpha=p+1}^{N} x_\alpha(g^{-1}w)^2\right\}.$$

If a real representation is replaced by a complex representation, then we have a similar result.

Theorem 5.4.1. *The Radon transform of $\tilde{f}$ in the case of cohomogeneity one is an isoparametric function on a unit sphere of W which induces an isoparametric hypersurface of a sphere with two distinct principal curvatures.*

5.4.2 The case of cohomogeneity greater than one

We obtain Radon transforms of $\tilde{f}$ on case-by-case computations.

- $(\mathrm{SU}(n)/\mathrm{SO}(n), \mathbf{C}^n)$ $n \geqq 3$.

Let $\{e_1, Je_1, \cdots, e_n, Je_n\}$ be an orthogonal basis of a real representation $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$ such that $\{w = e_1, \cdots, e_n\}$ is a basis of U and $\{Je_1, \cdots, Je_n\}$ is a basis of V. Since $\tilde{f} = (|s|^2 - |t|^2)^2 + 4g(s,t)^2$, by definition, we have

$$\tilde{f}(\pi(g)) = \left(\left|\pi_U\left(g^{-1}w\right)\right|^2 - \left|\pi_V\left(g^{-1}w\right)\right|^2\right)^2 + 4g\left(\pi_U\left(g^{-1}w\right), \pi_V\left(g^{-1}w\right)\right)^2,$$

where we identify U with V in a standard way and $g \in \mathrm{SU}(n)$. Let $\{x_1, y_1, \cdots, x_n, y_n\}$ be the standard coordinate functions with respect to $e_1, Je_1, \cdots, e_n, Je_n$ on W. It follows that

$$R(\tilde{f})(x,y) = \left(\sum_{i=1}^n x_i(g^{-1}w)^2 - \sum_{i=1}^n y_i(g^{-1}w)^2\right)^2 + 4\left(\sum_{i=1}^n x_i(g^{-1}w)y_i(g^{-1}w)\right)^2.$$

Theorem 5.4.2. *In the case of $(SU(n)/SO(n), \mathbf{C}^n)$ $(n \geqq 3)$, the Radon transform of $\tilde{f}$ is an isoparametric function defined by Nomizu [25] on a unit sphere of $\mathbf{C}^n$ which induces an isoparametric hypersurface of a sphere with four distinct principal curvatures.*

• $(\mathrm{Sp}(n)/\mathrm{U}(n), \mathbf{C}^{2n})$ $n \geqq 2$.

Let $\{e_1, je_1, \cdots, e_n, je_n\}$ be a unitary basis of a complex representation $\mathbf{C}^{2n}$ such that $\{w = e_1, \cdots, e_n\}$ is a basis of $U \cong \mathbf{C}^n$ and $\{je_1, \cdots, je_n\}$ is a basis of $V \cong \mathbf{C}^{n^*}$. Since $\tilde{f} = (|s|^2 - |t|^2)^2 + 4|(s,t)|^2$ by definition, we have

$$\tilde{f}(\pi(g)) = \left(\left|\pi_U\left(g^{-1}w\right)\right|^2 - \left|\pi_V\left(g^{-1}w\right)\right|^2\right)^2 + 4\left|\left(\pi_U\left(g^{-1}w\right), \pi_V\left(g^{-1}w\right)\right)\right|^2,$$

where $g \in \mathrm{Sp}(n)$. Let $\{z_1, w_1, \cdots, z_n, w_n\}$ be the standard coordinate functions with respect to $e_1, Je_1, \cdots, e_n, Je_n$ on W. It follows that

$$R(\tilde{f})(z,w) = \left(\sum_{i=1}^n |z_i(g^{-1}w)|^2 - \sum_{i=1}^n |w_i(g^{-1}w)|^2\right)^2 + 4\left|\sum_{i=1}^n z_i(g^{-1}w)w_i(g^{-1}w)\right|^2.$$

Theorem 5.4.3. *In the case of $(Sp(n)/U(n), \mathbf{C}^{2n})$ $(n \geqq 2)$, the Radon transform of $\tilde{f}$ is an isoparametric function on a unit sphere of $\mathbf{C}^{2n}$ which induces an isoparametric hypersurface of a sphere with four distinct principal curvatures.*

From [9, Satz in §6.1], we have

Theorem 5.4.4. *In each case, every isoparametric hypersurface of a sphere in a family defined by $R(\tilde{f})$ is homogeneous, in the sense that it is an orbit of the action of isometry group.*

• $(Gr_4(\mathbf{R}^9), S_9)$

We use an identification between S_9 and $\mathbf{H}^2 \oplus \mathbf{H}^2$ in the previous section. It follows from (5.3.13) that

$$R(\tilde{f})(u,v) = 2\left|h_{\mathbf{H}}(u,v)\right|^2.$$

Theorem 5.4.5. *The Radon transform of $\tilde{f}$ is an isoparametric function on a unit sphere of S_9 which induces a family of isoparametric hypersurfaces of a sphere with four distinct principal curvatures. Every isoparametric hypersurface in our family is homogeneous.*

We will postpone a proof until the last paragraph.

Since $\tilde{f}$ is also invariant under $\tilde{H}$, we can easily obtain a Radon transform of $\tilde{f}$ on $\tilde{H}\backslash G$, which is denoted by $\tilde{R}(\tilde{f})$. In each case, we also have a fibration $\tilde{\psi} : H\backslash G \to \tilde{H}\backslash G$ with totally geodesic fibers. More concretely, we have

$$S^{2n-1} \to \mathbf{C}P^{n-1}, \quad S^{4n-1} \to \mathbf{H}P^{n-1}, \quad \text{and} \quad S^{15} \to S^8.$$

Using the normalized Haar measure on $\tilde{H}$, we have

$$\tilde{\psi}^* \tilde{R}(\tilde{f}) = R(\tilde{f}).$$

Since $R(\tilde{f})$ is constant on the fiber of $\tilde{\psi} : H\backslash G \to \tilde{H}\backslash G$, it follows from Theorems 5.4.2, 5.4.3 and 5.4.5 that

Theorem 5.4.6. *The Radon transform $\tilde{R}(\tilde{f})$ is an isoparametric function on $\tilde{H}\backslash G$.*

We describe $\tilde{R}(\tilde{f})$ in the last case. To do so, we "normalize" $\tilde{f}$ to get an eigenfunction. Since $\pi_0(s \otimes t)$ is the corresponding section to $\alpha(w) \in \mathbf{R}^9$ with $|\alpha(w)|^2 = \frac{1}{8}$, it follows from the Remark after Theorem 5.3.13 that $\hat{f} := \tilde{f} - \frac{1}{18}$ is an eigenfunction. According to the SO(4) $\times$ SO(5) decomposition of $\mathbf{R}^9$, we put $(\tilde{u}, \tilde{v}) \in \mathbf{R}^4 \oplus \mathbf{R}^5 = \mathbf{R}^9$. Then Theorem 5.4.1 yields that $5|\tilde{u}|^2 - 4|\tilde{v}|^2$ is an isoparametric function. If α is restricted to the unit sphere of S_9, we have that $\tilde{\psi} = \alpha$, It follows from (5.3.14) that

$$\begin{aligned}\tilde{R}(\hat{f})(\alpha(u,v)) =& \frac{2}{72}\left[5\,|h_{\mathbf{H}}(u,v)|^2 - 4\left\{\frac{1}{4}(|u|^2 + |v|^2)^2 - |h_{\mathbf{H}}(u,v)|^2\right\}\right] \\ =& \frac{1}{36}\left\{9\,|h_{\mathbf{H}}(u,v)|^2 - (|u|^2 + |v|^2)^2\right\}.\end{aligned}$$

From [9, Satz in §6.4], Theorem 5.4.5 holds. We can directly check that $(|u|^2 + |v|^2)^2 - 9\,|h_{\mathbf{H}}(u,v)|^2$ is a harmonic function on S_9, but in [9], a polynomial $(|u|^2 + |v|^2)^2 - 8\,|h_{\mathbf{H}}(u,v)|^2$ is introduced as an isoparametric function, which is called a Cartan-Münzner polynomial.

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