A Study on the one-row colored sl_3 Jones polynomials and tails for pretzel links

明治大学大学院先端数理科学研究科

２０２２年度

博士学位請求論文

A study on the one-row colored $\mathfrak{sl}_3$ Jones polynomials and tails for pretzel links

プレッツェル絡み目に対する一行色付き$\mathfrak{sl}_3$ジョーンズ多項式とテイルに関する研究

学位請求者　先端メディアサイエンス専攻

川添　浩太郎

Acknowledments

I would like to express my gratitude to Professor Masaaki Suzuki for his advice and patience. I owe my deepest gratitude to the various financial support which I have received from Meiji University. Last but not least, I wish to thank my family for their warm support.

Preface

This paper is based on the author's paper, [Kaw]. We study one of the quantum invariants, the colored $\mathfrak{sl}_3$ Jones polynomials, in this paper. Jones discovered the Jones polynomial, an invariant of the link, in [Jon85]. The discovery of the Jones polynomial led to the discovery of many link invariants using the R-matrix, which is the solution of the Yang-Baxter equation. These invariants are understood as quantum invariants by treating the R-matrices in a unified manner. We use the Lie algebra $\mathfrak{g}$ and its irreducible representation V to treat R-matrices in a unified way. Thus, the quantum invariant of a link is obtained by the Lie algebra $\mathfrak{g}$ and its irreducible representation V. Note that it is not precisely the Lie algebra $\mathfrak{g}$, but the quantum group $U_q(\mathfrak{g})$ with the structure of a ribbon Hopf algebra obtained from the Lie algebra $\mathfrak{g}$. In particular, if $\mathfrak{g} = \mathfrak{sl}_2$ and $V = \mathbb{C}^2$, the quantum invariant of links is Jones polynomial. There are many examples of computing Jones polynomials of knots directly using the R-matrix. However, in general, it is difficult to compute the Jones polynomial of the link in that way.

Kauffman [Kau87] reformulated the Jones polynomial of a link L by using the Kauffman bracket skein relation. This allowed us to compute the Jones polynomial of the links graphically. Furthermore, as a generalization of the Jones polynomial, we constitute the colored $\mathfrak{sl}_2$ Jones polynomial $J_N^{\mathfrak{sl}_2}(L;q)$ by an $(N+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$. For a link L, we can also compute $J_N^{\mathfrak{sl}_2}(L;q)$ graphically by using the Kauffman bracket and the Jones-Wenzl projector [Wen87]. The Jones-Wenzel projector is an element of the A_1 web space and plays a crucial role in $J_N^{\mathfrak{sl}_2}(L;q)$. The linear skein theory is this method of calculating $J_N^{\mathfrak{sl}_2}(L;q)$. There are many examples of $J_N^{\mathfrak{sl}_2}(L;q)$ calculated by using the linear skein theory. We can see one example in [Lic97]. The explicit formulae in $J_N^{\mathfrak{sl}_2}(L;q)$ are helpful when we consider the property that there exists the limit of the colored Jones polynomial called the tail, and conjectures related to the quantum invariants of links such as the volume conjecture [Kas97], the Jones slope conjecture [Gar11], etc.

As regards the tail of links, first, Dasbach and Lin [DL06] showed that the first two coefficients and the last two coefficients of $J_{N+1}^{\mathfrak{sl}_2}(K;q)$ do not depend on N for alternating knots K. They also showed that the third and the third to last coefficients of $J_N^{\mathfrak{sl}_2}(K;q)$ on N for alternating knots do not depend on N provided $N \geq 3$. This result led them to predict that the coefficients of $J_N^{\mathfrak{sl}_2}(K;q)$ up to the k-th do not depend on N provided $N \geq k$; for alternating knots K, there exists a q-series $T^{\mathfrak{sl}_2}(K;q)$ in $\mathbb{Z}[[q]]$ such that

$$T^{\mathfrak{sl}_2}(K;q) - \hat{J}_N^{\mathfrak{sl}_2}(K;q) \in q^{N+1}\mathbb{Z}[[q]]$$

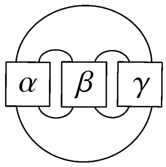
where $\hat{J}_N^{\mathfrak{sl}_2}(L;q)$ is a normalization with the minimum degree of $J_N^{\mathfrak{sl}_2}(L;q)$ and $T^{\mathfrak{sl}_2}(L;q)$ is called the tails of $J_N^{\mathfrak{sl}_2}(K;q)$. Armond [Arm13] showed the existence of the tails of the colored Jones $\mathfrak{sl}_3$ polynomials for adequate knots containing alternating knots. Garoufalidis and Lê [GL15] gave proof of the existence of more general stability of coefficients of $J_N^{\mathfrak{sl}_2}(K;q)$ for alternating knots. Armond and Dasbach [AD11] gave explicit tails for $(2,2m+1)$-torus knots and $(2,2m)$-torus links. Hajij [Haj14] gave explicit tails for 8_5 in the Rolfsen table of knots. Elhamdadi-Hajij [EH17] and Beirne [Bei19] gave explicit tails for pretzel knots $P(2k+1,2,2l+1)$. Kicilthy-Osburn [KO16], Beirne-Osburn [BO17], and Garoufalidis-Lê [GL15] gave the explicit tails for knots with small crossing numbers. Furthermore, Armond and Dasbach [AD17] proved that the tails of adequate links only depend on these reduced A-graphs.

Meanwhile, the colored $\mathfrak{sl}_3$ Jones polynomial $J_{(n_1,n_2)}^{\mathfrak{sl}_3}(L;q)$ is obtained by a link and an (n_1,n_2)-irreducible representation of $\mathfrak{sl}_3$. However, due to the complexity of the calculation, the explicit formulae for the colored $\mathfrak{sl}_3$ Jones polynomial only exists for a trefoil knot [Law03], $(2,2m+1)$-torus knots [GV17, GM13], two bridge links [Yua17], and $(2,2m)$-torus links [Yua17, Yua18b, Yua21]. For trefoil knot and $(2,2m+1)$-torus knots, representation theoretical methods, the Jones-Rosso formula [RJ93], gave the colored $\mathfrak{sl}_3$ Jones polynomials. In the colored $\mathfrak{sl}_3$ Jones polynomials, the A_2 bracket plays the same role as the Kauffman bracket, and the A_2 clasp [Kup96] plays the same role as the Jones-Wenzel projector. For $(2,2m)$-torus links and two bridge links, graphical techniques with the A_2 bracket and the A_2 claps gave the one-row colored $\mathfrak{sl}_3$ Jones polynomials obtained by an $(n,0)$-irreducible representation of $\mathfrak{sl}_3$. Kuperburg's linear skein theory [Kup96] is these graphical techniques. In [Kaw], the author calculated $J_{(n,0)}^{\mathfrak{sl}_3}(P(\alpha,\beta,\gamma);q)$ for three-parameter families of oriented pretzel links $P(\alpha,\beta,\gamma)$ except for those where α,β,γ are all odd by using Kuperberg's linear skein theory.

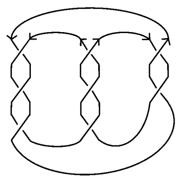
Besides, we can consider the tail concerning the colored $\mathfrak{sl}_3$ colored Jones polynomials of links. Garoufalidis and Voung [GV17] showed the stability of coefficients of $J_{(n_1,n_2)}^{\mathfrak{sl}_3}(L;q)$ for (a,b)-torus knots. Yuasa [Yua17, Yua18b, Yua20, Yua21] gave explicit tails of $J_{(n,0)}^{\mathfrak{sl}_3}(L;q)$ for $(2,2m)$-torus links and proved the existence of the tails of the one-row colored $\mathfrak{sl}_3$ Jones polynomials for minus adequate oriented links. Moreover, these explicit tails derived the Andrews-Gordon type identities for the (false) theta series. The author [Kaw] showed the existence of the tails of the one-row colored $\mathfrak{sl}_3$ Jones polynomials for alternating pretzel links $P(\downarrow 2\alpha+1 \downarrow, \uparrow 2\beta+1 \uparrow, \downarrow 2\gamma \uparrow)$.

Main results

We list the main results of this paper. Let α, β, γ be integers. Denote by $P(\alpha,\beta,\gamma)$ the pretzel link with 3 crossing regions given the following:

$P(\alpha,\beta,\gamma) =$ 

where a box marked with the letter α represents a right-handed (resp. left-handed) $|\alpha|$-twist if $\alpha > 0$ (resp. $\alpha < 0$). Boxes with other letters represent the same thing. The one-row colored $\mathfrak{sl}_3$ Jones polynomial of a link depends on the direction of the link. Therefore, it is necessary to distinguish links with different directions. For example, we consider a knot with the following directions using the arrow symbol.

$P(\downarrow 3 \downarrow, \uparrow 3 \uparrow, \downarrow 2 \uparrow) =$ 

In the following theorem, we give formulae for the one-row colored $\mathfrak{sl}_3$ Jones polynomial of pretzel links$P(\alpha,\beta,\gamma)$ except for those where α,β,γ are all odd.

Main Theorem 1 ([Kaw])**.** Let α,β,γ be non-zero integers. The one-row colored $\mathfrak{sl}_3$ Jones polynomials for pretzel links $P(\alpha,\beta,\gamma)$ are the following:

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)$$

$$= \sum_{0 \le k_{|\alpha|} \le k_{|\alpha|-1} \le \cdots \le k_1 \le n} \quad \sum_{0 \le l_{|\beta|} \le l_{|\beta|-1} \le \cdots \le l_1 \le n} \quad \sum_{0 \le m_{|\gamma|} \le m_{|\gamma|-1} \le \cdots \le m_1 \le n} \quad \sum_{s=\max\{k_{|\alpha|}, l_{|\beta|}\}}^{\min\{k_{|\alpha|}+l_{|\beta|}, n\}} \quad \sum_{t=\max\{s, m_{|\gamma|}\}}^{\min\{s+m_{|\gamma|}, n\}}$$

$$(q^{\frac{n^2+3n}{3}})^{2\alpha+2\beta+2\gamma} \phi(n, k_1, k_2, ..., k_{|\alpha|})_{q^{\epsilon_\alpha}} \phi(n, l_1, l_2, ..., l_{|\beta|})_{q^{\epsilon_\beta}} \phi(n, m_1, m_2, ..., m_{|\gamma|})_{q^{\epsilon_\gamma}}$$

$$\times \psi(n, s, k_{|\alpha|}, l_{|\beta|}) \psi(n, t, s, m_{|\gamma|}) q^{-(n-t)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})},$$

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)$$

$$= \sum_{0 \le k_{|\alpha|} \le k_{|\alpha|-1} \le \cdots \le k_1 \le n} \sum_{0 \le l_{|\beta|} \le l_{|\beta|-1} \le \cdots \le l_1 \le n} \sum_{0 \le m_{|\gamma|} \le m_{|\gamma|-1} \le \cdots \le m_1 \le n} \sum_{s=\max\{k_{|\alpha|}, l_{|\beta|}\}}^{\min\{k_{|\alpha|}+l_{|\beta|}, n\}} \sum_{a=s}^{n}$$

$$\sum_{t=\max\{a, m_{|\gamma|}\}}^{\min\{a+t+m_{|\gamma|}, n\}} \sum_{t=\max\{a, m_{|\gamma|}\}}^{\min\{a+t+m_{|\gamma|}, n\}} (q^{\frac{n^2+3n}{3}})^{-(\alpha+\beta-2\gamma)} \chi_{sign(\alpha)}(n, k_1, k_2, ..., k_{|\alpha|}) \chi_{sign(\beta)}(n, l_1, l_2, ..., l_{|\beta|})$$

$$\times \phi(n, m_1, m_2, ..., m_{|\gamma|})_{q^{\epsilon_\gamma}} \Omega(n, s, k_{|\alpha|}, l_{|\beta|}) \psi(n, t, a, m_{|\gamma|})$$

$$\times q^{-(n-t)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})}$$

where $q^{\epsilon_\gamma} = q^{\frac{\gamma}{|\gamma|}}$.

Pretzel knots $P(\downarrow \alpha \uparrow, \downarrow \beta \uparrow, \downarrow 2\gamma \uparrow)$ include 8_5 and 8_{19}. These knots are not $(2, 2m+1)$-torus knots and have the bridge index $br = 3$. In other words, Main Theorem 1 is a result involving knots that have not been computed so far. We also give the one-row $\mathfrak{sl}_3$ colored Jones polynomials for four and five parameter pretzel knots $8_{10}, 8_{15}, 8_{20}$, and 8_{21}. As a result, the one-row $\mathfrak{sl}_3$ colored Jones polynomials are determined for all knots with eight or fewer crossings except $8_{16}, 8_{17}, 8_{18}$. Moreover, from Main Theorem 1, we can see the stability of the coefficient of the one-row colored $\mathfrak{sl}_3$ Jones polynomials of $P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow)$ and $P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow)$. For instance, the one-row colored $\mathfrak{sl}_3$ Jones polynomial for $P(3,3,2) = 8_5$ multiplied by q^{3n^2+8n} is

$$\begin{aligned}
n = 1: &\quad 1 - q + q^2 - 2q^4 + q^5 - 2q^6 + q^7 + q^8 + q^{10}, \\
n = 2: &\quad 1 - q + q^3 - 2q^4 + 2q^6 - 2q^7 - 2q^8 + 4q^9 + q^{10} + \cdots, \\
n = 3: &\quad 1 - q - q^4 + q^5 + q^7 - 4q^8 - q^9 + 8q^{10} + \cdots, \\
n = 4: &\quad 1 - q - 2q^4 + 2q^5 + q^6 - 2q^8 - 4q^9 + 4q^{10} \cdots, \\
n = 5: &\quad 1 - q - 2q^4 + q^5 + 2q^6 + q^7 - 3q^8 - 4q^q + q^{10} + \cdots, \\
n = 6: &\quad 1 - q - 2q^4 + q^5 + q^6 + 2q^7 - 2q^8 - 2q^9 + 3q^{10} + \cdots, \\
n = 7: &\quad 1 - q - 2q^4 + q^5 + q^6 + q^7 - q^8 - q^9 + 2q^{10} + \cdots, \\
n = 8: &\quad 1 - q - 2q^4 + q^5 + q^6 + q^7 - 2q^8 + 3q^{10} + \cdots, \\
n = 9: &\quad 1 - q - 2q^4 + q^5 + q^6 + q^7 - 2q^8 - q^9 + 4q^{10} + \cdots, \\
n = 10: &\quad 1 - q - 2q^4 + q^5 + q^6 + q^7 - 2q^8 - q^9 + 3q^{10} + \cdots.
\end{aligned}$$

Main Theorem 2 . Let α, β, and γ be positive integers. For oriented pretzel links $P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow)$, there exists $\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)$ in $\mathbb{Z}[[q]]$ such that

$$\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q) - \hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)) \in q^{n+1}\mathbb{Z}[[q]].$$

Moreover, for oriented pretzel knots $P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow)$, there exists $\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)$ in $\mathbb{Z}[[q]]$ such that

$$\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q) - \hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha \downarrow, \uparrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)) \in q^{n+1}\mathbb{Z}[[q]].$$

Main Theorem 2 is a more generalized version of Theorem 5.3 in [Kaw].

Plan of the paper

This paper is organized as follows. In Chapter 1, we review Kurperberg's linear skein theory and the formulae for the A_2 clasp given by Ohtsuki and Yamada [OY97], Kim [Kim06, Kim07] and Yuasa [Yua17, Yua18b]. In Chapter 2, we derive a formula for m times-half twists where two strands with the same directions for Kurperberg's web space. These formulae were given by the author in [Kaw]. Then, we give the one-row $\mathfrak{sl}_3$ colored Jones polynomial for a three-parameter family of pretzel links $P(\alpha,\beta,\gamma)$. And finally, we give the one-row $\mathfrak{sl}_3$ colored Jones Polynomials for $8_{10}, 8_{15}, 8_{20}$ and 8_{21}. In Capter 3, we give the proof that the tail of $J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)$ and $J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)$ exist. We also give explicit formulae of the tails of $J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)$ and $J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)$. In the Appendix, we give the $2m+1$ times half twist formulae for two strands, which we do not use in the proof of Main Theorem 1 and Main Theorem 2.

Contents

Chapter 1

Preliminaries

In this chapter, we review Kuperberg's linear skein theory for A_2 web space and the formulae for the A_2 claps given by Ohtsuki and Yamada [OY97], Kim [Kim06, Kim07] and Yuasa [Yua17, Yua18b]. We use many of these formulas successfully in calculating the one row $\mathfrak{sl}_3$ colored Jones polynomials.

1.1 Kuperberg's linear skein theory

We use the following q-integer notations:

$$\{n\}_q = \{q^{\frac{n}{2}} - q^{-\frac{n}{2}}\}_q, \quad \{n\}_q! = \{n\}_q\{n-1\}_q \cdots \{1\}_q, \quad [n]_q = \frac{\{n\}}{\{1\}}, \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q$$

where n is a non-negative integer. And, the binomial coefficients of q-integer are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!} \tag{1.1.1}$$

where $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$.

Lemma 1.1.1 ([OY97])**.** Let a, b and c integers. We have the following formulae.

$$[a]_q[b]_q - [a-c]_q[b-c]_q = [a+b-c]_q[c]_q, \tag{1.1.2}$$

$$[a]_q[b-c]_q + [c]_q[a-b]_q = [b]_q[a-c]_q, \tag{1.1.3}$$

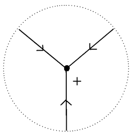
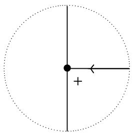
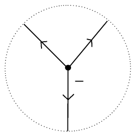
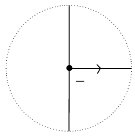
$$[a]_q[b]_q = \sum_{i=1}^{a}[a+b-(2i-1)]_q = \sum_{i=1}^{b}[a+b-(2i-1)]_q \tag{1.1.4}$$

Proof. We confirm (1.1.2) as follows;

$$
\begin{aligned}
[a]_q[b]_q-[a-c]_q[b-c]_q=&\frac{1}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}\{(q^{\frac{a}{2}}-q^{-\frac{a}{2}})(q^{\frac{b}{2}}-q^{-\frac{b}{2}})-(q^{\frac{a-c}{2}}-q^{-\frac{a-c}{2}})(q^{\frac{b-c}{2}}-q^{-\frac{b-c}{2}})\}\\
=&\frac{1}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}(q^{\frac{a+b}{2}}-q^{\frac{a+b-2c}{2}}-q^{-\frac{a+b-2c}{2}}+q^{-\frac{a+b}{2}})\\
=&[a+b-c]_q[c]_q.
\end{aligned}
$$

We can confirm the other formulae in the same way. □

We first define the A_2 web spaces. Let D be a disk with signed marked point (P,ϵ) on its boundary where P is a finite set such that $P\subset\partial D$ and $\epsilon: P\to\{+,-\}$ is a map. A bipartite uni-trivalent graph on D is an immersion of a directed graph such that every vertex is either univalent or trivalent and is divided into sink or source.

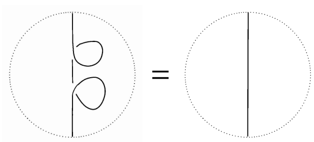
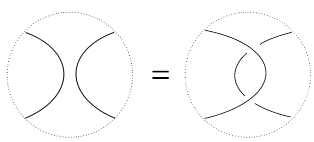
sink :  or  source :  or 

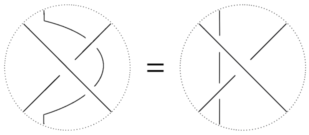
A tangled bipartite uni-trivalent graph diagram on D is a bipartite uni-trivalent graph on D satisfying:

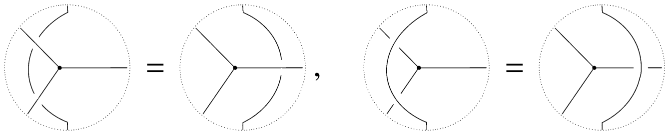
(1) the set of univalent vertices coincide with P,

(2) every crossing point is a transverse double point of two edges with under or over crossing data.

For tangled uni-trivalent graph diagrams G and G', we call G regular isotopic to G' on D if G is related to G' by a finite sequence of boundary-fixing isotopies and Reidemeister moves $(R1)-(R4)$ with some directions of edges.

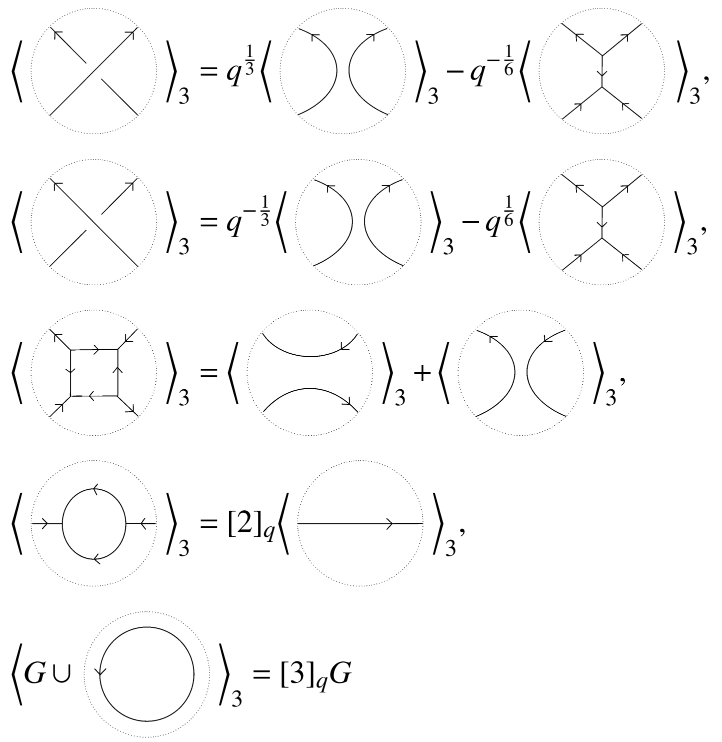
$(R1)$  $(R2)$ 

$(R3)$ 

$(R4)$ 

The tangled uni-trivalent graphs on D are regular isotopy classes of tangled trivalent graph diagrams on D. We denote $T(P,\epsilon)$ the set of tangled uni-trivalent graphs on D. The A_2 basis web has no crossing. Let $B(P,\epsilon)$ be the set of A_2 basis web. The A_2 web space $W(P,\epsilon)$ is a $\mathbb{Q}(q^{\frac{1}{6}})$-vector space spanned by $B(P,\epsilon)$. An element in $W(P,\epsilon)$ is called web.

Definition 1.1.2 (the A_2 bracket [Kup96])**.** We define a $\mathbb{Q}(q^{\frac{1}{6}})$ linear map $\langle\cdot\rangle_3 : \mathbb{Q}(q^{\frac{1}{6}})T(P,\epsilon) \to W(P,\epsilon)$ by the following:



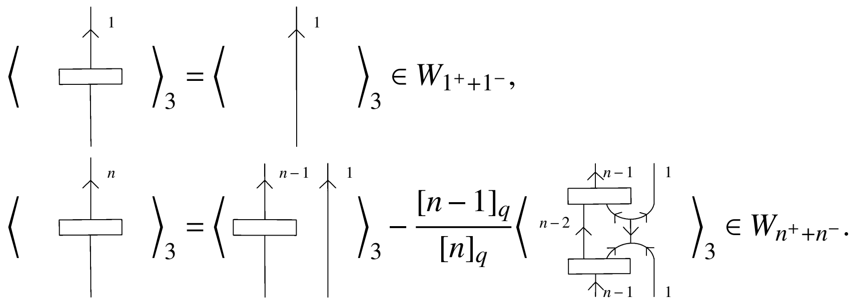
where G is a bipartite uni-tri valent graph.

We can easily show that this linear map is invariant under a regular isotopy. We next introduce the A_2 clasps of type $(n,0)$ according to [Kup96] and [OY97]. Let P be the set $\{p_1,p_2,...,p_{2n}\}$ where $p_1,p_2,...,p_{2n}$ are elements of P aligned counterclockwise on ∂D from p_1 in order of decreasing subscript of p_i, and ϵ is defined by

$$\epsilon(p_i) = \begin{cases} + & (1 \le i \le n) \\ - & (n+1 \le 2n) \end{cases}. \tag{1.1.5}$$

We denote $W(P;\epsilon)$ as $W_{n^+ + n^-}$.

Definition 1.1.3. Let n be a positive intger. We define the A_2 clasps of type $(n,0)$ by

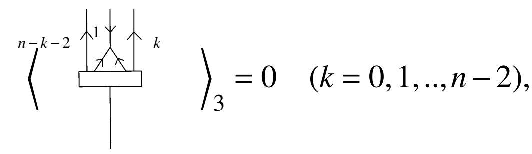


Here, a strand labeled by the number n implies the n parallel copies of the strand.

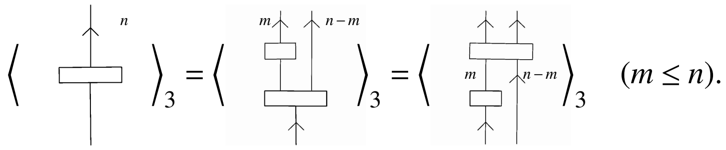
The following properties hold for A_2 clasps.

Lemma 1.1.4 ([OY97])**.** For any positive integers m and n, we have

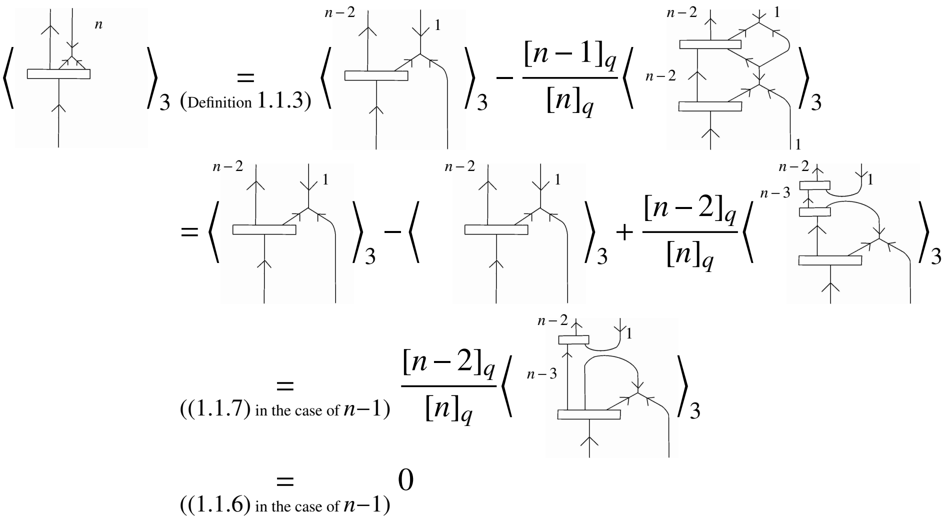
(1.1.6)



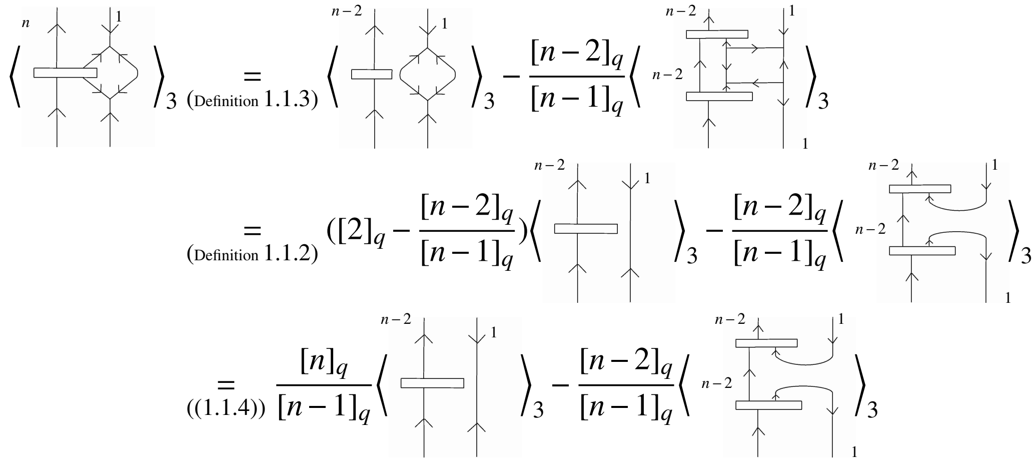
(1.1.7)



Proof. At the beginning, we prove (1.1.6) in the case of n assuming (1.1.6) and (1.1.7) in the case of $n-1$. By the induction hypothesis, it is sufficient to prove (1.1.6) in the case of n concerning $k = n-2$.



We use



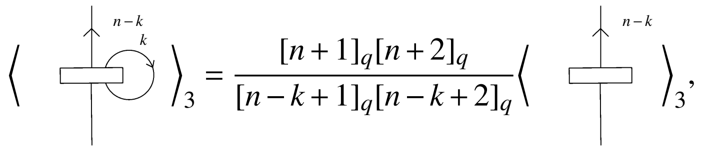
in the second equation. Then, we show (1.1.7) in the case of n assuming (1.1.6) in the case of n. From Definition 1.1.2, the A_2 clasps of type $(m,0)$ is the sum of the terms consisting of a top and bottom symmetric diagram composed of 3-valent graphs or m parallel strands. The terms consisting of a top and bottom symmetric diagram vanish by (1.1.6) in the case of n. Therefore we have (1.1.7) in the case of n.

□

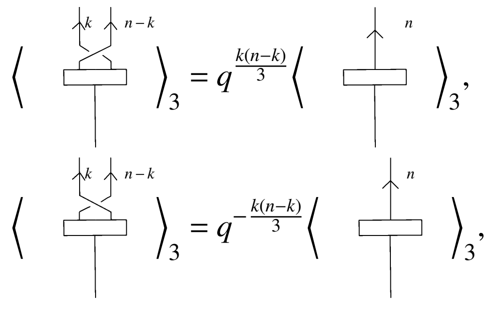
Ohtsuki, Yamda[OY97] and Yuasa[Yua17] gave formulae for A_2 clasps of type $(n,0)$.

Lemma 1.1.5 ([OY97][Yua17])**.** For $k = 0,1,...,n$, we have

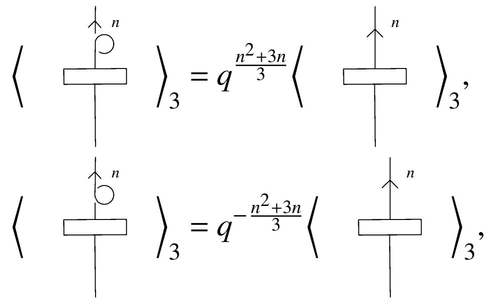
(1.1.8)



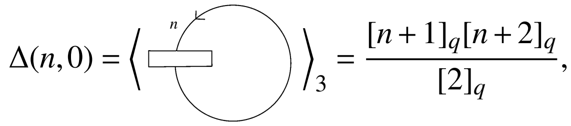
(1.1.9)



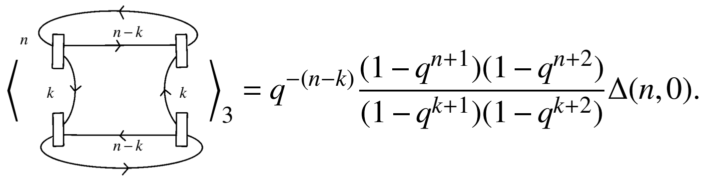
(1.1.10)



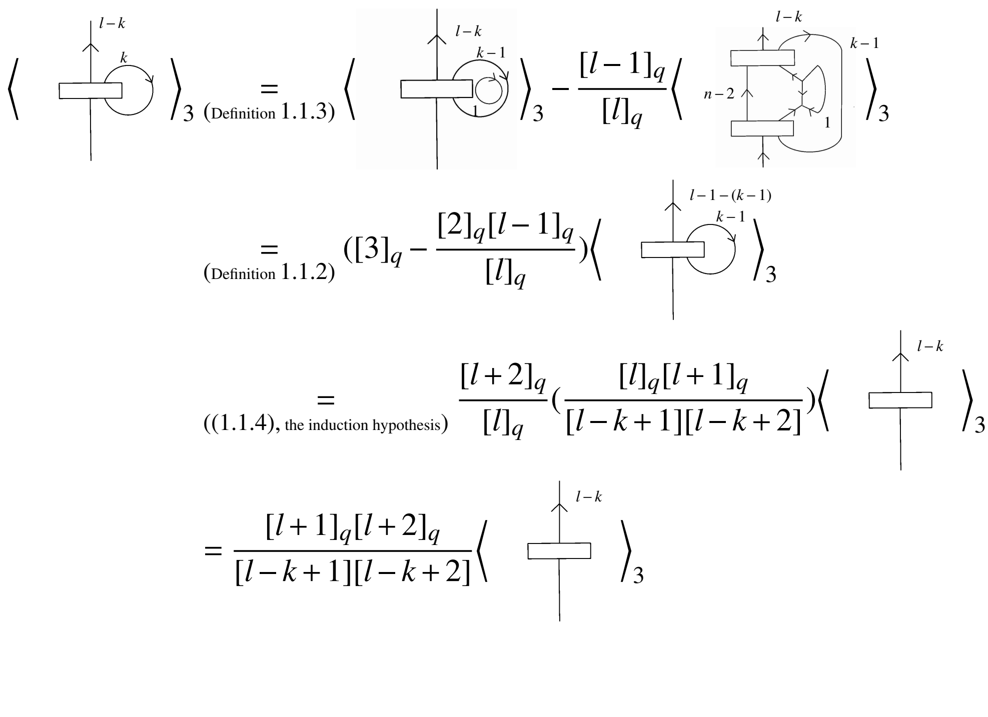
(1.1.11)



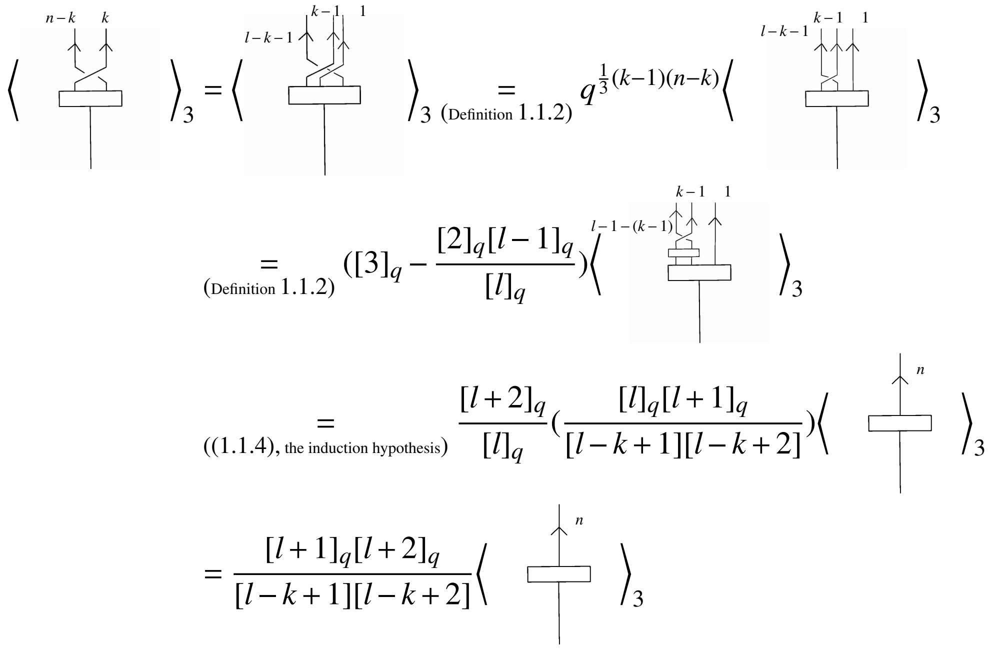
(1.1.12)



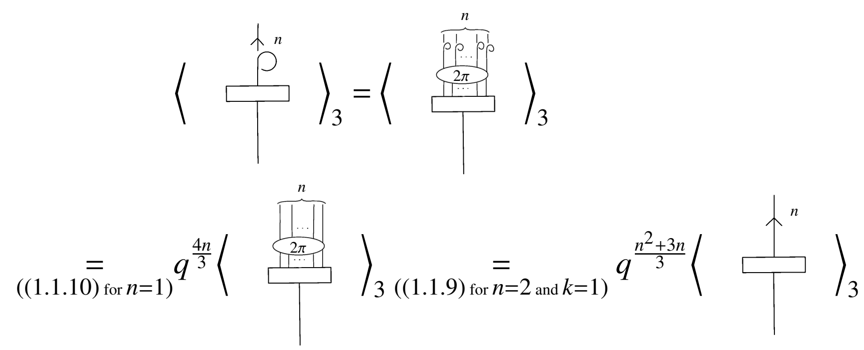
Proof. First, we show (1.1.8) by the induction on n. We can easily check in the case of $n = 1$. Assume (1.1.8) in the case of $n-1$.



Next, we prove (1.1.9) by the induction on n. If $n = 1$, then it is clear that (1.1.9) holds. Assume (1.1.9) in the case of $n-1$.



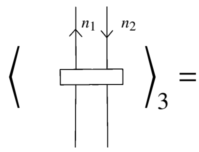
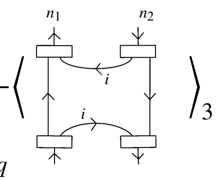
We can prove this formula in the case of the intersection being upside down in the same way. Then, we show (1.1.10). In the case of $n = 1$, it is true by Definition1.1.2. Then, we have



where there are $n(n-1)$ crossings in the 2π twisted strands. And finally, we can easily confirm (1.1.11) and (1.1.12) by (1.1.8) and (1.1.7). □

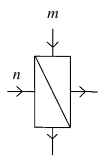
We also introduce the A_2 clasp of type (n_1, n_2) according to [OY97].

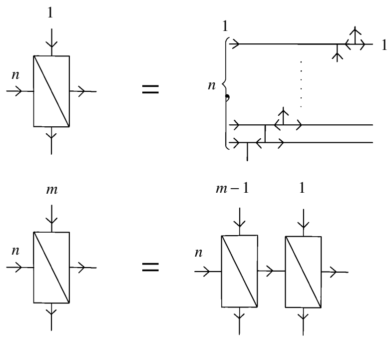
Definition 1.1.6 (the A_2 clasp of type (n_1, n_2) [OY97])**.** Let n_1 and n_2 be non-negative integers. We define the A_2 clasp of type (n_1, n_2) by


$$= \sum_{i=0}^{\min\{n_1,n_2\}} (-1)^i \frac{\begin{bmatrix} n_1 \\ i \end{bmatrix}_q \begin{bmatrix} n_2 \\ i \end{bmatrix}_q}{\begin{bmatrix} n_1+n_2+1 \\ i \end{bmatrix}_q}$$

$$\in W_{n_1^+ + n_2^- + n_1^- + n_2^+}.$$

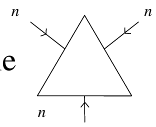
1.2 The formulae for the A_2 claps

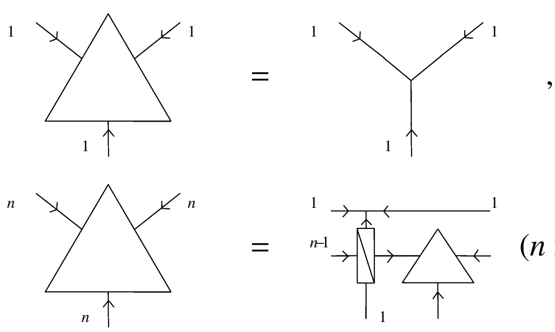
We use the following two types of web, stair-step and triangle, which we can see in [Kim06, Kim07, Yua17], Etc.

Definition 1.2.1. For positive integers n and m, stair-step  is defined by



$(m > 1)$.

For positive integers n, triangle  is defined by

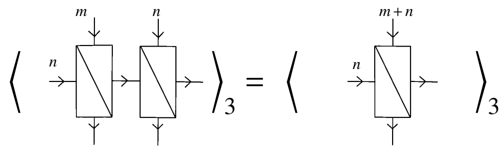


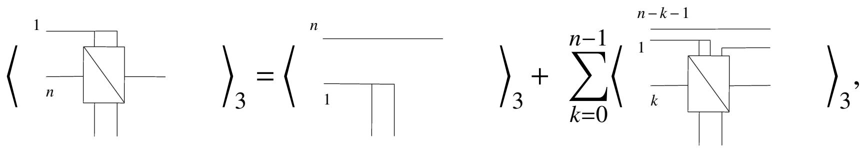
$(n > 1)$.

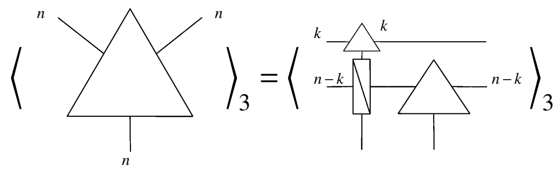
The opposite direction is defined in the same way.

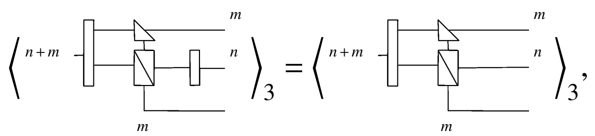
The following formulae help the calculation of the $\mathfrak{sl}_3$ colored Jones polynomial. Yuasa [Yua17, Yua18b, Yua21] and Kim [Kim06] gave them. We sometimes omit directions of edges of A_2 webs.

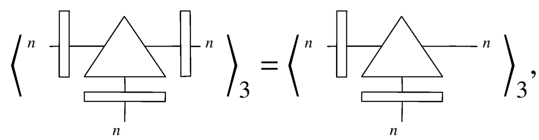
Lemma 1.2.2 ([Yua17, Yua18a, Yua21, Kim06]). For positive integers m and n, we have

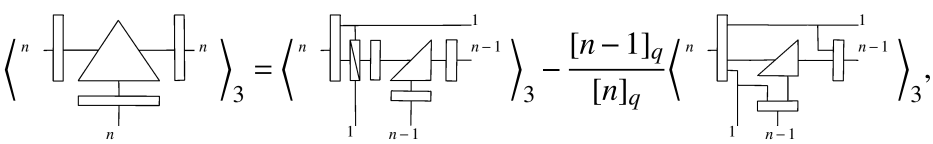
(1.2.1) 

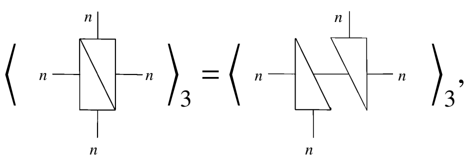
(1.2.2) 

(1.2.3)  $(1 \le k \le n)$,

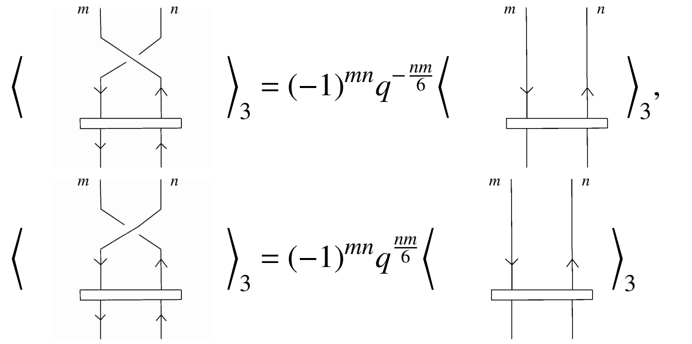
(1.2.4) 

(1.2.5) 

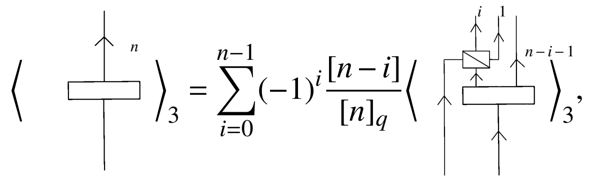
(1.2.6) 

(1.2.7) 

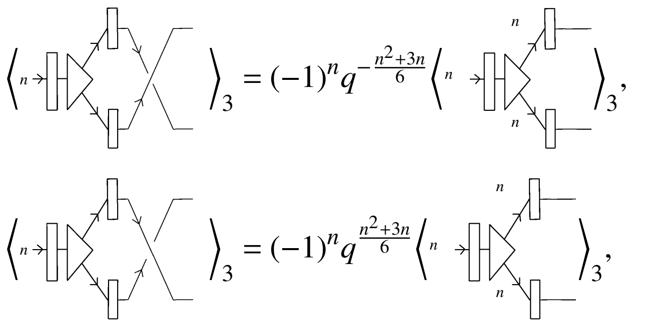
(1.2.8)



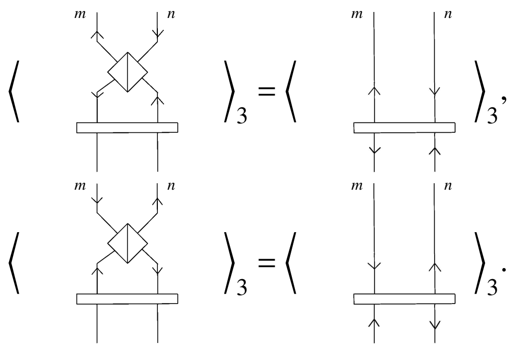
(1.2.9)



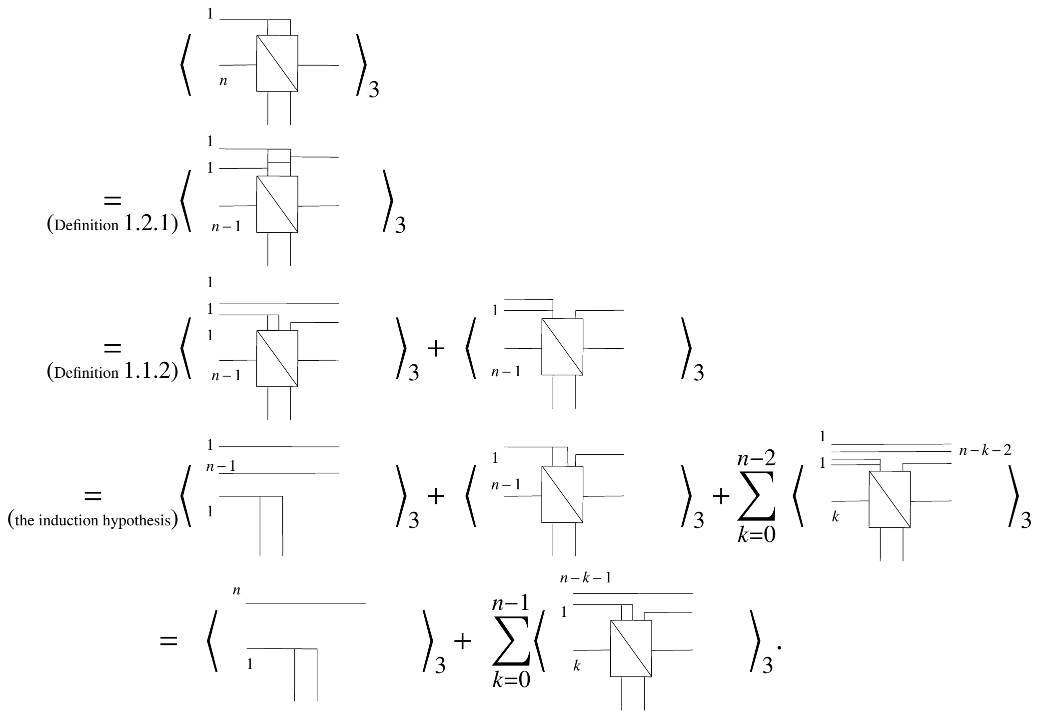
(1.2.10)



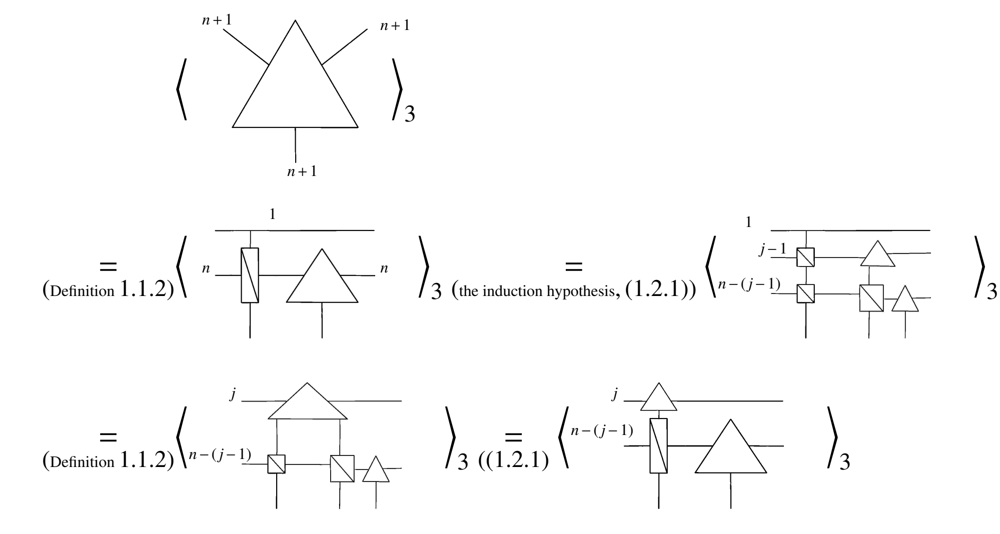
(1.2.11)



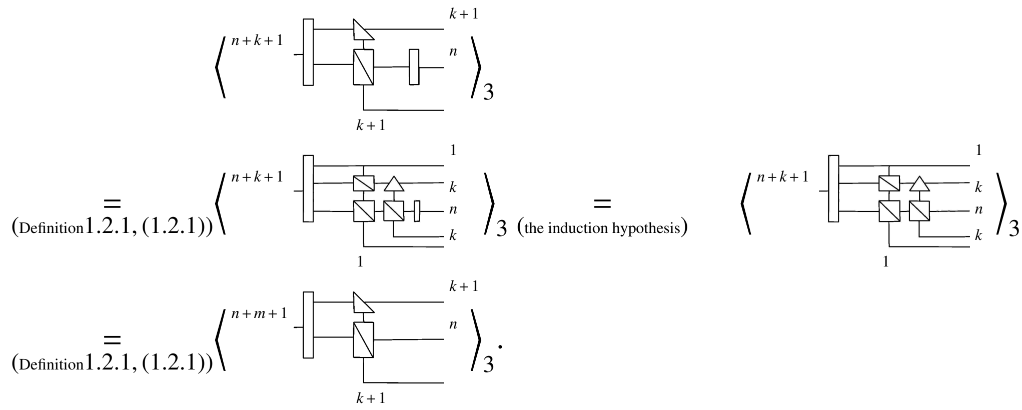
Proof. First, we have (1.2.1) by the definition of stair-step. Next, we show (1.2.2) by the induction on n. If $n = 1$, then we can easily confirm (1.2.2). Assume (1.2.2) in the case of n-1. We have



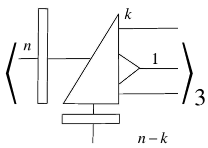
Then, let us show (1.2.3) by the induction n. When $n = 1$, (1.2.3) can be easily checked. If $n > 1$ and $0 \leq j \leq n$,



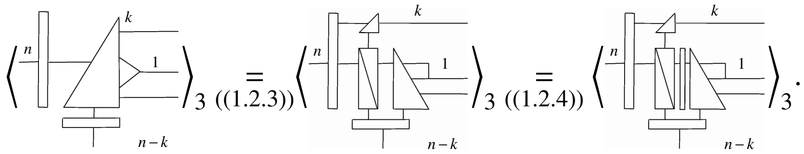
by the induction n. Furthermore, if $j = n+1$, then we can easily check (1.2.3). Therefore, (1.2.3) holds. Then, we show (1.2.4) by the induction m. In the case of $m = 1$, (1.2.4) holds by the definition of the A_2 clasp. In the case of $m = k+1$, we have



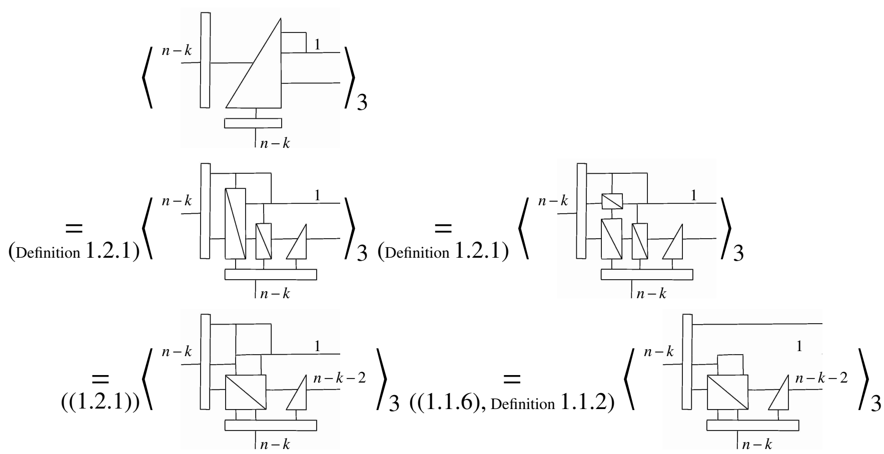
Then, we only have to prove

(1.2.12)  $= 0 \quad (k = 1, 2, ..., n-2)$

in order to show (1.2.5) because of Definition 1.1.3. The left-hand side of the above equation can be transformed as follows:

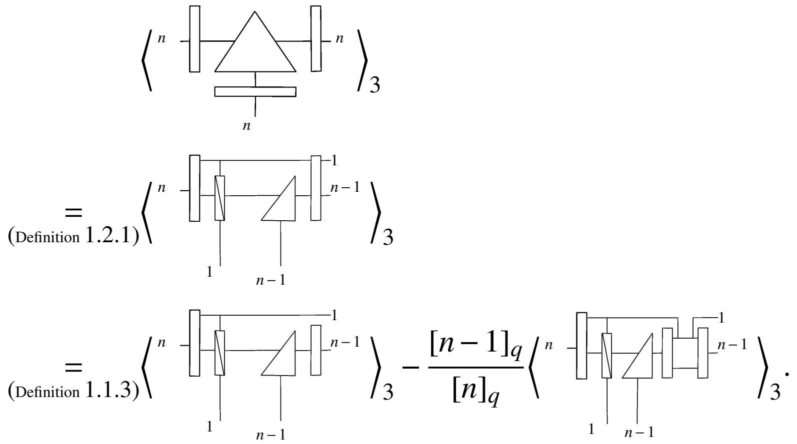


In addition, we obtain

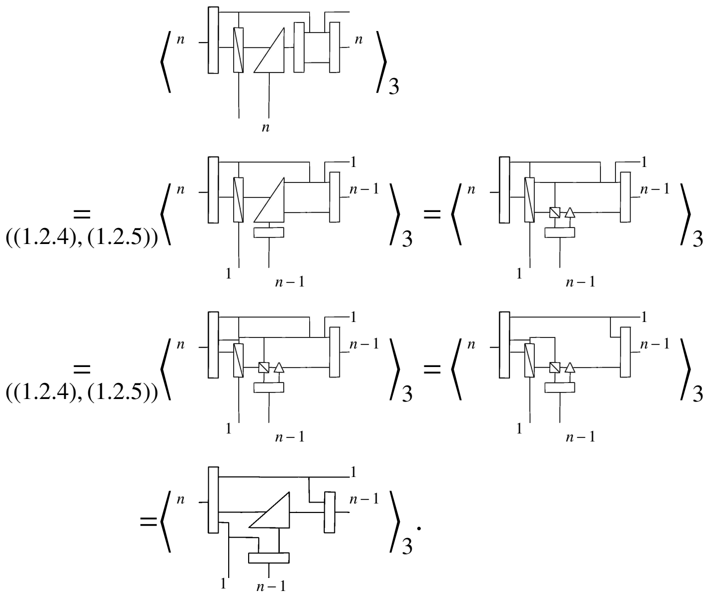


$\underset{((1.1.6),\ \text{Definition 1.1.2})}{=} 0.$

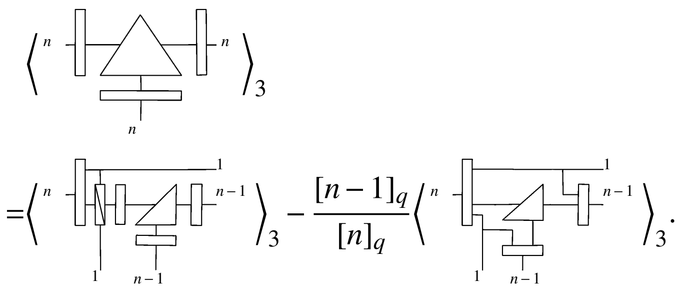
Therefore, since (1.2.12) holds, (1.2.5) is proved. Then, we show (1.2.6). We have



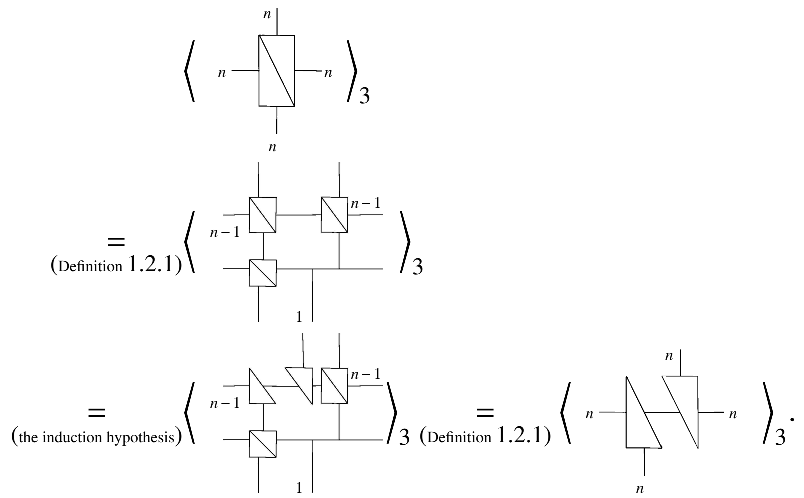
In the second term,



Thus, we can obtain



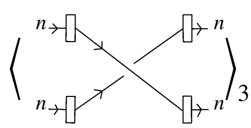
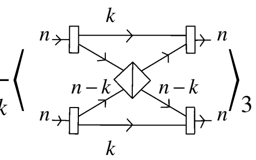
Then, we show (1.2.7) by the induction on n. For $n = 1$, it is clear by Definition 1.2.1. Assume (1.2.7) for $n-1$. Then,

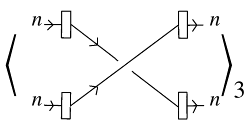
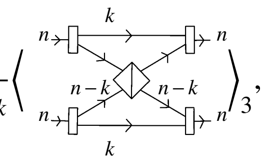


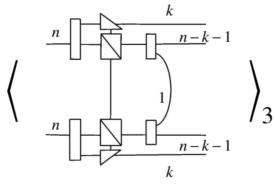
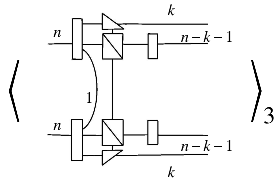
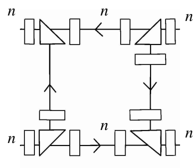
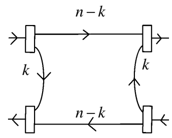
Then, (1.2.8) is obtained by Defintion 1.1.2 and (1.1.6). And finally, see (1.2.9) for [Kim07], (1.2.10) for [Yua18a], and (1.2.11) for [Yua21]. □

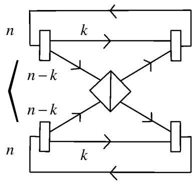
In [Yua17], Yuasa gave the half twists formulae for two strands, the m times full twists formula for two strands with opposite directions, and the A_2 bracket bubble skein expansion formula. We use these formulae to calculate the one-row $\mathfrak{sl}_3$ colored Jones polynomial for pretzel links in Chapter 2. See [Yua17] for proofs of these formulae.

Lemma 1.2.3 ([Yua17])**.** For positive integer n, we have

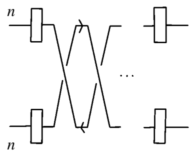
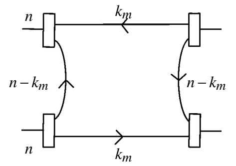
(1.2.13)  $= \sum_{k=0}^{n} (-1)^{n-k} q^{\frac{-n^2+3k^2}{6}} \frac{(q)_n}{(q)_k (q)_{n-k}}$ ,

(1.2.14)  $= \sum_{k=0}^{n} (-1)^{n-k} q^{\frac{n^2-6nk+3k^2}{6}} \frac{(q)_n}{(q)_k (q)_{n-k}}$ ,

(1.2.15)  $=$  $(n \geq 2,\ 0 \leq k \leq n-1)$, (1.2.16) $\Big\langle$  $\Big\rangle_3 = \sum_{k=0}^{n} \Big\langle$  $\Big\rangle_3,$

(1.2.17) $\Big\langle$  $\Big\rangle_3 = \dfrac{[n+1]_q[n+2]_q}{[n-k+2]_q}\Delta(n,0) \quad (0 \le k \le n).$

Theorem 1.2.4 (*m* times full twist formula [Yua17])**.** For a positive integer n, we have

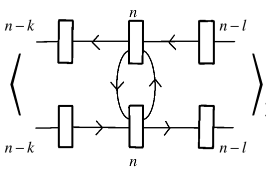
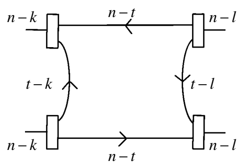
$\Big\langle$  $\Big\rangle_3 = \sum_{0 \le k_m \le k_{m-1} \le \cdots \le k_1 \le n} \phi(n,k_1,k_2,...,k_m)_{q^{\epsilon_m}} \Big\langle$  $\Big\rangle_3$

m times full twists

where

$$\phi(n,k_1,k_2,...,k_m)_{q^{\epsilon_m}} = \frac{(q^{\epsilon_m})^{-\frac{2m}{3}(n^2+3n)}(q^{\epsilon_m})^{n-k_m}(q^{\epsilon_m})^{\sum_{i=1}^{m}(k_i^2+2k_i)}(q^{\epsilon_m})_n^2}{(q^{\epsilon_m})_{n-k_1}(q^{\epsilon_m})_{k_1-k_2}\cdots(q^{\epsilon_m})_{k_{m-1}-k_m}(q^{\epsilon_m})_{k_m}^2}.$$

Theorem 1.2.5 (the A_2 bracket bubble skein expansion formula[Yua17])**.** Let n be a positive integer and k, l non-negative integers. For $n \ge k, l$, we have

$\Big\langle$  $\Big\rangle_3 = \sum_{t=\max\{k,l\}}^{\min\{k+l,n\}} \psi(n,t,k,l) \Big\langle$  $\Big\rangle_3$

where

$$\psi(n,t,k,l) = \frac{q^{(t+1)(t-k-l)+kl}(q)_k(q)_l(q)_{n-k}^2(q)_{n-l}^2(q)_{2n-t+2}}{(q)_n^2(q)_{n-t}^2(q)_{t-k}(q)_{t-l}(q)_{2n-k-l+2}(q)_{-t+k+l}}.$$

Chapter 2

Computed the one-row colored $\mathfrak{sl}_3$ Jones polynomials for pretzel knots and links

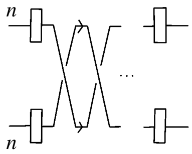
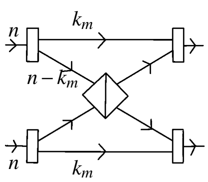
In this chapter, we first introduce two formulae for A_2 claps. These formulae were given by the author in [Kaw]. Then, we give proof of Main Theorem 1. And finally, we calculate the one-row colored $\mathfrak{sl}_3$ Jones polynomial of the knots $8_{10}, 8_{15}, 8_{20}$ and 8_{21}.

2.1 Twists formulae for the A_2 claps

We give m times half twists formulae for two strands with the same directions for A_2 bracket.

Proposition 2.1.1 (*m* times half twists formule [Kaw]). Let n be a positive integer and $k_0 = n$. For a positive integer m, we have

(2.1.1)

$\Big\langle$  $\Big\rangle_3 = \displaystyle\sum_{0\leq k_m\leq k_{m-1}\leq\cdots\leq k_1\leq n} \chi_1(n,k_1,k_2,...,k_m)\Big\langle$  $\Big\rangle_3$

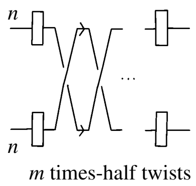
m times half twists

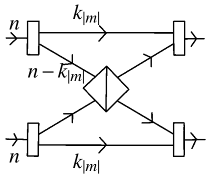
where

$$\chi_1(n,k_1,k_2,...,k_m) = (-1)^{nm}\frac{q^{-\frac{1}{6}(n^2+3n)m}q^{\frac{1}{2}(n-k_m)}(-1)^{\sum_{i=1}^{m}k_i}q^{\sum_{i=1}^{m}\frac{1}{2}(k_i^2+k_i)}(q)_n}{\prod_{i=1}^{m}(q)_{k_{i-1}-k_i}(q)_{k_m}}.$$

For a negative integer m, we have

(2.1.2)

$\Big\langle$ 

$\Big\rangle_3 = \sum_{0\le k_{|m|}\le k_{|m|-1}\le\cdots\le k_1\le n} \chi_{-1}(n,k_1,k_2,...,k_{|m|})\Big\langle$ 

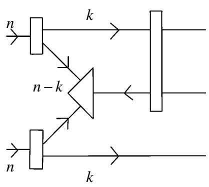
$\Big\rangle_3$

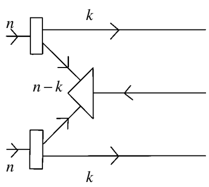
where

$$\chi_{-1}(n,k_1,k_2,...,k_{|m|}) = (-1)^{nm}\frac{q^{\frac{1}{6}(n^2+3n)m}q^{-\frac{1}{2}(n-k_{|m|})}(-1)^{\sum_{i=1}^{|m|}k_i}q^{\sum_{i=1}^{m}\frac{1}{2}(k_i^2-k_i)}q^{\sum_{i=1}^{|m|}k_{i-1}k_i}(q)_n}{\prod_{i=1}^{|m|}(q)_{k_{i-1}-k_i}(q)_{k_{|m|}}}.$$

We prepare the following lemma to prove Proposition 2.1.1.

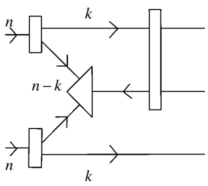
Lemma 2.1.2 ([Kaw])**.** Let n be a positive integer and k be a non-negative integer. For $0 \le k \le n$, we have

(2.1.3) $\Big\langle$ 

$\Big\rangle_3 = \Big\langle$ 

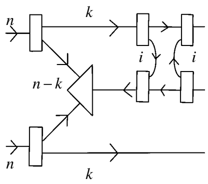
$\Big\rangle_3.$

Proof. By definition of A_2 clasp of type (n_1,n_2), we obtain

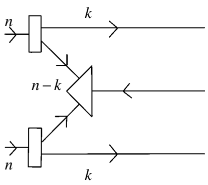
$\Big\langle$ 

$\Big\rangle_3$

$$= \sum_{i=0}^{\min\{n-k,k\}} (-1)^i \frac{\begin{bmatrix} n-k \\ i \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q}{\begin{bmatrix} n+i \\ i \end{bmatrix}_q}\Big\langle$$



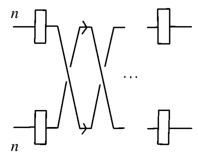
$\Big\rangle_3$

$= \Big\langle$ 

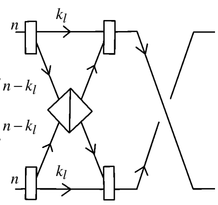
$\Big\rangle_3.$

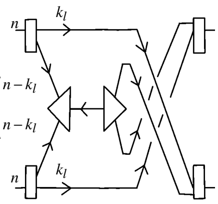
The right-hand side of the above equation is zero by (1.2.5), (1.1.6), and (1.1.7) except for $i = 0$. □

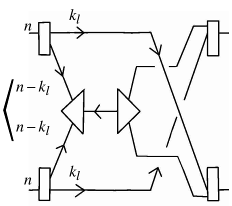
Proof of Proposition 2.1.1. We proceed by the induction on m. We can see that $(A.1.3)$ holds by (1.2.13) for $m = 1$. Assume $(A.1.3)$ holds when $m = l$ for some integers $l \geq 1$.

$\Big\langle$  $\Big\rangle_3$

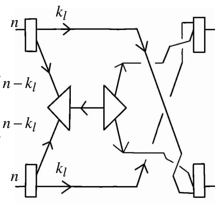
m times-half twists

$$= \sum_{0\le k_l\le k_{l-1}\le\cdots\le k_1\le n} \chi_1(n,k_1,k_2,...,k_l)\Big\langle$$  $\Big\rangle_3$

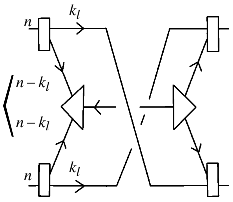
$$= \sum_{0\le k_l\le k_{l-1}\le\cdots\le k_1\le n} \chi_1(n,k_1,k_2,...,k_l)\Big\langle$$  $\Big\rangle_3$

$$\underset{((1.2.10))}{=} \sum_{0\le k_l\le k_{l-1}\le\cdots\le k_1\le n} \chi_1(n,k_1,k_2,...,k_l)(-1)^{n-k_l}q^{-\frac{1}{6}(n-k_l)^2-\frac{1}{2}(n-k_l)}\Big\langle$$  $\Big\rangle_3$

$$\underset{((1.1.9))}{=} \sum_{0\le k_l\le k_{l-1}\le\cdots\le k_1\le n} \chi_1(n,k_1,k_2,...,k_l)(-1)^{n-k_l}q^{-\frac{1}{6}(n-k_l)^2-\frac{1}{2}(n-k_l)}$$

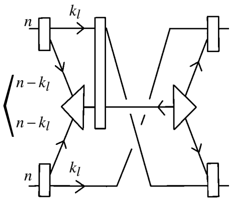
$$\times q^{-\frac{2}{3}(n-k_l)k_l}\Big\langle$$  $\Big\rangle_3$

$$\underset{(\text{Reidemeister move})}{=} \sum_{0\le k_l\le k_{l-1}\le\cdots\le k_1\le n} \chi_1(n,k_1,k_2,...,k_l)(-1)^{n-k_l}q^{-\frac{1}{6}(n-k_l)^2-\frac{1}{2}(n-k_l)}$$

$$\times q^{-\frac{2}{3}(n-k_l)k_l}\Big\langle$$


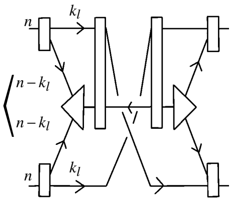
$$\Big\rangle_3$$

$$\underset{(\text{Lemma 2.1.2, (1.2.8)})}{=} \sum_{0\le k_l\le k_{l-1}\le\cdots\le k_1\le n} \chi_1(n,k_1,k_2,...,k_l)(-1)^{n-k_l}q^{-\frac{1}{6}(n-k_l)^2-\frac{1}{2}(n-k_l)}$$

$$\times q^{-\frac{1}{3}(n-k_l)k_l}\Big\langle$$


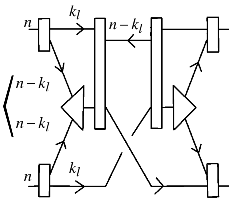
$$\Big\rangle_3$$

$$\underset{(\text{Lemma 2.1.2})}{=} \sum_{0\le k_l\le k_{l-1}\le\cdots\le k_1\le n} \chi_1(n,k_1,k_2,...,k_l)(-1)^{n-k_l}q^{-\frac{1}{6}(n-k_l)^2-\frac{1}{2}(n-k_l)}$$

$$\times q^{-\frac{1}{3}(n-k_l)k_l}\Big\langle$$


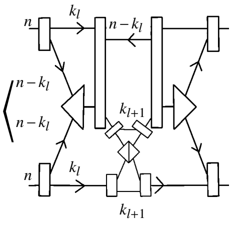
$$\Big\rangle_3$$

$$\underset{((1.2.8))}{=} \sum_{0\le k_l\le k_{l-1}\le\cdots\le k_1\le n} \chi_1(n,k_1,k_2,...,k_l)(-1)^{n-k_l}q^{-\frac{1}{6}(n-k_l)^2-\frac{1}{2}(n-k_l)}$$

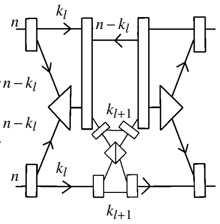
$$\times q^{-\frac{1}{3}(n-k_l)k_l}\Big\langle$$


$$\Big\rangle_3$$

$$\underset{((1.2.13))}{=} \sum_{0\le k_l\le\cdots\le k_1\le n} \chi_1(n,k_1,k_2,...,k_l)(-1)^{n-k_l}q^{-\frac{1}{6}(n-k_l)^2-\frac{1}{2}(n-k_l)}q^{-\frac{1}{3}(n-k_l)k_l}$$

$$\times \sum_{k_{l+1}=0}^{k_l}(-1)^{k_l-k_{l-1}}q^{\frac{-k_l^2+3k_{l+1}^2}{6}}\frac{(q)_{k_l}}{(q)_{k_l-k_{l+1}}(q)_{k_{l+1}}}\Big\langle$$


$$\Big\rangle_3$$

$$= \sum_{0\le k_{l+1}\le k_l\le\cdots\le k_1\le n} \chi_1(n,k_1,k_2,\dots,k_{l+1})\Big\langle$$  $$\Big\rangle_3$$

In the eighth equation, it produces coefficients $(-1)^{k_l(n-k_1)}q^{\frac{k_l(n-k_l)}{6}}$ and $(-1)^{k_l(n-k_1)}q^{-\frac{k_l(n-k_l)}{6}}$. We use

$$\chi_1(n,k_1,k_2,\dots,k_l)(-1)^{n-k_l}q^{-\frac{1}{6}(n-k_l)^2-\frac{1}{2}(n-k_l)}q^{-\frac{1}{3}(n-k_l)k_l}\sum_{k_{l+1}=0}^{k_l} q^{\frac{-k_l^2+3k_{l+1}^2}{6}}\frac{(q)_{k_l}}{(q)_{k_l-k_{l+1}}(q)_{k_{l+1}}}$$

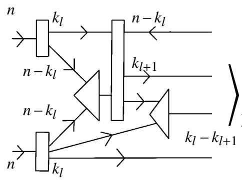
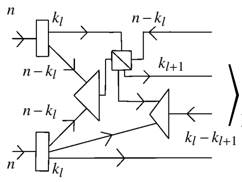
$$=(-1)^{nl}\frac{q^{-\frac{1}{6}(n^2+3n)l}q^{\frac{1}{2}(n-k_l)}(-1)^{\sum_{i=1}^{l}k_i}q^{\sum_{i=1}^{l}\frac{1}{2}(k_i^2+k_i)}(q)_n}{\prod_{i=1}^{l+1}(q)_{k_{i-1}-k_i}(q)_{k_l}}(-1)^{n-k_l}q^{-\frac{1}{6}(n-k_l)^2-\frac{1}{2}(n-k_l)}q^{-\frac{1}{3}(n-k_l)k_l}$$

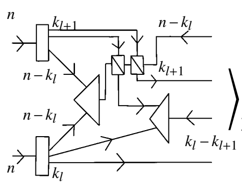
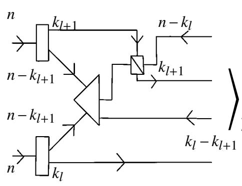
$$\times\sum_{k_{l+1}=0}^{k_l}(-1)^{k_l-k_{l-1}}q^{\frac{-k_l^2+3k_{l+1}^2}{6}}\frac{(q)_{k_l}}{(q)_{k_l-k_{l+1}}(q)_{k_{l+1}}}$$

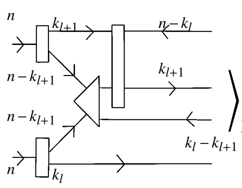
$$=\sum_{k_{l+1}=0}^{k_l}(-1)^{n(l+1)}\frac{q^{-\frac{1}{6}(n^2+3n)(l+1)}q^{\frac{1}{2}(n-k_{l+1})}(-1)^{\sum_{i=1}^{l+1}k_i}q^{\sum_{i=1}^{l+1}\frac{1}{2}(k_i^2+k_i)}(q)_n}{\prod_{i=1}^{l+1}(q)_{k_{i-1}-k_i}(q)_{k_l}}$$

$$=\sum_{k_{l+1}=0}^{k_l}\chi_1(n,k_1,k_2,\dots,k_{l+1}).$$

for the last equation. We can see that

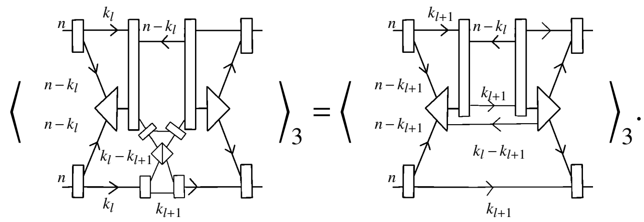
$$\Big\langle$$  $$\Big\rangle_3 \underset{(\text{Lemma 2.1.2, (1.2.11)})}{=} \Big\langle$$  $$\Big\rangle_3$$

(2.1.4) $$\underset{(\text{Definition 1.2.1})}{=}\Big\langle$$  $$\Big\rangle_3 \underset{((1.2.7))}{=} \Big\langle$$  $$\Big\rangle_3$$

$$\underset{(\text{Lemma 2.1.2, (1.2.11)})}{=}\Big\langle$$  $$\Big\rangle_3.$$

By using (2.1.4), we have

(2.1.5)



We remove the A_2 clasps by using Lemma 2.1.2 and (2.1.4) in (*RHS* of (2.1.5)). Therefore, (2.1.1) holds for $n = l+1$. We can see (2.1.2) in a similar way. □

We use the same definition for the one-row $\mathfrak{sl}_3$ colored Jones polynomial as we do in [Yua17].

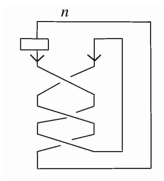
Definition 2.1.3. Let L be an oriented link, and $\bar{L}$ a diagram of L. The one-row $\mathfrak{sl}_3$ colored Jones polynomial for L is defined by

$$J^{\mathfrak{sl}_3}_{(n,0)}(L;q) = (q^{\frac{n^2+3n}{3}})^{-w(\bar{L})}\langle L(n,0)\rangle_3/\Delta(n,0)$$

where $w(\bar{L})$ is the writhing number of $\bar{L}$ and $L(n,0)$ replaces a part of $\bar{L}$ with A_2 clasp of type $(n,0)$.

For example,

$$\vec{\vec{T}}(2,3)(n,0) =$$



Let a map $sign : \mathbb{Z} \to \{-1,0,1\}$ be defined by

$$sign(x) = \begin{cases} 1 & (x>0) \\ 0 & (x=0) \\ -1 & (x<0) \end{cases}.$$

Corollary 2.1.4 ([Kaw])**.** The one-row colored $\mathfrak{sl}_3$ Jones polynomials for $(2,m)$-torus links $T(2,m)$ are the following:

(2.1.6)

$$J^{\mathfrak{sl}_3}_{(n,0)}(\vec{\vec{T}}(2,m);q) = (q^{\frac{n^2+3n}{3}})^{-m}\sum_{0\le k_{|m|}\le k_{|m|-1}\le\cdots\le k_1\le n}\chi_{sign(m)}q^{\frac{n+k_{|m|}}{2}}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q)(1-q^{n-k_{|m|}+2})}.$$

Proof. we can obtain (2.1.6) from Definition 2.1.3, Proposition 2.1.1 and (1.2.17). □

Remark 2.1.5. For $m = 1\ 7$, n=$1,2,...,10$, Corollary 2.1.4 are equal to Theorem 2.1 in [GM13] and Theorem 5.7 in [Yua17] for $T(2,m)$ by using Mathematica. For example,

$J^{\mathfrak{sl}_3}_{(1,0)}(T(2,3);q) = q^{-2}+q^{-4}-q^{-6},$
$J^{\mathfrak{sl}_3}_{(2,0)}(T(2,3);q) = q^{-4}+q^{-7}+q^{8}-q^{-12}-q^{-13}+q^{-15},$
$J^{\mathfrak{sl}_3}_{(3,0)}(T(2,3);q) = q^{-6}+q^{-10}+q^{-11}-q^{-13}+q^{-15}-2q^{17}-q^{-18}-q^{22}+q^{-23}+q^{-24}$
$+q^{-25}-q^{-27},$
$J^{\mathfrak{sl}_3}_{(4,0)}(T(2,3);q) = q^{-8}+q^{-13}+q^{-14}-q^{-17}+q^{-19}+q^{-20}-q^{-21}-2q^{-22}-q^{-23}+q^{-25}$
$-q^{-27}-q^{-28}+q^{-30}+2q^{-31}+q^{-35}+q^{-36}-q^{38}-q^{-39}-q^{-40}+q^{-42},$

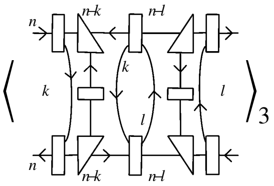
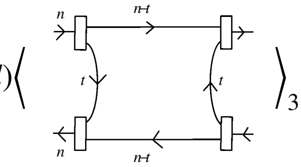
$\vdots$

$J^{\mathfrak{sl}_3}_{(10,0)}(T(2,3);q) = q^{-20}+q^{-31}+q^{-32}-q^{-41}+q^{-43}+q^{-44}-q^{-51}-2q^{-52}-q^{-53}+q^{-55}$
$+q^{-56}-q^{-62}-2q^{-63}-2q^{-64}-q^{-65}+q^{-67}+q^{-68}+q^{-70}+q^{-71}+q^{-72}-2q^{-74}-2q^{-75}$
$-2q^{-76}-q^{-77}+2q^{-79}+2q^{-80}+2q^{-81}+2q^{-82}+2q^{-83}+q^{-84}-q^{-85}-2q^{-86}-2q^{-87}$
$-2q^{-88}-q^{-89}+q^{-90}+2q^{-91}+3q^{-92}+3q^{-93}+3q^{-94}+2q^{-95}-q^{-96}-2q^{-97}-3q^{-98}$
$-3q^{-99}-3q^{-100}-q^{-101}+q^{-102}+3q^{-103}+3q^{-104}+3q^{-105}+2q^{-106}-q^{-113}-3q^{-112}$
$-4q^{-111}-4q^{-110}-3q^{-109}-2q^{-108}-q^{-107}-2q^{-108}-3q^{-109}-4q^{-110}-4q^{-111}-3q^{-112}$
$-q^{-113}+2q^{-114}+3q^{-115}+4q^{-116}+3q^{-117}-q^{-119}-2q^{-120}-3q^{-121}-4q^{-122}-3q^{-123}$
$-2q^{-124}+q^{-125}+4q^{-126}+5q^{-127}+5q^{-128}+2q^{-129}+q^{-130}-q^{-131}-2q^{-132}-3q^{-133}$
$-3q^{-134}-2q^{-135}+3q^{-137}+5q^{-138}+4q^{-139}+2q^{-140}+q^{-141}-2q^{-143}-3q^{-144}-4q^{-145}$
$-4q^{-146}-2q^{-147}+3q^{-149}+3q^{-150}+q^{-151}+q^{-152}-q^{-154}-3q^{-155}-3q^{-156}-3q^{-157}$
$-2q^{-158}+2q^{-160}+2q^{-163}+2q^{-164}+q^{-165}-q^{-167}-2q^{-168}-q^{-169}+4q^{-161}+2q^{-162}$
$-q^{-170}+q^{-171}+2q^{-172}+q^{173}-q^{-177}-q^{-178}-2q^{-179}-2q^{-180}-q^{-181}+q^{-183}+q^{-184}$
$+q^{-185}+q^{-186}+q^{-187}+q^{-188}-q^{-191}-q^{192}-q^{-193}+q^{-195}.$

The following proposition is important when we calculate the one-row colored $\mathfrak{sl}_3$ Jones polynomials of pretzel links.

Proposition 2.1.6 ([Kaw])**.** Let n be a positive integer and k, l non-negative integers.

For $n \geq k, l$, we have

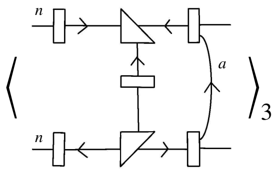
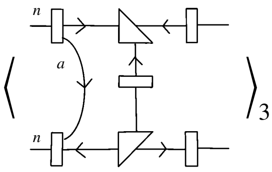
 $= \sum_{t=\max\{k,l\}}^{\min\{k+l,n\}} \sum_{a=t}^{n} \Omega(n,t,k,l)$ 

where

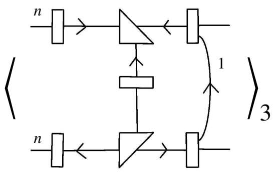
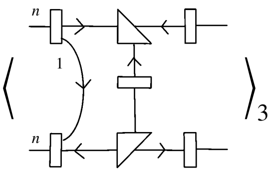
$$\Omega(n,k,l,t) = \frac{q^{-\frac{k+l}{2}+t} q^{(t+1)(t-k-l)+kl}(1-q^{n+1-k})(1-q^{n+1-l})(q)_k(q)_l(q)^2_{n-k}(q)^2_{n-l}(q)_{2n-t+2}}{(1-q^{n+1-t})^2(q)^2_n(q)^2_{n-t}(q)_{t-k}(q)_{t-l}(q)_{2n-k-l+2}(q)_{-t+k+l}}.$$

We first prove the following lemma.

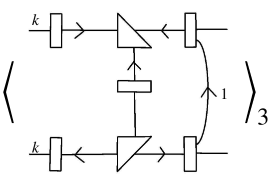
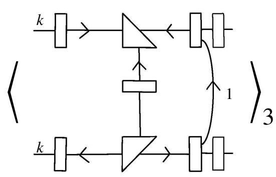
Lemma 2.1.7 ([Kaw])**.** Let n and a be positeve integers. We have

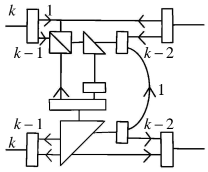
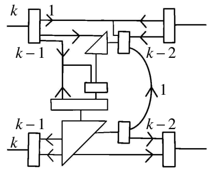
 $= \frac{[n+1]_q}{[n-a+1]_q}$ 

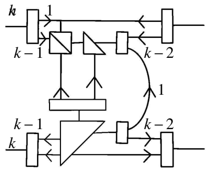
Proof. First, we prove that it holds for $a = 1$.

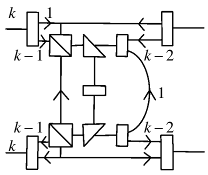
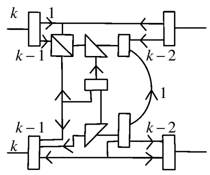
(2.1.7)  $= \frac{[n+1]_q}{[n]_q}$ 

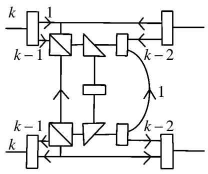
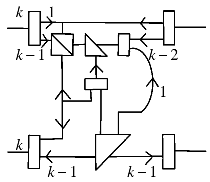
The equation (2.1.7) is the same lemma in [Yua20], but we prove it differently. We proceed by the induction on n. We can see that (2.1.7) holds by an easy calculation for $n = 2$. Assume (2.1.7) holds when $n \leq k-1$ for some integer $k \geq 3$. Note that (1.2.5) and (1.2.4) should be used appropriately and many times. We obtain

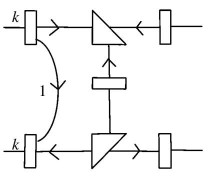
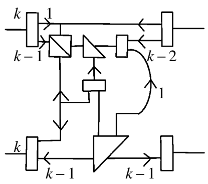
 $=$ 

$\underset{((1.2.6))}{=} \Big\langle$  $\Big\rangle_3 - \dfrac{[k-1]_q}{[k]_q} \Big\langle$  $\Big\rangle_3$

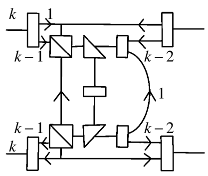
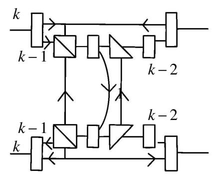
$\underset{((1.1.6),(1.1.7))}{=} \Big\langle$  $\Big\rangle_3$

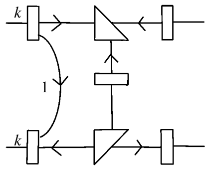
$\underset{((1.2.6))}{=} \Big\langle$  $\Big\rangle_3 - \dfrac{[k-1]_q}{[k]_q} \Big\langle$  $\Big\rangle_3$

$\underset{(\text{Definition 1.2.1, (1.2.9)})}{=} \Big\langle$  $\Big\rangle_3 - (-1)^{k-2} \dfrac{1}{[k]_q} \Big\langle$  $\Big\rangle_3$

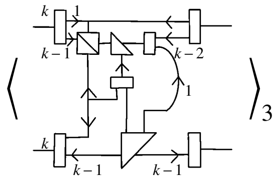
$= \dfrac{[k]_q}{[k-1]_q} \Big\langle$  $\Big\rangle_3 - (-1)^{k-2} \dfrac{1}{[k]_q} \Big\langle$  $\Big\rangle_3.$

We apply (1.2.10) with label $k-1$ in the fourth equation and

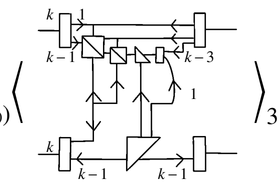
$\Big\langle$  $\Big\rangle_3 \underset{(\text{the induction hypothesis})}{=} \Big\langle$  $\Big\rangle_3$

$\underset{((1.2.15))}{=} \dfrac{[k]_q}{[k-1]_q} \Big\langle$  $\Big\rangle_3.$

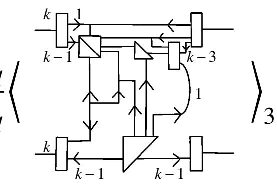
in the last equation. For the second term of the last equation,



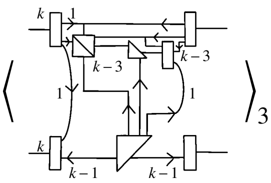
$\underset{((1.2.6))}{=}$



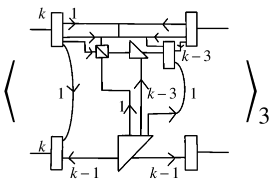
$- \dfrac{[k-2]_q}{[k-1]_q}$



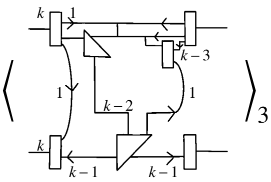
$= - \dfrac{[k-2]_q}{[k-1]_q}$



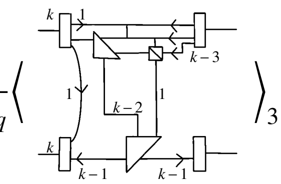
$\underset{(\text{Definition 1.2.1})}{=} - \dfrac{[k-2]_q}{[k-1]_q}$



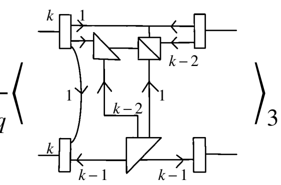
$\underset{(\text{Definition 1.2.1})}{=} - \dfrac{[k-2]_q}{[k-1]_q}$



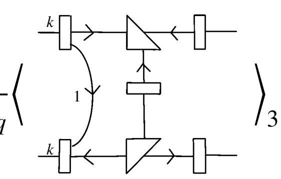
$\underset{(\text{Definition 1.2.1, (1.2.9)})}{=} -(-1)^{k-3} \dfrac{1}{[k-1]_q}$



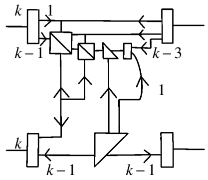
$\underset{(\text{Definition 1.2.1})}{=} -(-1)^{k-3} \dfrac{1}{[k-1]_q}$



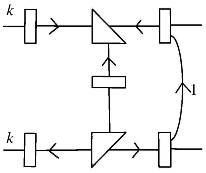
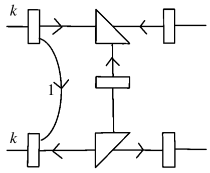
$\underset{(\text{the induction hypothesis})}{=} -(-1)^{k-3} \dfrac{1}{[k-1]_q}$



We use (1.2.10) with the label $k-2$ in the fifth equation. Regarding second equality, we have

$$\Big\langle$$  $$\Big\rangle_3 = 0$$

by using Definition 1.2.1 and (1.1.7). Hence,

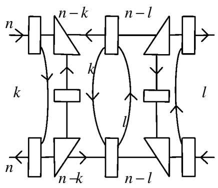
$$\Big\langle$$  $$\Big\rangle_3 = \frac{[k]_q^2 - 1}{[k]_q[k-1]_q}\Big\langle$$  $$\Big\rangle_3$$

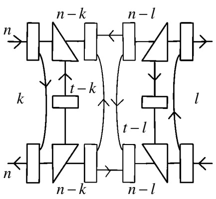
We can check easily the following equation by Lemma 1.1.1.

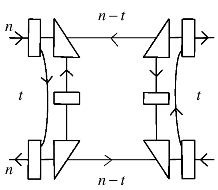
(2.1.8) $$\frac{[k]_q^2 - 1}{[k]_q[k-1]_q} = \frac{[k+1]_q}{[k]_q}$$

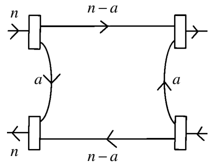
Therefore, (2.1.7) holds for $n = k$. In addtion, the equation formula (2.1.7) holds for $k = 1$ by definition of A_2 bracket. We can prove easily that if we use (2.1.8) repeatedly, then Lemma2.1.7 holds. □

*Proof of Proposition*2.1.6. We have

$$\Big\langle$$  $$\Big\rangle_3$$

$$\underset{(\text{Theorem1.2.5})}{=} \sum_{t=\max\{k,l\}}^{\min\{k+l,n\}} \psi(n,t,k,l)\Big\langle$$  $$\Big\rangle_3$$

$$\underset{((1.1.6),\ \text{Lemma 2.1.7})}{=} \sum_{t=\max\{k,l\}}^{\min\{k+l,n\}} \frac{[n-k+1]_q[n-l+1]_q}{[n+1-t]_q^2}\psi(n,t,k,l)\Big\langle$$  $$\Big\rangle_3$$

$$\underset{(\text{(1.2.16)})}{=}\sum_{t=\max\{k,l\}}^{\min\{k+l,n\}}\sum_{a=t}^{n}\Omega(n,t,k,l)\Big\langle$$  $$\Big\rangle_3.$$

□

2.2 Proof of Main Theorem 1

Recall Main Theorem 1.

Main Theorem 1 ([Kaw])**.** Let α,β,γ be non-zero integers. The one-row colored $\mathfrak{sl}_3$ Jones polynomials for pretzel links $P(\alpha,\beta,\gamma)$ are the following:

(2.2.1)

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha\uparrow,\downarrow 2\beta\uparrow,\downarrow 2\gamma\uparrow);q)$$
$$=\sum_{0\le k_{|\alpha|}\le k_{|\alpha|-1}\le\cdots\le k_1\le n}\ \sum_{0\le l_{|\beta|}\le l_{|\beta|-1}\le\cdots\le l_1\le n}\ \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n}\ \sum_{s=\max\{k_{|\alpha|},l_{|\beta|}\}}^{\min\{k_{|\alpha|}+l_{|\beta|},n\}}\ \sum_{t=\max\{s,m_{|\gamma|}\}}^{\min\{s+m_{|\gamma|},n\}}$$
$$(q^{\frac{n^2+3n}{3}})^{2\alpha+2\beta+2\gamma}\phi(n,k_1,k_2,...,k_{|\alpha|})_{q^{\epsilon_\alpha}}\phi(n,l_1,l_2,...,l_{|\beta|})_{q^{\epsilon_\beta}}\phi(n,m_1,m_2,...,m_{|\gamma|})_{q^{\epsilon_\gamma}}$$
$$\times\psi(n,s,k_{|\alpha|},l_{|\beta|})\psi(n,t,s,m_{|\gamma|})q^{-(n-t)}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})},$$

(2.2.2)

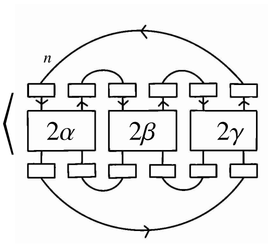
$$J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow\alpha\downarrow,\uparrow\beta\uparrow,\downarrow 2\gamma\uparrow);q)$$
$$=\sum_{0\le k_{|\alpha|}\le k_{|\alpha|-1}\le\cdots\le k_1\le n}\ \sum_{0\le l_{|\beta|}\le l_{|\beta|-1}\le\cdots\le l_1\le n}\ \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n}\ \sum_{s=\max\{k_{|\alpha|},l_{|\beta|}\}}^{\min\{k_{|\alpha|}+l_{|\beta|},n\}}\ \sum_{a=s}^{n}$$
$$\sum_{t=\max\{a,m_{|\gamma|}\}}^{\min\{a+t+m_{|\gamma|},n\}}\ \sum_{t=\max\{a,m_{|\gamma|}\}}^{\min\{a+t+m_{|\gamma|},n\}}(q^{\frac{n^2+3n}{3}})^{-(\alpha+\beta-2\gamma)}\chi_{sign(\alpha)}(n,k_1,k_2,...,k_{|\alpha|})\chi_{sign(\beta)}(n,l_1,l_2,...,l_{|\beta|})$$
$$\times\phi(n,m_1,m_2,...,m_{|\gamma|})_{q^{\epsilon_\gamma}}\Omega(n,s,k_{|\alpha|},l_{|\beta|})\psi(n,t,a,m_{|\gamma|})$$
$$\times q^{-(n-t)}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})}$$

where $q^{\epsilon_\gamma}=q^{\frac{\gamma}{|\gamma|}}$.

Proof. We first prove (2.2.1).

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)$$

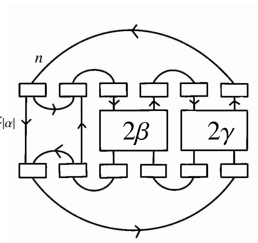
$$\underset{(\text{Definition 2.1.3})}{=} (q^{\frac{n^2+3n}{3}})^{2\alpha+2\beta+2\gamma} \Big\langle$$



$$\Big\rangle_3 / \Delta(n,0)$$

$$\underset{(\text{Theorem 1.2.5})}{=} (q^{\frac{n^2+3n}{3}})^{2\alpha+2\beta+\gamma 2} \sum_{0 \le k_{|\alpha|} \le k_{|\alpha|-1} \le \cdots \le k_1 \le n} \phi(n, k_1, k_2, ..., k_{|\alpha|})_{q^{\epsilon\alpha}}$$

$$\times \Big\langle$$

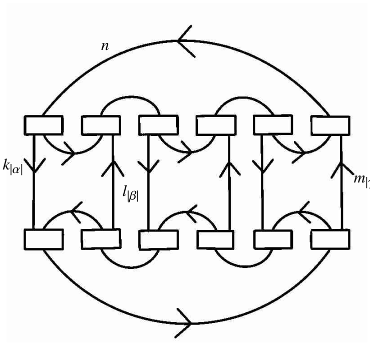


$$\Big\rangle_3 / \Delta(n,0)$$

$$\underset{(\text{Theorem 1.2.5})}{=} \sum_{0 \le k_{|\alpha|} \le k_{|\alpha|-1} \le \cdots \le k_1 \le n} \quad \sum_{0 \le l_{|\beta|} \le l_{|\beta|-1} \le \cdots \le l_1 \le n} \quad \sum_{0 \le m_{|\gamma|} \le m_{|\gamma|-1} \le \cdots \le m_1 \le n}$$

$$(q^{\frac{n^2+3n}{3}})^{2\alpha+2\beta+2\gamma} \phi(n, k_1, k_2, ..., k_{|\alpha|})_{q^{\epsilon\alpha}} \phi(n, l_1, l_2, ..., l_{|\beta|})_{q^{\epsilon\beta}} \phi(n, m_1, m_2, ..., m_{|\gamma|})_{q^{\epsilon\gamma}}$$

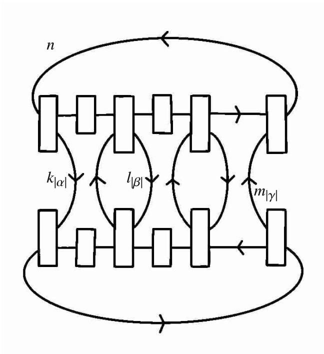
$$\times \Big\langle$$



$$\Big\rangle_3 / \Delta(n,0)$$

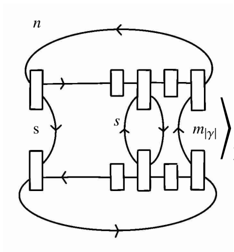
$$\underset{((1.1.7))}{=} \sum_{0 \le k_{|\alpha|} \le k_{|\alpha|-1} \le \cdots \le k_1 \le n} \quad \sum_{0 \le l_{|\beta|} \le l_{|\beta|-1} \le \cdots \le l_1 \le n} \quad \sum_{0 \le m_{|\gamma|} \le m_{|\gamma|-1} \le \cdots \le m_1 \le n}$$

$$(q^{\frac{n^2+3n}{3}})^{2\alpha+2\beta+2\gamma} \phi(n, k_1, k_2, ..., k_{|\alpha|})_{q^{\epsilon\alpha}} \phi(n, l_1, l_2, ..., l_{|\beta|})_{q^{\epsilon\beta}} \phi(n, m_1, m_2, ..., m_{|\gamma|})_{q^{\epsilon\gamma}}$$

$\times \Big\langle$  $\Big\rangle_3 / \Delta(n,0)$

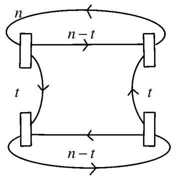
$$\underset{(\text{Theorem 1.2.5})}{=} \sum_{0\le k_{|\alpha|}\le k_{|\alpha|-1}\le\cdots\le k_1\le n} \sum_{0\le l_{|\beta|}\le l_{|\beta|-1}\le\cdots\le l_1\le n} \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n} \sum_{s=\max\{k_{|\alpha|},l_{|\beta|}\}}^{\min\{k_{|\alpha|}+l_{|\beta|},n\}}$$

$$(q^{\frac{n^2+3n}{3}})^{2\alpha+2\beta+2\gamma}\phi(n,k_1,k_2,...,k_{|\alpha|})_{q^{\epsilon_\alpha}}\phi(n,l_1,l_2,...,l_{|\beta|})_{q^{\epsilon_\beta}}\phi(n,m_1,m_2,...,m_{|\gamma|})_{q^{\epsilon_\gamma}}$$

$\times\psi(n,s,k_{|\alpha|},l_{|\beta|})\Big\langle$  $\Big\rangle_3/\Delta(n,0)$

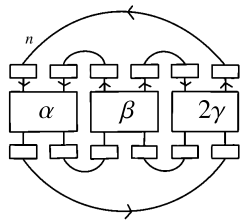
$$\underset{(\text{Theorem 1.2.5})}{=} \sum_{0\le k_{|\alpha|}\le k_{|\alpha|-1}\le\cdots\le k_1\le n} \sum_{0\le l_{|\beta|}\le l_{|\beta|-1}\le\cdots\le l_1\le n} \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n} \sum_{s=\max\{k_{|\alpha|},l_{|\beta|}\}}^{\min\{k_{|\alpha|}+l_{|\beta|},n\}} \sum_{t=\max\{s,m_{|\gamma|}\}}^{\min\{s+m_{|\gamma|},n\}}$$

$$(q^{\frac{n^2+3n}{3}})^{2\alpha+2\beta+2\gamma}\phi(n,k_1,k_2,...,k_{|\alpha|})_{q^{\epsilon_\alpha}}\phi(n,l_1,l_2,...,l_{|\beta|})_{q^{\epsilon_\beta}}\phi(n,m_1,m_2,...,m_{|\gamma|})_{q^{\epsilon_\gamma}}$$

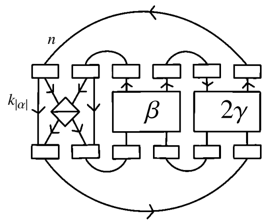
$\times\psi(n,s,k_{|\alpha|},l_{|\beta|})\psi(n,t,s,m_{|\gamma|})\Big\langle$  $\Big\rangle_3/\Delta(n,0).$

Then, we show (2.2.2).

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow\alpha\downarrow,\uparrow\beta\uparrow,\downarrow 2\gamma\uparrow);q)$$

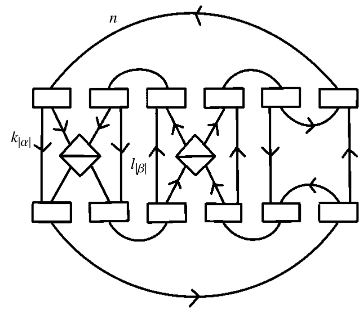
$\underset{(\text{Definition 2.1.3})}{=} (q^{\frac{n^2+3n}{3}})^{-(\alpha+\beta-2\gamma)}\Big\langle$  $\Big\rangle_3/\Delta(n,0)$

$$\underset{(\text{Proposition 2.1.1})}{=} (q^{\frac{n^2+3n}{3}})^{-(\alpha+\beta-2\gamma)} \sum_{0\le k_{|\alpha|}\le k_{|\alpha|-1}\le\cdots\le k_1\le n} \chi_{sign(\alpha)}(n,k_1,k_2,...,k_{|\alpha|})$$

$$\times\Big\langle$$  $$\Big\rangle_3/\Delta(n,0)$$

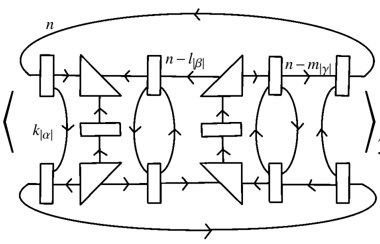
$$\underset{(\text{Proposition 2.1.1,Theorem 1.2.4})}{=} (q^{\frac{n^2+3n}{3}})^{-(\alpha+\beta-2\gamma)} \sum_{0\le k_{|\alpha|}\le k_{|\alpha|-1}\le\cdots\le k_1\le n} \sum_{0\le l_{|\beta|}\le l_{|\beta|-1}\le\cdots\le l_1\le n} \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n}$$

$$\times\chi_{sign(\alpha)}(n,k_1,k_2,...,k_{|\alpha|})\chi_{sign(\beta)}(n,l_1,l_2,...,l_{|\beta|})\phi(n,m_1,m_2,...,m_{|\gamma|})_{q^{\epsilon\gamma}}$$

$$\times\Big\langle$$  $$\Big\rangle_3/\Delta(n,0)$$

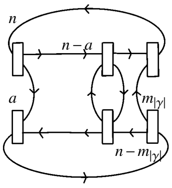
$$\underset{((1.1.6))}{=} (q^{\frac{n^2+3n}{3}})^{-(\alpha+\beta-2\gamma)} \sum_{0\le k_{|\alpha|}\le k_{|\alpha|-1}\le\cdots\le k_1\le n} \sum_{0\le l_{|\beta|}\le l_{|\beta|-1}\le\cdots\le l_1\le n} \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n}$$

$$\times\chi_{sign(\alpha)}(n,k_1,k_2,...,k_{|\alpha|})\chi_{sign(\beta)}(n,l_1,l_2,...,l_{|\beta|})\phi(n,m_1,m_2,...,m_{|\gamma|})_{q^{\epsilon\gamma}}$$

$$\times\Big\langle$$  $$\Big\rangle_3/\Delta(n,0)$$

$$\underset{(\text{Propositon 2.1.6})}{=} (q^{\frac{n^2+3n}{3}})^{-(\alpha+\beta-2\gamma)} \sum_{0\le k_{|\alpha|}\le k_{|\alpha|-1}\le\cdots\le k_1\le n} \sum_{0\le l_{|\beta|}\le l_{|\beta|-1}\le\cdots\le l_1\le n} \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n}$$

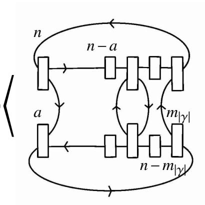
$$\sum_{s=\max\{k_{|\alpha|},l_{|\beta|}\}}^{\min\{k_{|\alpha|}+l_{|\beta|},n\}} \sum_{a=s}^{n} \chi_{sign(\alpha)}(n,k_1,k_2,...,k_{|\alpha|})\chi_{sign(\beta)}(n,l_1,l_2,...,l_{|\beta|})$$

$$\phi(n,m_1,m_2,...,m_{|\gamma|})_{q^{\epsilon_\gamma}}\Big\langle$$ 

$$\Big\rangle_3/\Delta(n,0)$$

$$\underset{((1.1.6))}{=} (q^{\frac{n^2+3n}{3}})^{-(\alpha+\beta-2\gamma)} \sum_{0\le k_{|\alpha|}\le k_{|\alpha|-1}\le\cdots\le k_1\le n}\ \sum_{0\le l_{|\beta|}\le l_{|\beta|-1}\le\cdots\le l_1\le n}\ \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n}$$

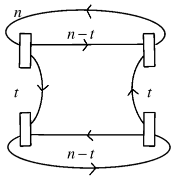
$$\sum_{s=\max\{k_{|\alpha|},l_{|\beta|}\}}^{\min\{k_{|\alpha|}+l_{|\beta|},n\}}\ \sum_{a=s}^{n}\chi_{sign(\alpha)}(n,k_1,k_2,...,k_{|\alpha|})\chi_{sign(\beta)}(n,l_1,l_2,...,l_{|\beta|})$$

$$\times\phi(n,m_1,m_2,...,m_{|\gamma|})_{q^{\epsilon_\gamma}}\Omega(n,s,k_{|\alpha},l_{|\beta|})\Big\langle$$ 

$$\Big\rangle_3/\Delta(n,0)$$

$$\underset{(\text{Theorem 1.2.5})}{=} (q^{\frac{n^2+3n}{3}})^{-(\alpha+\beta-2\gamma)} \sum_{0\le k_{|\alpha|}\le k_{|\alpha|-1}\le\cdots\le k_1\le n}\ \sum_{0\le l_{|\beta|}\le l_{|\beta|-1}\le\cdots\le l_1\le n}\ \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n}$$

$$\sum_{s=\max\{k_{|\alpha|},l_{|\beta|}\}}^{\min\{k_{|\alpha|}+l_{|\beta|},n\}}\ \sum_{a=s}^{n}\ \sum_{t=\max\{s,m_\gamma\}}^{\min\{s+m_\gamma,n\}}\chi_{sign(\alpha)}(n,k_1,k_2,...,k_{|\alpha|})\chi_{sign(\beta)}(n,l_1,l_2,...,l_{|\beta|})$$

$$\times\phi(n,m_1,m_2,...,m_{|\gamma|})_{q^{\epsilon_{|\gamma|}}}\Omega(n,s,k_{|\alpha|},l_{|\beta|})\psi(n,a,m_{|\gamma|},t)\Big\langle$$ 

$$\Big\rangle_3/\Delta(n,0)$$

$$\underset{((1.1.12))}{=} \sum_{0\le k_{|\alpha|}\le k_{|\alpha|-1}\le\cdots\le k_1\le n}\ \sum_{0\le l_{|\beta|}\le l_{|\beta|-1}\le\cdots\le l_1\le n}\ \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n}\ \sum_{s=\max\{k_{|\alpha|},l_{|\beta|}\}}^{\min\{k_{|\alpha|}+l_{|\beta|},n\}}$$

$$\sum_{a=s}^{n}\ \sum_{t=\max\{a,m_{|\gamma|}\}}^{\min\{a+t+m_{|\gamma|},n\}}(q^{\frac{n^2+3n}{3}})^{-(\alpha+\beta-2\gamma)}\chi_{sign(\alpha)}(n,k_1,k_2,...,k_{|\alpha|})\chi_{sign(\beta)}(n,l_1,l_2,...,l_{|\beta|})$$

$$\times\phi(n,m_1,m_2,...,m_{|\gamma|})_{q^{\epsilon_\gamma}}\Omega(n,s,k_{|\alpha|},l_{|\beta|})\psi(n,t,a,m_{|\gamma|})q^{-(n-t)}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})}.$$

□

Remark 2.2.1. For n=1, 2, ..., 10, Theorem 2.1.1 is equal to Theorem 5.7 in [Yua17] for

$4_1, 6_2$ and 8_2 by using Mathematica. For example,

$$J^{\mathfrak{sl}_3}_{(1,0)}(4_1;q) = q^3 - q + 1 - q^{-1} + q^{-3},$$
$$J^{\mathfrak{sl}_3}_{(1,0)}(4_1;q) = q^8 - q^6 - q^5 + q^4 + q^3 - 2q^2 + 3 - 2q^{-2} + q^{-3} + q^{-4} - q^{-5} - q^{-6} + q^{-8},$$
$$J^{\mathfrak{sl}_3}_{(3,0)}(4_1;q) = q^{15} - q^{13} - q^{12} - q^{11} + 2q^{10} + q^9 - 2q^7 - q^6 + 4q^5 + 2q^4 - 2q^3 - 4q^2 + 5 - 4q^{-2}$$
$$- 2q^{-3} + 2q^{-4} + 4q^{-5} - q^{-6} - 2q^{-7} + q^{-9} + 2q^{-10} - q^{-11} - q^{-12} - q^{-13} + q^{-15},$$
$$\vdots$$
$$J^{\mathfrak{sl}_3}_{(10,0)}(4_1;q) = q^{120} - q^{118} - q^{117} - q^{116} + q^{113} + q^{112} + q^{111} + q^{110} + q^{109} + 2q^{108} + q^{107} - q^{106}$$
$$- 3q^{105} - 4q^{104} - 3q^{103} - 3q^{102} - q^{101} + q^{99} + 2q^{98} + 5q^{97} + 8q^{96} + 6q^{95} + 4q^{94} - 2q^{93}$$
$$- 5q^{92} - 7q^{91} - 8q^{90} - 6q^{89} - 5q^{88} - 5q^{87} + q^{86} + 8q^{85} + 14q^{84} + 15q^{83} + 11q^{82} + 4q^{81}$$
$$- 5q^{80} - 10q^{79} - 14q^{78} - 14q^{77} - 20q^{76} - 15q^{75} - 2q^{74} + 11q^{73} + 24q^{72} + 27q^{71} + 25q^{70}$$
$$+ 15q^{69} + 2q^{68} - 9q^{67} - 20q^{66} - 34q^{65} - 38q^{64} - 27q^{63} - 7q^{62} + 15q^{61} + 35q^{60} + 43q^{59}$$
$$+ 42q^{58} + 30q^{57} + 11q^{56} - 8q^{55} - 41q^{54} - 58q^{53} - 60q^{52} - 41q^{51} - 11q^{50} + 21q^{49} + 51q^{48}$$
$$+ 64q^{47} + 65q^{46} + 48q^{45} + 20q^{44} - 27q^{43} - 66q^{42} - 84q^{41} - 79q^{40} - 50q^{39} - 7q^{38} + 37q^{37}$$
$$+ 75q^{36} + 92q^{35} + 88q^{34} + 62q^{33} + 3q^{32} - 52q^{31} - 95q^{30} - 107q^{29} - 91q^{28} - 49q^{27} + 7q^{26}$$
$$+ 61q^{25} + 104q^{24} + 116q^{23} + 103q^{22} + 44q^{21} - 23q^{20} - 82q^{19} - 120q^{18} - 121q^{17} - 91q^{16}$$
$$- 35q^{15} + 31q^{14} + 90q^{13} + 129q^{12} + 133q^{11} + 86q^{10} + 19q^9 - 52q^8 - 106q^7 - 134q^6$$
$$- 120q^5 - 76q^4 - 10q^3 + 61q^2 + 116q + 147 + 116q^{-1} + 61q^{-2} - 10q^{-3} - 76q^{-4} - 120q^{-5}$$
$$- 134q^{-6} - 106q^{-7} - 52q^{-8} + 19q^{-9} + 86q^{-10} + 133q^{-11} + 129q^{-12} + 90q^{-13} + 31q^{-14}$$
$$- 35q^{-15} - 91q^{-16} - q^{-17} - q^{-18} - 82q^{-19} - 23q^{-20} + 44q^{-21} + 103q^{-22} + 116q^{-23}$$
$$+ 104q^{-24} + 61q^{-25} + 7q^{-26} - 49q^{-27} - 91q^{-28} - 107q^{-29} - 95q^{-30} - 52q^{-31} + 3q^{-32}$$
$$+ 62q^{-33} + 88q^{-34} + 92q^{-35} + 75q^{-36} + 37q^{-37} - 7q^{-38} - 50q^{39} - 79q^{-40} - 84q^{-41}$$
$$- 66q^{-42} - 27q^{-43} + 20q^{-44} + 48q^{-45} + 65q^{-46} + 64q^{-47} + 51q^{-48}21q^{-49} - 11q^{-50}$$
$$- 41q^{-51} - 60q^{-52} - 58q^{-53} - 41q^{-54} - 8q^{-55} + 11q^{-56} + 30q^{-57} + 42q^{-58} + 43q^{-59}$$
$$+ 35q^{-60} + 15q^{-61} - 7q^{-62} - 27q^{-63} - 38q^{-64} - 34q^{-65} - 20q^{-66} - 9q^{67} + 2q^{-68}$$
$$+ 15q^{-69} + 25q^{-70} + 27q^{-71} + 24q^{-72} + 11q^{-73} - 2q^{-74} - 15q^{-75} - 20q^{-76} - 14q^{-77}$$
$$- 14q^{-78} - 10q^{-79} - 5q^{-80} + 4q^{-81} + 11q^{-82} + 15q^{-83} + 14q^{-84} + 8q^{-85} + 1q^{-86} - 5q^{-87}$$
$$- 5q^{-88} - 6q^{-89} - 8q^{-90} - 7q^{-91} - 5q^{-92} - 2q^{-93} + 4q^{-94} + 6q^{-95} + 8q^{-96} + 5q^{-97}$$
$$+ -2q^{-98} + 1q^{-99} - q^{-101} - 3q^{-102} - 3q^{-103} - 4q^{-104} - 3q^{-105} - q^{-106} + q^{-107} + 2q^{-108}$$
$$+ q^{-109} + q^{-110} + q^{-111} + q^{-112} + q^{-113} - q^{-116} - q^{-117} - q^{-118} + q^{-120}.$$

2.3 The knots 8_{10}, 8_{15}, 8_{20} and 8_{21}

The knots $8_{10}, 8_{15}, 8_{20}$ and 8_{21} are four or five parameters pretzel knots. For these knots, we can calculate the one-row colored $\mathfrak{sl}_3$ Jones polynomials by combining a similar method as three-parameter pretzel knots and links.

Theorem 2.3.1 ([Kaw]). The one-row colored $\mathfrak{sl}_3$ Jones polynomials for $P(-3,-2,3,-1) = 8_{10}$, $P(3,-1,-2,-1,3) = 8_{15}$, $P(3,-2,-3,1) = 8_{20}$ and $P(3,3,-1,2) = 8_{21}$ are the following:

(2.3.1)

$$
\begin{aligned}
&J^{\mathfrak{sl}_3}_{(n,0)}(P(-3,-2,3,-1);q)\\
&=\sum_{0\le k_3\le k_2\le k_1\le n}\ \sum_{0\le l_2\le l_1\le n}\ \sum_{0\le p_3\le p_2\le p_1\le n}\ \sum_{m=0}^{n}\ \sum_{t=\max\{k_3,l_2\}}^{\min\{k_3+l_2,n\}}\ \sum_{a=t}^{n}\\
&\quad \sum_{s=\max\{p_3,m\}}^{\min\{p_3+m,n\}}\ \sum_{b=s}^{n}\ \sum_{u=\max\{a,b\}}^{\min\{a+b,n\}} (q^{\frac{n^2+3n}{3}})^3 \chi_{-1}(n,k_0,k_1,k_2)\chi_{-1}(n,l_0,l_1)_{q^{-1}}\chi_1(n,p_1,p_2,p_3)\\
&\quad \times\chi_{-1}(n,m)\Omega(n,t,k_3,l_2)\Omega(n,s,p_3,m)\psi(n,u,a,b)q^{-(n-u)}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{u+1})(1-q^{u+2})},
\end{aligned}
$$

(2.3.2)

$$
\begin{aligned}
&J^{\mathfrak{sl}_3}_{(n,0)}(P(3,-1,-2,-1,3);q)\\
&=\sum_{0\le k_3\le k_2\le k_1\le n}\ \sum_{0\le l\le n}\ \sum_{0\le m_2\le m_1}\ \sum_{0\le p\le n}\ \sum_{0\le r_3\le r_2\le r_1\le n}\ \sum_{t=\max\{k_3,l\}}^{\min\{k_3+l,n\}}\ \sum_{a=t}^{n}\ \sum_{s=\max\{p,r_3\}}^{\min\{p+r_3,n\}}\ \sum_{b=s}^{n}\\
&\quad \sum_{u=\max\{a,m_1\}}^{\min\{a+m_2,n\}}\ \sum_{v=\max\{b,m_2\}}^{\min\{b+m_2,n\}} (q^{\frac{n^2+3n}{3}})^{-2}\chi_1(n,k_1,k_2,k_3)\chi_{-1}(n,l)\phi(n,m_1,m_2)\chi_{-1}(n,p)\\
&\quad \times\chi_1(n,r_1,r_2,r_3)\Omega(n,t,k_3,l)\Omega(n,s,p,r_3)\psi(n,u,a,m_2)\psi(n,v,u,b)q^{-(n-v)}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{v+1})(1-q^{v+2})},
\end{aligned}
$$

(2.3.3)

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(3,-2,-3,1);q),$$

$$= \sum_{0\le k_3\le k_2\le k_1\le n} \sum_{0\le l_2\le l_1\le n} \sum_{0\le p_3\le p_2\le p_1\le n} \sum_{m=0}^{n} \sum_{t=\max\{k_3,l_2\}}^{\min\{k_3+l_2,n\}} \sum_{a=t}^{n} \sum_{s=\max\{p_3,m\}}^{\min\{p_3+m,n\}}$$

$$\sum_{b=s}^{n} \sum_{u=\max\{a,b\}}^{\min\{a+b,n\}} q^{\frac{n^2+3n}{3}} \chi_1(n,k_1,k_2,k_3)\chi_{-1}(n,l_1,l_2)_{q^{-1}}\chi_{-1}(n,p_1,p_2,p_3)$$

$$\times\chi_1(n,m)\Omega(n,t,n-k_3,l_2)\Omega(n,s,n-p_2,n-m)\psi(n,u,t+a,s+b)q^{-(n-u)}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{u+1})(1-q^{u+2})},$$

(2.3.4)

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(3,3,-1,-2);q),$$

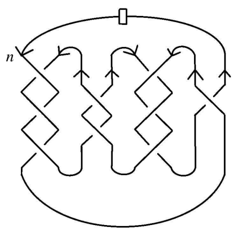
$$= \sum_{0\le k_3\le k_2\le k_1\le n} \sum_{0\le l_3\le l_2\le l_1\le n} \sum_{0\le p\le n} \sum_{0\le m_2\le m_1\le n} \sum_{t=\max\{k_3,l_3\}}^{\min\{k_3+l_3,n\}} \sum_{a=t}^{n} \sum_{s=\max\{p,m\}}^{\min\{p+m,n\}} \sum_{b=s}^{n}$$

$$\sum_{u=\max\{a,b\}}^{\min\{t+s,n\}} (q^{\frac{n^2+3n}{3}})^{-3}\chi_1(n,k_1,k_2,k_3)\chi_1(n,l_1,l_2,l_3)\chi_{-1}(n,p)\chi_{-1}(n,m_1,m_2)$$

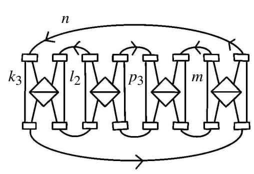
$$\times\Omega(n,t,k_3,l_3)\Omega(n,s,p,m_2)\psi(n,u,a,b)q^{-(n-u)}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{u+1})(1-q^{u+2})}.$$

Proof. We first prove (2.3.1).

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(-3,-2,3,-1);q)$$

$$=(q^{\frac{n^2+3n}{3}})^3\Big\langle$$  $$\Big\rangle_3/\Delta(n,0)$$

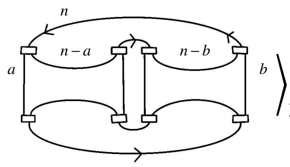
$$\underset{(\text{Proposition 2.1.1})}{=} \sum_{0\le k_3\le k_2\le k_1\le n} \sum_{0\le l_2\le l_1\le n} \sum_{0\le p_3\le p_2\le p_1\le n} \sum_{m=0}^{n} (q^{\frac{n^2+3n}{3}})^3\chi_{-1}(n,k_1,k_2,k_3)\chi_{-1}(n,l_1,l_2)_{q^{-1}}$$

$$\chi_1(n,p_1,p_2,p_3)\chi_{-1}(n,m)\Big\langle$$  $$\Big\rangle_3/\Delta(n,0)$$

$$\underset{\text{(Proposition 2.1.6)}}{=} \sum_{0\le k_3\le k_2\le k_1\le n} \sum_{0\le l_2\le l_1\le n} \sum_{0\le p_3\le p_2\le p_1\le n} \sum_{m=0}^{n} \sum_{t=\max\{k_3,l_2\}}^{\min\{k_3+l_2,n\}} \sum_{a=t}^{n}$$

$$\sum_{s=\max\{p_3,m\}}^{\min\{p_3+m,n\}} \sum_{b=s}^{n} (q^{\frac{n^2+3n}{3}})^3 \chi_{-1}(n,k_0,k_1,k_2)\chi_{-1}(n,l_0,l_1)_{q^{-1}}\chi_1(n,p_1,p_2,p_3)$$

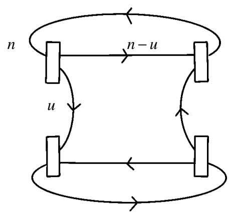
$$\times\chi_{-1}(n,m)\Omega(n,t,k_3,l_2)\Omega(n,s,p_3,m)\Big\langle \quad \Big\rangle_3$$



$$\underset{\text{(Theorem 1.2.5)}}{=} \sum_{0\le k_3\le k_2\le k_1\le n} \sum_{0\le l_2\le l_1\le n} \sum_{0\le p_3\le p_2\le p_1\le n} \sum_{m=0}^{n} \sum_{t=\max\{k_3,l_2\}}^{\min\{k_3+l_2,n\}} \sum_{a=t}^{n}$$

$$\sum_{s=\max\{p_3,m\}}^{\min\{p_3+m,n\}} \sum_{b=s}^{n} \sum_{u=\max\{a,b\}}^{\min\{a+b,n\}} (q^{\frac{n^2+3n}{3}})^3 \chi_{-1}(n,k_0,k_1,k_2)\chi_{-1}(n,l_0,l_1)_{q^{-1}}\chi_1(n,p_1,p_2,p_3)$$

$$\times\chi_{-1}(n,m)\Omega(n,t,k_3,l_2)\Omega(n,s,p_3,m)\psi(n,u,a,b)\Big\langle \quad \Big\rangle_3$$



$$\underset{\text{((1.1.12))}}{=} \sum_{0\le k_3\le k_2\le k_1\le n} \sum_{0\le l_2\le l_1\le n} \sum_{0\le p_3\le p_2\le p_1\le n} \sum_{m=0}^{n} \sum_{t=\max\{k_3,l_2\}}^{\min\{k_3+l_2,n\}} \sum_{a=t}^{n}$$

$$\sum_{s=\max\{p_3,m\}}^{\min\{p_3+m,n\}} \sum_{b=s}^{n} \sum_{u=\max\{a,b\}}^{\min\{a+b,n\}} (q^{\frac{n^2+3n}{3}})^3 \chi_{-1}(n,k_0,k_1,k_2)\chi_{-1}(n,l_0,l_1)_{q^{-1}}\chi_1(n,p_1,p_2,p_3)$$

$$\times\chi_{-1}(n,m)\Omega(n,t,k_3,l_2)\Omega(n,s,p_3,m)\psi(n,u,a,b)q^{-(n-u)}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{u+1})(1-q^{u+2})}.$$

We can prove (2.3.2), (2.3.3) and (2.3.4) in a similar way to (2.3.1). □

Remark 2.3.2. According to Main Theorem 1, Main Theorem 2, and Theorem 5.7 in [Yua17], the one-row colored $\mathfrak{sl}_3$ Jones polynomial is determined for all knots with eight or fewer crossings except $8_{16}, 8_{17}, 8_{18}$.

Chapter 3

The tails of the one-row colored $\mathfrak{sl}_3$ Jones polynomials for pretzel knots and links

In this chapter, we first compute the minimum degree of the one-row colored $\mathfrak{sl}_3$ Jones polynomials for $P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow)$ and $P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow)$. These give a normalization of the one-row colored $\mathfrak{sl}_3$ Jones polynomials. Then, we show Main Theorem 2.

3.1 The minimum degree of the one-row colored $\mathfrak{sl}_3$ Jones polynomials for pretzel links

Definition 3.1.1. Let f Laurant polynomial in $Z[q^{\pm}]$. We define *mindeg*(f) to be the minimum degree of f expressed as $Z[[q^{\pm}]]$.

Definition 3.1.2. For any power series $f(q)$, we define $\hat{f}(q)$ by

$$\hat{f}(q) = \pm q^{-\mathrm{mindeg}(f(q))} f(q) = \sum_{i=0}^{\infty} a_i q^i \in \mathbb{Z}[[q]].$$

In the above normalization, we determine a_0 to be positive.

Corollary 3.1.3. For positive integers α, β and γ, we have

$$\mathrm{mindeg}(J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)) = -\frac{1}{2}(\alpha + \beta)n^2 - \frac{1}{2}(3\alpha + 3\beta - 2)n, \tag{3.1.1}$$

$$\mathrm{mindeg}(J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)) = 2n \tag{3.1.2}$$

Proof. First, we show (3.1.1).

$$\begin{aligned}
&\mathrm{mindeg}(J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow\alpha\downarrow,\uparrow\beta\uparrow,\downarrow 2\gamma\uparrow);q))\\
=&\mathrm{mindeg}(\sum_{0\le k_\alpha\le k_{\alpha-1}\le\cdots\le k_1\le n}\ \sum_{0\le l_\beta\le l_{\beta-1}\le\cdots\le l_1\le n}\ \sum_{0\le m_\gamma\le m_{\gamma-1}\le\cdots\le m_1\le n}\ \sum_{s=\max\{k_\alpha,l_\beta\}}^{\min\{k_\alpha+l_\beta,n\}}\ \sum_{a=s}^{n}\\
&\sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n\}}(q^{\frac{n^2+3n}{3}})^{-(\alpha+\beta-2\gamma)}\frac{q^{-\frac{1}{6}(n^2+3n)\alpha}q^{\frac{1}{2}(n-k_\alpha)}q^{\sum_{i=1}^{\alpha}\frac{1}{2}(k_i^2+k_i)}(q)_n}{(\prod_{i=1}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}}\\
&\times\frac{q^{-\frac{1}{6}(n^2+3n)\beta}q^{\frac{1}{2}(n-l_\beta)}q^{\sum_{i=1}^{\beta}\frac{1}{2}(l_i^2+l_i)}(q)_n}{(\prod_{i=1}^{\beta}(q)_{l_{i-1}-l_i})(q)_{k_\beta}}\times\frac{(q)^{-\frac{2\gamma}{3}(n^2+3n)}(q)^{n-m_\gamma}(q)^{\sum_{i=1}^{\gamma}(m_i^2+2m_i)}(q)_n^2}{(\prod_{i=1}^{\gamma}(q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2}\\
&\times\frac{q^{-\frac{k_\alpha+l_\beta}{2}+s}q^{(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(1-q^{n+1-k_\alpha})(1-q^{n+1-l_\beta})(q)_{k_\alpha}(q)_{l_\beta}(q)^2_{n-k_\alpha}(q)^2_{n-l_\beta}(q)_{2n-s+2}}{(1-q^{n+1-s})^2(q)_n^2(q)_{n-s}^2(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{2n-k_\alpha-l_\beta+2}(q)_{-s+k_\alpha+l_\beta}}\\
&\times\frac{q^{(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}(q)^2_{n-a}(q)^2_{n-m_\gamma}(q)_{2n-t+2}}{(q)_n^2(q)_{n-t}^2(q)_{t-a}(q)_{t-m_\gamma}(q)_{2n-a-m_\gamma+2}(q)_{-t+a+m_\gamma}}q^{-(n-t)}\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})})\\
=&\mathrm{mindeg}(q^{-\frac{1}{2}(\alpha+\beta)n^2-\frac{1}{2}(3\alpha+3\beta-2)n}\sum_{0\le k_\alpha\le k_{\alpha-1}\le\cdots\le k_1\le n}\ \sum_{0\le l_\beta\le l_{\beta-1}\le\cdots\le l_1\le n}\ \sum_{0\le m_\gamma\le m_{\gamma-1}\le\cdots\le m_1\le n}\\
&\sum_{s=\max\{k_\alpha,l_\beta\}}^{\min\{k_\alpha+l_\beta,n\}}\ \sum_{a=s}^{n}\ \sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n\}} q^{\frac{1}{2}(-k_\alpha)}q^{\sum_{i=1}^{\alpha}\frac{1}{2}(k_i^2+k_i)}q^{\frac{1}{2}(-l_\beta)}q^{\sum_{i=1}^{\beta}\frac{1}{2}(l_i^2+l_i)}q^{-m_\gamma}q^{\sum_{i=1}^{\gamma}(m_i^2+2m_i)}\\
&\times q^{-\frac{k_\alpha+l_\beta}{2}+s}q^{(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}q^{(t+1)(t-a-m_\gamma)+am_\gamma}q^{t})\\
=&\mathrm{mindeg}(q^{-\frac{1}{2}(\alpha+\beta)n^2-\frac{1}{2}(3\alpha+3\beta-2)n}\sum_{0\le k_\alpha\le k_{\alpha-1}\le\cdots\le k_1\le n}\ \sum_{0\le l_\beta\le l_{\beta-1}\le\cdots\le l_1\le n}\ \sum_{0\le m_\gamma\le m_{\gamma-1}\le\cdots\le m_1\le n}\\
&\sum_{s=\max\{k_\alpha,l_\beta\}}^{\min\{k_\alpha+l_\beta,n\}}\ \sum_{a=s}^{n}\ \sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n\}} q^{\sum_{i=1}^{\alpha}\frac{1}{2}(k_i^2+k_i)}q^{\sum_{i=1}^{\beta}\frac{1}{2}(l_{i-1}^2+l_i)}q^{m_\gamma^2+m_\gamma}q^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)}q^{\frac{k_\alpha^2+l_\beta^2-(k_\alpha+l_\beta)}{2}}\\
&\times q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}q^{t+(t+1)(t-a-m_\gamma)+am_\gamma})\\
=&-\frac{1}{2}(\alpha+\beta)n^2-\frac{1}{2}(3\alpha+3\beta-2)n
\end{aligned}$$

We use $\frac{1}{1-q} = 1 + q + q^2 + \cdots$ in the second equality. If $l_\beta \le k_\alpha \le s$, then

$$\begin{aligned}(3.1.3)\qquad & s + (s+1)(s-(k_\alpha + l_\beta)) + k_\alpha l_\beta \\ =& s^2 + s(2-(k_\alpha + l_\beta) - k_\alpha - l_\beta + k_\alpha l_\beta \\ =& (s-k_\alpha+1)(s-l_\beta+1) - 1 \\ \ge& (k_\alpha - l_\beta + 1) - 1 \ge 0\end{aligned}$$

The same is true for $k_\alpha \le l_\beta \le s$. In the above last inequality, we have equality if and only if $l_\beta = k_\alpha = s$ holds. By the same method as the above, if $a \le m_\gamma \le t$ and $m_\gamma \le a \le t$, then we can obtain

$$(3.1.4)\qquad t + (t+1)(t-(a+m_\gamma)) + am_\gamma \ge 0.$$

The equality sign is valid for $a = m_\gamma = t$ and $m_\gamma = a = t$. By (3.1.3), (3.1.4), $k_\alpha^2 - k_\alpha \ge 0$ and $l_\beta^2 - l_\beta \ge 0$, the fourth equality holds. Then, we can show (3.1.2) in a similar way.

□

3.2 Proof of Main Theorem 2

Recall Main Theorem 2.

Main Theorem 2 . Let α, β, and γ be positive integers. For oriented pretzel links $P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow)$, there exists $\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)$ in $\mathbb{Z}[[q]]$ such that

$$\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q) - \hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)) \in q^{n+1}\mathbb{Z}[[q]].$$

Moreover, for oriented pretzel knots $P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow)$, there exists $\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)$ in $\mathbb{Z}[[q]]$ such that

$$\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q) - \hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha \downarrow, \uparrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)) \in q^{n+1}\mathbb{Z}[[q]].$$

Remark 3.2.1. For a simple Lie algebra, $\mathfrak{g}$, Le[Le00] proved the integrality theorem for a quantum $\mathfrak{g}$ invariant. This says $\hat{J}^{\mathfrak{sl}_3}_{(n,0)}(L; q)$ belongs to $\mathbb{Z}[[q]]$.

Definition 3.2.2. For two power series $\hat{f}(q)$, $\hat{g}(q) \in \mathbb{Z}[[q]]$, we define $\hat{f}(q) \equiv_{n+1} \hat{g}(q)$ by $\hat{f}(q) = \hat{g}(q)$ in $\mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]]$ for all n.

Proof of Main Theorem 2. We first proveMain Theorem 2 for $P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow)$. There exists $\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)$ if and only if

$$(3.2.1)\qquad \hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)) \equiv_{n+1} \hat{J}^{\mathfrak{sl}_3}_{(n+1,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q))$$

for all n. Let α and β both be odd or even. To begin with, we obtain the following by Main Theorem 1 and Corollary 3.1.3:

(3.2.2)

$$
\begin{aligned}
&\hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow\alpha\downarrow,\uparrow\beta\uparrow,\downarrow 2\gamma\uparrow);q))\\
=&q^{\frac{1}{2}(\alpha+\beta)n^2+\frac{1}{2}(3\alpha+3\beta-2)n}J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow\alpha\downarrow,\uparrow\beta\uparrow,\downarrow 2\gamma\uparrow);q))\\
=&\sum_{0\leq k_\alpha\leq k_{\alpha-1}\leq\cdots\leq k_1\leq n}\ \sum_{0\leq l_\beta\leq l_{\beta-1}\leq\cdots\leq l_1\leq n}\ \sum_{0\leq m_\gamma\leq m_{\gamma-1}\leq\cdots\leq m_1\leq n}\ \sum_{s=\max\{k_\alpha,l_\beta\}}^{\min\{k_\alpha+l_\beta,n\}}\ \sum_{a=s}^{n}\ \sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n\}}\\
&\times\frac{(-1)^{\sum_{i=1}^{\alpha}k_i}q^{\frac{1}{2}k_\alpha^2}q^{\sum_{i=1}^{\alpha-1}\frac{1}{2}(k_i^2+k_i)}(q)_n}{(\prod_{i=1}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}}\times\frac{(-1)^{\sum_{i=1}^{\beta}l_i}q^{\frac{1}{2}l_\beta^2}q^{\sum_{i=1}^{\beta-1}\frac{1}{2}(l_i^2+l_i)}(q)_n}{(\prod_{i=1}^{\beta}(q)_{l_{i-1}-l_i})(q)_{k_\beta}}\\
&\times\frac{(q)^{m_\gamma^2+m_\gamma}(q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)}(q)_n^2}{(\prod_{i=1}^{\gamma}(q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2}\\
&\times\frac{q^{-\frac{k_\alpha+l_\beta}{2}+s}q^{(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(1-q^{n+1-k_\alpha})(1-q^{n+1-l_\beta})(q)_{k_\alpha}(q)_{l_\beta}(q)_{n-k_\alpha}^2(q)_{n-l_\beta}^2(q)_{2n-s+2}}{(1-q^{n+1-s})^2(q)_n^2(q)_{n-s}^2(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{2n-k_\alpha-l_\beta+2}(q)_{-s+k_\alpha+l_\beta}}\\
&\times\frac{q^{(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}(q)_{n-a}^2(q)_{n-m_\gamma}^2(q)_{2n-t+2}}{(q)_n^2(q)_{n-t}^2(q)_{t-a}(q)_{t-m_\gamma}(q)_{2n-a-m_\gamma+2}(q)_{-t+a+m_\gamma}}q^t\frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})}\\
=&\sum_{0\leq k_\alpha\leq k_{\alpha-1}\leq\cdots\leq k_1\leq n}\ \sum_{0\leq l_{2\beta+1}\leq l_{2\beta}\leq\cdots\leq l_1\leq n}\ \sum_{0\leq m_\gamma\leq m_{\gamma-1}\leq\cdots\leq m_1\leq n}\ \sum_{s=\max\{k_\alpha,l_\beta\}}^{\min\{k_\alpha+l_\beta,n\}}\ \sum_{a=s}^{n}\ \sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n\}}\\
&\times\frac{(-1)^{\sum_{i=1}^{\alpha}k_i}q^{\sum_{i=1}^{\alpha-1}\frac{1}{2}(k_i^2+k_i)}(q)_n}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{n-k_1}(q)_{k_\alpha}}\times\frac{(-1)^{\sum_{i=1}^{\beta}l_i}q^{\sum_{i=1}^{\beta-1}\frac{1}{2}(l_i^2+l_i)}(q)_n}{(\prod_{i=2}^{\beta}(q)_{l_{i-1}-l_i})(q)_{n-l_1}(q)_{k_\beta}}\\
&\times\frac{(q)^{m_\gamma^2+m_\gamma}(q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)}(q)_n^2}{(\prod_{i=2}^{\gamma}(q)_{m_{i-1}-m_i})(q)_{n-m_1}(q)_{m_\gamma}^2}\\
&\times\frac{q^{\frac{(k_\alpha^2-k_\alpha)+(l_\beta^2-l_\beta)}{2}}q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(1-q^{n+1-k_\alpha})(1-q^{n+1-l_\beta})}{(1-q^{n+1-s})^2}
\end{aligned}
$$

$$
\begin{aligned}
(3.2.3)\qquad &\times \frac{(q)_{k_\alpha}(q)_{l_\beta}(q)^2_{n-k_\alpha}(q)^2_{n-l_\beta}(q)_{2n-s+2}}{(q)^2_n(q)^2_{n-s}(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{2n-k_\alpha-l_\beta+2}(q)_{-s+k_\alpha+l_\beta}} \\
&\times \frac{q^{t+(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}(q)^2_{n-a}(q)^2_{n-m_\gamma}(q)_{2n-t+2}}{(q)^2_n(q)^2_{n-t}(q)_{t-a}(q)_{t-m_\gamma}(q)_{2n-a-m_\gamma+2}(q)_{-t+a+m_\gamma}} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})}
\end{aligned}
$$

where $q^{\sum_{i=1}^{0}\frac{1}{2}(k_i^2+k_i)} = 1$, $q^{\sum_{i=1}^{0}\frac{1}{2}(l_i^2+l_i)} = 1$, and $q^{\sum_{i=1}^{0}(m_i^2+2m_i)} = 1$. Now,

$$
\begin{aligned}
(1-q^{n+1})(1-q^{n+2}) &= 1-q^{n+2}-q^{n+1}+q^{2n+3} \\
&\equiv_{n+1} 1.
\end{aligned}
$$

In $s-k_\alpha \geq 0$ and $s-l_\beta \geq 0$,

$$
\begin{aligned}
q^{s^2+s}(1-q^{n+1-k_\alpha})(1-q^{n+1-l_\beta}) &= q^{s^2+s}-q^{n+1+s^2+s-l_\beta}-q^{n+1+s^2+s-k_\alpha}+q^{2n+2+(s^2-k_\alpha)+(s-l_\beta)} \\
&\equiv_{n+1} q^{s^2+s}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{q^s}{(1-q^{n+1-s})} &= q^s(1+q^{n+1-s}+q^{2(n+1-s)}+\cdots) \\
&= q^s+q^{n+1}+\text{(higher order terms)} \\
&\equiv_{n+1} q^s.
\end{aligned}
$$

By $k_1 > 0$ and $\frac{1}{2}(k_1^2-k_1) \geq 0$, we have

$$
\begin{aligned}
q^{\frac{1}{2}(k_1^2+k_1)}\frac{(q)_n}{(q)_{n-k_1}} &= q^{\frac{1}{2}(k_1^2+k_1)}\frac{\prod_{i=1}^{n}(1-q^i)}{\prod_{j=1}^{n-k_1}(1-q^j)} \\
(3.2.4)\qquad &= q^{\frac{1}{2}(k_1^2+k_1)}\prod_{i=n-k_1+1}^{n}(1-q^i) \\
&= \prod_{i=0}^{k_1-1}(q^{\frac{1}{2}(k_1^2+k_1)}-q^{i+n+1+\frac{1}{2}(k_1^2-k_1)}) \\
&\equiv_{n+1} q^{\frac{1}{2}(k_1^2+k_1)}.
\end{aligned}
$$

If $k_1 = 0$, then (3.2.4) is obvious. Similarly,

$$
q^{\frac{1}{2}(l_1^2+l_1)}\frac{(q)_n}{(q)_{n-l_1}} \equiv_{n+1} q^{\frac{1}{2}(l_1^2+l_1)} \quad (\frac{l_1^2-l_1}{2} \geq 0),
$$

$$q^{m_1}\frac{(q)_n}{(q)_{n-m_1}} \equiv_{n+1} q^{m_1},$$

$$q^{t^2+t}\frac{(q)_{2n-t+2}}{(q)_{2n-a-m_\gamma+2}} \equiv_{n+1} q^{t^2+t} \quad (a+m_\gamma \geq t, t-a \geq 0, t^2-m_\gamma \geq 0),$$

and

$$q^{s^2+s}\frac{(q)_{2n-s+2}}{(q)_{2n-k_\alpha-l_\beta+2}} \equiv_{n+1} q^{s^2+s} \quad (k_\alpha+l_\beta \geq s, s-k_\alpha \geq 0, s^2-l_\beta \geq 0).$$

Using $k_\alpha > 0$ and $s-k_\alpha \geq 0$, the fact that

$$\begin{aligned} q^s\frac{(q)_{n-k_\alpha}}{(q)_n} &= q^s\frac{\prod_{i=1}^{n-k_\alpha}(1-q^i)}{\prod_{j=1}^{n}(1-q^j)} \\ &= \frac{q^s}{\prod_{j=n-k_\alpha+1}^{n}(1-q^j)} \\ &= \frac{q^s}{\prod_{j=0}^{k_\alpha-1}(1-q^{j+n+1-k_{2\alpha+1}})} \\ &= q^s(1+q^{n+1-k_\alpha}+(\text{higher order terms})) \\ &\equiv_{n+1} q^s. \end{aligned} \tag{3.2.5}$$

In the case of $k_\alpha = 0$, we can easily conform (3.2.5). Likewise,

$$q^s\frac{(q)_{n-l_\beta}}{(q)_{n-s}} \equiv_{n+1} q^s \quad (s-l_\beta \geq 0),$$

$$q^t\frac{(q)_{n-a}}{(q)_n} \equiv_{n+1} q^t \quad (t-a \geq 0),$$

$$q^t\frac{(q)_{n-m_\gamma}}{(q)_{n-a}} \equiv_{n+1} q^t \quad (t-m_\gamma \geq 0).$$

Hence,

(3.2.6)

$$\hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow\alpha\downarrow,\uparrow\beta\uparrow,\downarrow 2\gamma\uparrow);q))$$

$$\equiv_{n+1} \sum_{0\le k_\alpha\le k_{\alpha-1}\le\cdots\le k_1\le n}\ \sum_{0\le l_\beta\le l_{\beta-1}\le\cdots\le l_1\le n}\ \sum_{0\le m_\gamma\le m_{\gamma-1}\le\cdots\le m_1\le n}\ \sum_{s=\max\{k_\alpha,l_\beta\}}^{\min\{k_\alpha+l_\beta,n\}}\ \sum_{a=s}^{n}$$

$$\sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n\}} \frac{(-1)^{\sum_{i=1}^{\alpha}k_i}q^{\sum_{i=1}^{\alpha-1}\frac{1}{2}(k_i^2+k_i)}}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}} \times \frac{(-1)^{\sum_{i=1}^{\beta}l_i}q^{\sum_{i=1}^{\beta-1}\frac{1}{2}(l_i^2+l_i)}}{(\prod_{i=2}^{\beta}(q)_{l_{i-1}-l_i})(q)_{k_\beta}}$$

$$\times \frac{(q)^{m_\gamma^2+m_\gamma}(q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)}(q)_n}{(\prod_{i=2}^{\gamma}(q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2} \frac{q^{\frac{(k_\alpha^2-k_\alpha)+(l_\beta^2-l_\beta)}{2}}q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}$$

$$\times \frac{q^{t+(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})}$$

In (3.2.6), we have

(3.2.7)
$$\begin{aligned}&\frac{1}{2}(k_1^2+k_1+l_1^2+l_1)-(k_\alpha+l_\beta)\\ &=\frac{1}{2}\{(k_1^2-k_\alpha)+(k_1-k_\alpha)+(l_1^2-l_\beta)+(l_1-l_\beta)\}\\ &\ge 0.\end{aligned}$$

If $k_\alpha+l_\beta\ge n+1$, then

$$\frac{1}{2}(k_1^2+k_1+l_1^2+l_1)\ge n+1$$

by (3.2.7). Then,

$$q^{\frac{1}{2}(k_1^2+k_1+l_1^2+l_1)}\Big(\sum_{s=\max\{k_\alpha,l_\beta\}}^{n}\times\Big(\sum_{a=s}^{n}\frac{q^{\frac{(k_\alpha^2-k_\alpha)+(l_\beta^2-l_\beta)}{2}}q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}$$

$$\times\Big(\sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n\}}\frac{q^{t+(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})}\Big)\Big)\Big)$$

$$\equiv_{n+1} q^{\frac{1}{2}(k_1^2+k_1+l_1^2+l_1)}\Big(\sum_{s=\max\{k_\alpha,l_\beta\}}^{n}\times\Big(\sum_{a=s}^{n}\frac{(q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}$$

$$\times\Big(\sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n\}}\frac{(q)_a(q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})}\Big)\Big)\Big)$$
$$= q^{n+1}+(\text{high order terms})$$
$$\equiv_{n+1} 0.$$

In the first congruence, we apply

$$(k_\alpha^2-k_\alpha)\geq 0, \tag{3.2.8}$$
$$(l_\beta^2-l_\beta)\geq 0, \tag{3.2.9}$$
$$s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta\geq 0, \tag{3.2.10}$$
$$t+(t+1)(t-a-m_\gamma)+am_\gamma\geq 0, \tag{3.2.11}$$

and (3.2.7). As regards (3.2.10) and (3.2.11), see the proof of Corollary3.1.3. We use $\frac{1}{1-q}=1+q+q^2+\cdots$ for the second congruence. Thus, we have

(3.2.12)

$$\hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow\alpha\downarrow,\uparrow\beta\uparrow,\downarrow 2\gamma\uparrow);q))$$
$$\equiv_{n+1}\sum_{0\leq k_\alpha\leq k_{\alpha-1}\leq\cdots\leq k_1\leq n}\ \sum_{0\leq l_\beta\leq l_{\beta-1}\leq\cdots\leq l_1\leq n}\ \sum_{0\leq m_\gamma\leq m_{\gamma-1}\leq\cdots\leq m_1\leq n}$$
$$\times\frac{(-1)^{\sum_{i=1}^{\alpha}k_i}q^{\sum_{i=1}^{\alpha-1}\frac{1}{2}(k_i^2+k_i)}}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}}\times\frac{(-1)^{\sum_{i=1}^{\beta}l_i}q^{\sum_{i=1}^{\beta-1}\frac{1}{2}(l_i^2+l_i)}}{(\prod_{i=2}^{\beta}(q)_{l_{i-1}-l_i})(q)_{k_\beta}}\times\frac{(q)^{m_\gamma^2+m_\gamma}(q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)}(q)_n}{(\prod_{i=2}^{\gamma}(q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2}$$
$$\times\Big(\sum_{s=\max\{k_\alpha,l_\beta\}}^{k_\alpha+l_\beta}\times\Big(\sum_{a=s}^{n}q^{\frac{(k_\alpha^2-k_\alpha)+(l_\beta^2-l_\beta)}{2}}\frac{q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}$$
$$\times\Big(\sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n\}}\frac{q^{t+(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})}\Big)\Big)\Big).$$

In (3.2.13), if $a+m_\gamma\geq n+1$, then

$$\begin{aligned}&t+(t+1)(t-a-m_\gamma)+am_\gamma+m_\gamma+m_\gamma^2\\&=(t-a+1)(t-m_\gamma+1)-1+m_\gamma+m_\gamma^2\\&\geq t-m_\gamma+1-1+m_\gamma+m_\gamma^2\\&=t+m_\gamma^2\geq n+1\end{aligned}$$

by using $t \geq \max\{a, m_\gamma\}$ and $m_\gamma^2 \geq m_\gamma$. Thus, we have

$$q^{m_\gamma^2+m_\gamma} \sum_{t=\max\{a,m_\gamma\}}^{n} \frac{q^{t+(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})}$$

$$\sum_{t=\max\{a,m_\gamma\}}^{n} q^{m_\gamma^2+m_\gamma} q^{t+(t+1)(t-a-m_\gamma)+am_\gamma} + (\text{high oredr terms})$$

$$= q^{n+1} + (\text{high order terms})$$

$$\equiv_{n+1} 0.$$

Therefore,

(3.2.13)

$$\hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q))$$

$$\equiv_{n+1} \sum_{0 \leq k_\alpha \leq k_{\alpha-1} \leq \cdots \leq k_1 \leq n} \sum_{0 \leq l_\beta \leq l_{\beta-1} \leq \cdots \leq l_1 \leq n} \sum_{0 \leq m_\gamma \leq m_{\gamma-1} \leq \cdots \leq m_1 \leq n}$$

$$\times \frac{(-1)^{\sum_{i=1}^{\alpha} k_i} q^{\sum_{i=1}^{\alpha-1} \frac{1}{2}(k_i^2+k_i)}}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}} \times \frac{(-1)^{\sum_{i=1}^{\beta} l_i} q^{\sum_{i=1}^{\beta-1} \frac{1}{2}(l_i^2+l_i)}}{(\prod_{i=2}^{\beta}(q)_{l_{i-1}-l_i})(q)_{k_\beta}} \times \frac{(q)^{m_\gamma^2+m_\gamma}(q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)}(q)_n}{(\prod_{i=2}^{\gamma}(q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2}$$

$$\times \Big(\sum_{s=\max\{k_\alpha,l_\beta\}}^{k_\alpha+l_\beta} \times \Big(\sum_{a=s}^{n} \frac{q^{\frac{(k_\alpha^2-k_\alpha)+(l_\beta^2-l_\beta)}{2}} q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}$$

$$\times \Big(\sum_{t=\max\{a,m_\gamma\}}^{a+m_\gamma} \frac{q^{t+(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})} \Big)\Big)\Big).$$

Then, consider the same as for $\hat{J}^{\mathfrak{sl}_3}_{(n+1,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q))$,

$$\hat{J}^{\mathfrak{sl}_3}_{(n+1,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q))$$

$$= q^{\frac{1}{2}(\alpha+\beta)(n+1)^2+\frac{1}{2}(3\alpha+3\beta-2)(n+1)} J^{\mathfrak{sl}_3}_{(n+1,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q))$$

$$= \sum_{0 \leq k_\alpha \leq k_{\alpha-1} \leq \cdots \leq k_1 \leq n+1} \sum_{0 \leq l_\beta \leq l_{\beta-1} \leq \cdots \leq l_1 \leq n+1} \sum_{0 \leq m_\gamma \leq m_{\gamma-1} \leq \cdots \leq m_1 \leq n+1} \sum_{s=\max\{k_\alpha,l_\beta\}}^{\min\{k_\alpha+l_\beta,n+1\}}$$

$$\times \sum_{a=s}^{n+1} \sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n+1\}} \frac{(-1)^{\sum_{i=1}^{\alpha} k_i} q^{\frac{1}{2}k_\alpha^2} q^{\sum_{i=1}^{\alpha-1} \frac{1}{2}(k_i^2+k_i)}(q)_{n+1}}{(\prod_{i=1}^{2\alpha+1}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}}$$

$$\times \frac{(-1)^{\sum_{i=1}^{\beta} l_i} q^{\frac{1}{2}l_\beta^2} q^{\sum_{i=1}^{\beta-1}\frac{1}{2}(l_i^2+l_i)}(q)_{n+1}}{(\prod_{i=1}^{\beta}(q)_{l_{i-1}-l_i})(q)_{k_\beta}} \times \frac{(q)^{m_\gamma^2+m_\gamma}(q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)}(q)_{n+1}^2}{(\prod_{i=1}^{\gamma}(q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2}$$

$$\times \frac{q^{-\frac{-k_\alpha+l_\beta}{2}} q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(1-q^{n+2-k_\alpha})(1-q^{n+2-l_\beta})}{(1-q^{n+2-s})^2}$$

$$\times \frac{(q)_{k_\alpha}(q)_{l_\beta}(q)_{n+1-k_\alpha}^2(q)_{n+1-l_\beta}^2(q)_{2(n+1)-s+2}}{(q)_{n+1}^2(q)_{n+1-s}^2(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{2(n+1)-k_\alpha-l_\beta+2}(q)_{-s+k_\alpha+l_\beta}}$$

$$\times \frac{q^{(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}(q)_{n+1-a}^2(q)_{n+1-m_\gamma}^2(q)_{2(n+1)-t+2}}{(q)_{n+1}^2(q)_{n+1-t}^2(q)_{t-a}(q)_{t-m_\gamma}(q)_{2(n+1)-a-m_\gamma+2}(q)_{-t+a+m_\gamma}} q^t \frac{(1-q^{n+2})(1-q^{n+3})}{(1-q^{t+1})(1-q^{t+2})}$$

$$= \sum_{0\le k_\alpha\le k_{\alpha-1}\le\cdots\le k_1\le n+1} \sum_{0\le l_\beta\le l_{\beta-1}\le\cdots\le l_1\le n+1} \sum_{0\le m_\gamma\le m_{\gamma-1}\le\cdots\le m_1\le n+1} \sum_{s=\max\{k_\alpha,l_\beta\}}^{\min\{k_\alpha+l_\beta,n+1\}} \sum_{a=s}^{n+1}$$

$$\sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n+1\}} \frac{(-1)^{\sum_{i=1}^{\alpha} k_i} q^{\sum_{i=1}^{\alpha-1}\frac{1}{2}(k_i^2+k_i)}(q)_{n+1}}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{n+1-k_1}(q)_{k_\alpha}}$$

$$\times \frac{(-1)^{\sum_{i=1}^{\beta} l_i} q^{\sum_{i=1}^{\beta-1}\frac{1}{2}(l_i^2+l_i)}(q)_{n+1}}{(\prod_{i=2}^{\beta}(q)_{l_{i-1}-l_i})(q)_{n+1-l_1}(q)_{k_\beta}} \times \frac{(q)^{m_\gamma^2+m_\gamma}(q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)}(q)_{n+1}^2}{(\prod_{i=2}^{\gamma}(q)_{m_{i-1}-m_i})(q)_{n+1-m_1}(q)_{m_\gamma}^2}$$

$$\times \frac{q^{\frac{(k_\alpha^2-k_\alpha)+(l_\beta^2-l_\beta)}{2}} q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(1-q^{n+2-k_\alpha})(1-q^{n+2-l_\beta})}{(1-q^{n+2-s})^2}$$

$$\times \frac{(q)_{k_\alpha}(q)_{l_\beta}(q)_{n+1-k_\alpha}^2(q)_{n+1-l_\beta}^2(q)_{2(n+1)-s+2}}{(q)_{n+1}^2(q)_{n+1-s}^2(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{2(n+1)-k_\alpha-l_\beta+2}(q)_{-s+k_\alpha+l_\beta}}$$

$$\times \frac{q^{(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}(q)_{n+1-a}^2(q)_{n+1-m_\gamma}^2(q)_{2(n+1)-t+2}}{(q)_{n+1}^2(q)_{n+1-t}^2(q)_{t-a}(q)_{t-m_\gamma}(q)_{2(n+1)-a-m_\gamma+2}(q)_{-t+a+m_\gamma}} q^t \frac{(1-q^{n+2})(1-q^{n+3})}{(1-q^{t+1})(1-q^{t+2})}$$

$$\equiv_{n+1} \sum_{0\le k_\alpha\le k_{\alpha-1}\le\cdots\le k_1\le n+1} \sum_{0\le l_\beta\le l_{\beta-1}\le\cdots\le l_1\le n+1} \sum_{0\le m_\gamma\le m_{\gamma-1}\le\cdots\le m_1\le n+1} \sum_{s=\max\{k_\alpha,l_\beta\}}^{\min\{k_\alpha+l_\beta,n+1\}} \sum_{a=s}^{n+1}$$

$$\sum_{t=\max\{a,m_\gamma\}}^{\min\{a+m_\gamma,n+1\}} \frac{(-1)^{\sum_{i=1}^{\alpha} k_i} q^{\sum_{i=1}^{\alpha-1}\frac{1}{2}(k_i^2+k_i)}}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}} \times \frac{(-1)^{\sum_{i=1}^{\beta} l_i} q^{\sum_{i=1}^{\beta-1}\frac{1}{2}(l_i^2+l_i)}}{(\prod_{i=2}^{\beta}(q)_{l_{i-1}-l_i})(q)_{k_\beta}}$$

$$\times \frac{(q)^{m_\gamma^2+m_\gamma}(q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)}(q)_{n+1}}{(\prod_{i=2}^{\gamma}(q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2} \times \frac{q^{\frac{(k_\alpha^2-k_\alpha)+(l_\beta^2-l_\beta)}{2}} q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}$$

$$\times \frac{q^{t+(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})}$$

$$\equiv_{n+1} \sum_{0\le k_\alpha\le k_{\alpha-1}\le\cdots\le k_1\le n+1} \sum_{0\le l_\beta\le l_{\beta-1}\le\cdots\le l_1\le n+1} \sum_{0\le m_\gamma\le m_{\gamma-1}\le\cdots\le m_1\le n+1}$$

$$\times \frac{(-1)^{\sum_{i=1}^{\alpha}k_i} q^{\sum_{i=1}^{\alpha-1}\frac{1}{2}(k_i^2+k_i)}}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}} \times \frac{(-1)^{\sum_{i=1}^{\beta}l_i} q^{\sum_{i=1}^{\beta-1}\frac{1}{2}(l_i^2+l_i)}}{(\prod_{i=2}^{\beta}(q)_{l_{i-1}-l_i})(q)_{k_\beta}} \times \frac{(q)^{m_\gamma^2+m_\gamma}(q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)}(q)_{n+1}}{(\prod_{i=2}^{\gamma}(q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2}$$

$$\times \Bigg(\sum_{s=\max\{k_\alpha,l_\beta\}}^{k_\alpha+l_\beta} \Bigg(\sum_{a=s}^{n+1} \frac{q^{\frac{(k_\alpha^2-k_\alpha)+(l_\beta^2-l_\beta)}{2}} q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}$$

$$\times \sum_{t=\max\{a,m_\gamma\}}^{a+m_\gamma} \frac{q^{t+(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})}\Bigg)\Bigg)\Bigg).$$

If $k_1 = n+1$, then we have

$$\frac{(-1)^{\sum_{i=1}^{\alpha}k_i} q^{\sum_{i=1}^{\alpha-1}\frac{1}{2}(k_i^2+k_i)}}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}}$$

$$(3.2.14) \qquad =(-1)^{n+1} q^{\frac{(n+1)^2+n+1}{2}} \frac{(-1)^{\sum_{i=2}^{\alpha}k_i} q^{\sum_{i=2}^{\alpha-1}\frac{1}{2}(k_i^2+k_i)}}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}}$$

$$= (-1)^{n+1} q^{n+1+\frac{1}{2}(n^2+n)} \frac{(-1)^{\sum_{i=2}^{\alpha}k_i} q^{\sum_{i=2}^{\alpha-1}\frac{1}{2}(k_i^2+k_i)}}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}}.$$

Thus, when $k_1 = n+1$,

$$\hat{J}^{\mathfrak{sl}_3}_{(n+1,0)}(P(\downarrow\alpha\downarrow,\uparrow\beta\uparrow,\downarrow 2\gamma\uparrow);q))$$

$$\equiv_{n+1} \sum_{0\le k_\alpha\le k_{\alpha-1}\le\cdots\le k_2\le n+1} \sum_{0\le l_\beta\le l_{\beta-1}\le\cdots\le l_1\le n+1} \sum_{0\le m_\gamma\le m_{\gamma-1}\le\cdots\le m_1\le n+1}$$

$$\times \frac{(-1)^{\sum_{i=1}^{\alpha} k_i} q^{\sum_{i=1}^{\alpha-1} \frac{1}{2}(k_i^2+k_i)}}{(\prod_{i=2}^{\alpha} (q)_{k_{i-1}-k_i})(q)_{k_\alpha}} \times \frac{(-1)^{\sum_{i=1}^{\beta} l_i} q^{\sum_{i=1}^{\beta-1} \frac{1}{2}(l_i^2+l_i)}}{(\prod_{i=2}^{\beta} (q)_{l_{i-1}-l_i})(q)_{k_\beta}} \times \frac{(q)^{m_\gamma^2+m_\gamma} (q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)} (q)_{n+1}}{(\prod_{i=2}^{\gamma} (q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2}$$

$$\times \Big(\sum_{s=\max\{k_\alpha, l_\beta\}}^{k_\alpha+l_\beta} \Big(\sum_{a=s}^{n+1} \frac{q^{\frac{(k_\alpha^2-k_\alpha)+(l_\beta^2-l_\beta)}{2}} q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta} (q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}$$

$$\times \sum_{t=\max\{a, m_\gamma\}}^{a+m_\gamma} \frac{q^{t+(t+1)(t-a-m_\gamma)+am_\gamma} (q)_a (q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})} \Big)\Big)\Big)$$

$$= \sum_{0 \le k_\alpha \le k_{\alpha-1} \le \cdots \le k_2 \le n+1} (-1)^{n+1} q^{n+1+\frac{1}{2}(n^2+n)} \Big(\frac{(-1)^{\sum_{i=1}^{\alpha} k_i} q^{\sum_{i=1}^{\alpha-1} \frac{1}{2}(k_i^2+k_i)}}{(\prod_{i=2}^{\alpha} (q)_{k_{i-1}-k_i})(q)_{k_\alpha}}$$

$$\times \sum_{0 \le l_\beta \le l_{\beta-1} \le \cdots \le l_1 \le n+1} \frac{(-1)^{\sum_{i=1}^{\beta} l_i} q^{\sum_{i=1}^{\beta-1} \frac{1}{2}(l_i^2+l_i)}}{(\prod_{i=2}^{\beta} (q)_{l_{i-1}-l_i})(q)_{k_\beta}}$$

$$\times \Big(\sum_{0 \le m_\gamma \le m_{\gamma-1} \le \cdots \le m_1 \le n+1} \frac{(q)^{m_\gamma^2+m_\gamma} (q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)} (q)_{n+1}}{(\prod_{i=2}^{\gamma} (q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2}$$

$$\times \Big(\sum_{s=\max\{k_\alpha, l_\beta\}}^{k_\alpha+l_\beta} \Big(\sum_{a=s}^{n+1} \frac{q^{\frac{(k_\alpha^2-k_\alpha)+(l_\beta^2-l_\beta)}{2}} q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta} (q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}$$

$$\times \sum_{t=\max\{a, m_\gamma\}}^{a+m_\gamma} \frac{q^{t+(t+1)(t-a-m_\gamma)+am_\gamma} (q)_a (q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})} \Big)\Big)\Big)$$

$$\equiv_{n+1} \sum_{0 \le k_\alpha \le k_{\alpha-1} \le \cdots \le k_2 \le n+1} (-1)^{n+1} q^{n+1+\frac{1}{2}(n^2+n)} \frac{(-1)^{\sum_{i=1}^{\alpha} k_i}}{(\prod_{i=2}^{\alpha} (q)_{k_{i-1}-k_i})(q)_{k_\alpha}}$$

$$\times \sum_{0 \le l_\beta \le l_{\beta-1} \le \cdots \le l_1 \le n+1} \frac{(-1)^{\sum_{i=1}^{\beta} l_i}}{(\prod_{i=2}^{\beta} (q)_{l_{i-1}-l_i})(q)_{k_\beta}} \times \sum_{0 \le m_\gamma \le m_{\gamma-1} \le \cdots \le m_1 \le n+1} \frac{(q)_{n+1}}{(\prod_{i=2}^{\gamma} (q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2}$$

$$\times \Big(\sum_{s=\max\{k_\alpha,l_\beta\}}^{k_\alpha+l_\beta} \Big(\sum_{a=s}^{n+1} \frac{(q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}$$
$$\times \sum_{t=\max\{a,m_\gamma\}}^{a+m_\gamma} \frac{(q)_a(q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})}\Big)\Big)\Big)$$
$$= q^{n+1} + (\text{high order terms})$$
$$\equiv_{n+1} 0.$$

We use (3.2.14) in the second equation and (3.2.10) and (3.2.11) in the third congruence and $\frac{1}{1-q} = 1 + q + q^2 + \cdots$ for the forth equation. By a similar calculation for $l_1 = n+1$ and $m_1 = n+1$, we obtain

(3.2.15)
$$\hat{J}^{\mathfrak{sl}_3}_{(n+1,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q))$$
$$\equiv_{n+1} \sum_{0\le k_\alpha \le k_{\alpha-1}\le\cdots\le k_2\le n} \sum_{0\le l_\beta\le l_{\beta-1}\le\cdots\le l_1\le n} \sum_{0\le m_\gamma\le m_{\gamma-1}\le\cdots\le m_1\le n}$$
$$\times \frac{(-1)^{\sum_{i=1}^{\alpha} k_i} q^{\sum_{i=1}^{\alpha-1}\frac{1}{2}(k_i^2+k_i)}}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)_{k_\alpha}} \times \frac{(-1)^{\sum_{i=1}^{\beta} l_i} q^{\sum_{i=1}^{\beta-1}\frac{1}{2}(l_i^2+l_i)}}{(\prod_{i=2}^{\beta}(q)_{l_{i-1}-l_i})(q)_{k_\beta}} \times \frac{(q)^{m_\gamma^2+m_\gamma}(q)^{\sum_{i=1}^{\gamma-1}(m_i^2+2m_i)}(q)_{n+1}}{(\prod_{i=2}^{\gamma}(q)_{m_{i-1}-m_i})(q)_{m_\gamma}^2}$$
$$\times \Big(\sum_{s=\max\{k_\alpha,l_\beta\}}^{k_\alpha+l_\beta} \Big(\sum_{a=s}^{n+1} \frac{q^{\frac{(k_\alpha^2-k_\alpha)+(l_\beta^2-l_\beta)}{2}} q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}$$
$$\times \sum_{t=\max\{a,m_\gamma\}}^{a+m_\gamma} \frac{q^{t+(t+1)(t-a-m_\gamma)+am_\gamma}(q)_a(q)_{m_\gamma}}{(q)_{t-a}(q)_{t-m_\gamma}(q)_{-t+a+m_\gamma}(1-q^{t+1})(1-q^{t+2})}\Big)\Big)\Big).$$

According to (3.2.13) and (3.2.15),

$$\hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)) \equiv_{n+1} \hat{J}^{\mathfrak{sl}_3}_{(n+1,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)).$$

Similarly, we can prove Main Theorem 2 for $P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow)$ when one of α and β is odd and the other is even. Note that

$$\hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q)) = -q^{\frac{1}{2}(\alpha+\beta)n^2+\frac{1}{2}(3\alpha+3\beta-2)n} J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow); q))$$

in this case. Moreover, we can show Main Theorem 2 for $P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow)$ in the same way we prove Main Theorem 2 for $P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow)$. □

We obtain explicit formulae of the tails of $P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow)$.

Definition 3.2.3. Let power series $f_n(q)$, $F(q) \in \mathbb{Z}[[q]]$ for $n \geq 1$. We define

$$\lim_{n\to\infty} f_n(q) = F(q)$$

by $f_n(q) \equiv_{n+1} F(q)$.

Theorem 3.2.4 (Kawasoe)**.** Let α, β and γ be positive integers. The tail of the one-row colored $\mathfrak{sl}_3$ Jones polynomials for pretzel links $P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow)$ are the followng:

(3.2.16)

$$\begin{aligned}
&\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)\\
&= \lim_{n\to\infty} \hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q))\\
&=(q)^3_\infty \sum_{0\leq k_\alpha\leq k_{\alpha-1}\leq\cdots\leq k_1} \sum_{0\leq l_\beta\leq l_{\beta-1}\leq\cdots\leq l_1} \sum_{0\leq m_\gamma\leq m_{\gamma-1}\leq\cdots\leq m_1}\\
&\times \frac{(q)^{-k_\alpha}(q)^{\sum_{i=1}^{\alpha}(k_i^2+2k_i)}}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)^2_{k_\alpha}} \times \frac{(q)^{-l_\beta}(q)^{\sum_{i=1}^{\beta}(l_i^2+2l_i)}}{(\prod_{i=2}^{\beta}(q)_{l_{i-1}-l_i})(q)^2_{l_\beta}} \times \frac{(q)^{-m_\gamma}(q)^{\sum_{i=1}^{\gamma}(m_i^2+2m_i)}}{(\prod_{i=2}^{\gamma}(q)_{m_{i-1}-m_i})(q)^2_{m_\gamma}}\\
&\times\Bigg(\sum_{s=\max\{k_\alpha,l_\beta\}}^{k_\alpha+l_\beta} \frac{q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}\\
&\times\Bigg(\sum_{t=\max\{s,m_\gamma\}}^{s+m_\gamma} \frac{q^{t+(t+1)(t-s-m_\gamma)+sm_\gamma}(q)_s(q)_{m_\gamma}}{(q)_{t-s}(q)_{t-m_\gamma}(q)_{-t+s+m_\gamma}(1-q^{t+1})(1-q^{t+2})}\Bigg)\Bigg).
\end{aligned}$$

Proof. Main Theorem 2 ensures the existence of the limit of $\hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q))$. Then, we have

(3.2.17)

$$\begin{aligned}
&\hat{J}^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q))\\
&=q^{-2n} J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q))\\
&\equiv_{n+1} \sum_{0\leq k_\alpha\leq k_{\alpha-1}\leq\cdots\leq k_1\leq n} \sum_{0\leq l_\beta\leq l_{\beta-1}\leq\cdots\leq l_1\leq n} \sum_{0\leq m_\gamma\leq m_{\gamma-1}\leq\cdots\leq m_1\leq n}\\
&\times \frac{(q)^{-k_\alpha}(q)^{\sum_{i=1}^{\alpha}(k_i^2+2k_i)}(q)_n}{(\prod_{i=2}^{\alpha}(q)_{k_{i-1}-k_i})(q)^2_{k_\alpha}} \times \frac{(q)^{-l_\beta}(q)^{\sum_{i=1}^{\beta}(l_i^2+2l_i)}(q)_n}{(\prod_{i=2}^{\beta}(q)_{l_{i-1}-l_i})(q)^2_{l_\beta}} \times \frac{(q)^{-m_\gamma}(q)^{\sum_{i=1}^{\gamma}(m_i^2+2m_i)}(q)_n}{(\prod_{i=2}^{\gamma}(q)_{m_{i-1}-m_i})(q)^2_{m_\gamma}}\\
&\times\Bigg(\sum_{s=\max\{k_\alpha,l_\beta\}}^{k_\alpha+l_\beta} \frac{q^{s+(s+1)(s-k_\alpha-l_\beta)+k_\alpha l_\beta}(q)_{k_\alpha}(q)_{l_\beta}}{(q)_{s-k_\alpha}(q)_{s-l_\beta}(q)_{-s+k_\alpha+l_\beta}}
\end{aligned}$$

$$\times\Bigg(\sum_{t=\max\{s,m_\gamma\}}^{s+m_\gamma}\frac{q^{t+(t+1)(t-s-m_\gamma)+sm_\gamma}(q)_s(q)_{m_\gamma}}{(q)_{t-s}(q)_{t-m_\gamma}(q)_{-t+s+m_\gamma}(1-q^{t+1})(1-q^{t+2})}\Bigg)\Bigg)$$

in the same way that we got (3.2.13). Then, we can obtain (3.2.16) by using (3.2.17). □

Remark 3.2.5. The one-row colored $\mathfrak{sl}_3$ Jones polynomial for $P(2,2,2)$ multiplied by q^{-2n} is

$$\begin{aligned}
n=1: &\quad 1-2q+q^2+2q^3-q^4+3q^5+2q^6+2q^8+q^9,\\
n=2: &\quad 1-2q-q^2+6q^3-7q^5+2q^6+7q^7-3q^8-6q^9+5q^{10}+\cdots,\\
n=3: &\quad 1-2q-q^2+4q^3+4q^4-3q^5-13q^6+2q^7+18q^8+6q^9-14q^{10}\cdots,\\
n=4: &\quad 1-2q-q^2+4q^3+2q^4+q^5-9q^6-10q^7+8q^8+18q^9+18q^{10}\cdots,\\
n=5: &\quad 1-2q-q^2+4q^3+2q^4-q^5-5q^6-6q^7-4q^8+11q^9+25q^{10}+\cdots,\\
n=6: &\quad 1-2q-q^2+4q^3+2q^4-q^5-7q^6-2q^7-q^9+18q^{10}+\cdots,\\
n=7: &\quad 1-2q-q^2+4q^3+2q^4-q^5-7q^6-4q^7+4q^8+3q^9+6q^{10}+\cdots,\\
n=8: &\quad 1-2q-q^2+4q^3+2q^4-q^5-7q^6-4q^7+2q^8+7q^9+10q^{10}+\cdots,\\
n=9: &\quad 1-2q-q^2+4q^3+2q^4-q^5-7q^6-4q^7+2q^8+5q^9+14q^{10}+\cdots,\\
n=10: &\quad 1-2q-q^2+4q^3+2q^4-q^5-7q^6-4q^7+2q^8+5q^9+12q^{10}+\cdots.
\end{aligned}$$

Using Mathematica, we computed the first 30 terms of $\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow 2\uparrow,\downarrow 2\uparrow,\downarrow 2\uparrow);q)$:

$$\begin{aligned}
&\mathcal{T}^{\mathfrak{sl}_3}(P(\downarrow 2\uparrow,\downarrow 2\uparrow,\downarrow 2\uparrow);q)\\
\equiv_{31}&1-2q-q^2+4q^3+2q^4-q^5-7q^6-4q^7+2q^8+5q^9+12q^{10}+5q^{11}-3q^{12}-10q^{13}\\
&-13q^{14}-15q^{15}-6q^{16}+9q^{17}+18q^{18}+23q^{19}+18q^{20}+18q^{21}-3q^{22}-21q^{23}\\
&-33q^{24}-39q^{25}-32q^{26}-18q^{27}-3q^{28}+21q^{29}+48q^{30}.
\end{aligned}$$

Appendix A

Appendix

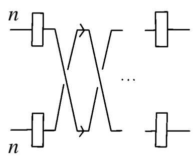
In this chapter, we give the $2m+1$ times half twists formulae for two strands.

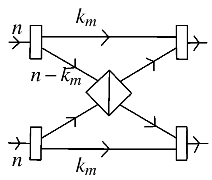
A.1 The $2m+1$ times half twists formulae for two strands with the same directions

We can also compute $T(2, 2m+1)$, $P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow)$ and $P(\downarrow 2\alpha \uparrow, \downarrow 2\beta \uparrow, \downarrow 2\gamma \uparrow)$ by using the following proposition.

Proposition A.1.1 (Kawasoe)**.** Let n be a positive integer and $k_0 = n$. For a positive integer m, we have

(A.1.1)

$\Big\langle$ 

$\Big\rangle_3 = \displaystyle\sum_{0 \le k_m \le k_{m-1} \le \cdots \le k_0 \le n} \xi_1(n, k_0, k_1, ..., k_m) \Big\langle$ 

$\Big\rangle_3$

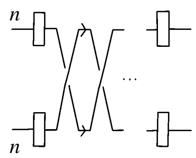
2m + 1 times half twists

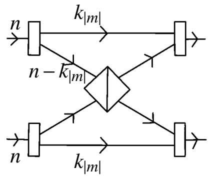
where

$$\xi_1(n, k_0, k_1, ..., k_m) = (-1)^{n-k_0} \frac{q^{-\frac{1}{3}(n^2+3n)m} q^{\frac{1}{6}(-n^2+3k_0^2)} q^{\sum_{i=1}^{m}(k_i^2+k_i)} (q)_n q^{\frac{1}{2}(k_0-k_m)} (q)_n (q)_{k_0}}{(q)_{n-k_0} \prod_{i=1}^{m} (q)_{k_{i-1}-k_i} (q)_{k_m^2}}.$$

For a negative integer m, we have

(A.1.2)

$\Big\langle$ 

$\Big\rangle_3 = \sum_{0 \le k_{|m|} \le k_{|m|-1} \le \cdots \le k_0 \le n} \xi_{-1}(n, k_0, k_1, ..., k_{|m|}) \Big\langle$ 

$\Big\rangle_3$

$2m+1$ times-half twists

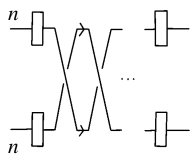
where

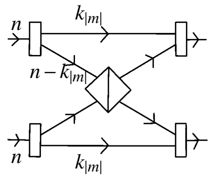
$$\xi_{-1}(n, k_0, k_1, ..., k_{|m|}) = (-1)^{n-k_0} \frac{q^{\frac{1}{3}(n^2+3n)m} q^{\frac{1}{6}(n^2-6nk_0+3k_0^2)} q^{\sum_{i=1}^{|m|}(-k_i^2-k_i)} (q)_n q^{-\frac{1}{2}(k_0-k_{|m|})} (q)_n (q)_{k_0}}{(q)_{n-k_0} \prod_{i=1}^{|m|} (q)_{k_{i-1}-k_i} (q)_{k_{|m|}^2}}$$

Proof. We can prove Proposition A.1.1 in the same way as Proposition 2.1.1 by using the following theorem. □

Theorem A.1.2 ([Yua21])**.** Let n be a non-zero integer and $k_0 = n$. For a integer m, we have

(A.1.3)

$\Big\langle$ 

$\Big\rangle_3 = \sum_{0 \le k_{|m|} \le k_{|m|-1} \le \cdots \le k_0 \le n} \kappa(n, k_0, k_1, ..., k_{|m|})_{q^{\epsilon_m}} \Big\langle$ 

$\Big\rangle_3$

m times half twists

where

$$\kappa(n, k_0, k_1, ..., k_{|m|})_{q^{\epsilon_m}} = \frac{q^{-\frac{1}{3}(n^2+3n)m} q^{\sum_{i=1}^{|m|}(k_i^2+k_i)} (q)_n q^{\frac{1}{2}(k_0-k_m)} (q)_{k_0}}{\prod_{i=1}^{|m|} (q)_{k_{i-1}-k_i} (q)_{k_{|m|}}}$$

and $q^{\epsilon_m} = q^{\frac{|m|}{m}}$.

Concerning (2.2.2) in Main Theorem 1, we can also compute the one-row colored $\mathfrak{sl}_3$ Jones polynomials for pretzel links $P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow)$ by using Proposition A.1.1 and Theorem A.1.2 instead of Propositon 2.1.1.

Theorem A.1.3 (Kawasoe)**.** Let α, β and γ be non-zero integers. The one-row colored $\mathfrak{sl}_3$ Jones polynomials for pretzel links $P(\downarrow \alpha \downarrow, \uparrow \beta \uparrow, \downarrow 2\gamma \uparrow)$ are the following:

(A.1.4)

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha+1 \downarrow, \uparrow 2\beta+1 \uparrow, \downarrow 2\gamma \uparrow); q)$$
$$= \sum_{0\le k_{|2\alpha+1|}\le k_{|2\alpha|}\le\cdots\le k_1\le n} \sum_{0\le l_{|2\beta+1|}\le l_{|2\beta|}\le\cdots\le l_1\le n} \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n} \sum_{s=\max\{k_{|2\alpha+1|}, l_{|2\beta+1|}\}}^{\min\{k_{|2\alpha+1|}+l_{|2\beta+1|}, n\}}$$
$$\sum_{a=s}^{n} \sum_{t=\max\{a,s\}}^{\min\{a+s,n\}} (q^{\frac{n^2+3n}{3}})^{-(2\alpha+2\beta-2\gamma+2)} \xi_{sign(2\alpha+1)}(n, k_1, k_2, ..., k_{|2\alpha+1|})$$
$$\times \xi_{sing(2\beta+1)}(n, l_1, l_2, ..., l_{|2\beta+1|}) \phi(n, m_1, m_2, ..., m_{|\gamma|})_{q^{\epsilon_\gamma}} \Omega(n, s, k_{|2\alpha+1|}, l_{|2\beta+1|})$$
$$\times \psi(n, a, m_{|\gamma|}, t) q^{-(n-t)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})},$$

(A.1.5)

$$J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha+1 \downarrow, \uparrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)$$
$$= \sum_{0\le k_{|2\alpha+1|}\le k_{|2\alpha|}\le\cdots\le k_1\le n} \sum_{0\le l_{2|\beta|}\le l_{2|\beta|-1}\le\cdots\le l_1\le n} \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n} \sum_{s=\max\{k_{|2\alpha+1|}, l_{2|\beta|}\}}^{\min\{k_{|2\alpha+1|}+l_{2|\beta|}\}}$$
$$\sum_{a=s}^{n} \sum_{t=\{a, m_\gamma\}}^{\min\{a+\gamma,n\}} (q^{\frac{n^2+3n}{3}})^{-(-2\alpha-2\beta+2\gamma-1)} \xi_{sign(2\alpha+1)}(n, k_1, k_2, ..., k_{|2\alpha+1|}) \kappa(n, l_1, l_2, ..., l_{2|\beta|})_{q^{\epsilon_\beta}}$$
$$\times \phi(n, m_1, m_2, ..., m_{|\gamma|})_{q^{\epsilon_{|\gamma|}}} \Omega(n, k_{|2\alpha+1|}, l_{2|\beta|}, t) \psi(n, t, a, m_{|\gamma|}) q^{-(n-t)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})},$$

(A.1.6)

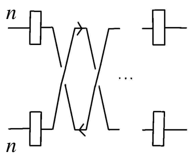
$$J^{\mathfrak{sl}_3}_{(n,0)}(P(\downarrow 2\alpha+1 \downarrow, \uparrow 2\beta \uparrow, \downarrow 2\gamma \uparrow); q)$$
$$= \sum_{0\le k_{|2\alpha+1|}\le k_{|2\alpha|}\le\cdots\le k_1\le n} \sum_{0\le l_{2|\beta|}\le l_{2|\beta|-1}\le\cdots\le l_1\le n} \sum_{0\le m_{|\gamma|}\le m_{|\gamma|-1}\le\cdots\le m_1\le n} \sum_{s=\max\{k_{|2\alpha+1|}, l_{2|\beta|}\}}^{\min\{k_{|2\alpha+1|}+l_{2|\beta|}\}} \sum_{a=0}^{n-s}$$
$$\sum_{t=\{a,s\}}^{\min\{a+s,n\}} (q^{\frac{n^2+3n}{3}})^{-(-2\alpha-2\beta+2\gamma-1)} \chi_{sign(2\alpha+1)}(n, k_1, k_2, ..., k_{|2\alpha+1|}) \kappa(n, l_1, l_2, ..., l_{2|\beta|})_{q^{\epsilon_\beta}}$$
$$\times \phi(n, m_1, m_2, ..., m_{|\gamma|})_{q^{\epsilon_{|\gamma|}}} \Omega(n, k_{|2\alpha+1|}, l_{2|\beta|}, t) \psi(n, t, a, m_{|\gamma|}) q^{-(n-t)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{t+1})(1-q^{t+2})}.$$

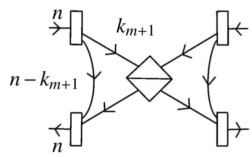
Proof. We can show *Theorem A*.1.3 in a similar to proving Main Theorem 1. □

Remark A.1.4. When we use compute he one-row colored $\mathfrak{sl}_3$ Jones polynomials for pretzel links $P(\downarrow\alpha\downarrow, \uparrow\beta\uparrow, \downarrow 2\gamma\uparrow)$ by using Mathematica, we can obtain the results more quickly using Theorem A.1.3 than Main Theorem 1.

A.2 The $2m+1$ times half twists formulae for two strands with opposite directions

Proposition A.2.1. Let n be a positive integer and $k_0 = n$. For a positive integer m, we have
(A.2.1)

$\Big\langle$ 

$\Big\rangle_3 = \displaystyle\sum_{0\le k_{m+1}\le k_m\le\cdots\le k_1\le n} \iota(n,k_1,k_2,...,k_{m+1})\Big\langle$ 

$\Big\rangle_3$

2m + 1 times half twists

where

$$\iota(n,k_1,k_2,...,k_{m+1}) = q^{-\frac{4}{3}m(n^2+3n)+n^2+4n}(-1)^{k_{m+1}} \frac{q^{\frac{1}{6}(4k_m^2-6k_mk_{m-1}+3k_{m+1}^2)}q^{\sum_{i=1}^{m}(k_i^2+2k_i)}}{\prod_{j=1}^{m+1}(q)_{k_{j-1}-k_j}(q)_{k_m}(q)_{k_{m+1}}}.$$

Proof. We can easily show this by using Lemma 1.2.3 and Theorem 1.2.4. □

Remark A.2.2. We expect to use Proposition A.2.1 to compute pretzel knots for which all three parameters not included in the result of Main Theorem 1 are odd.

Bibliography

[AD11] Cody Aromnd and Oliver T. Dasbach, *Rogers-ramanujan type identities and the head and tail of the colored Jones polynomial*, arXiv:1106.3948 (2011).

[AD17] Cody Armond and Oliver T. Dasbach, *The head and tail of the colored jones polynomial for adequate knots*, Proc. Amer. Math.Soc **145** (2017), no. 3, 1357–1367, MR3589331.

[Arm13] Cody Armond, *The head and tail conjecure for alternating knots*, Algebr. Geom. Topol. **13** (2013), no. 5, 2809–2826, MR3116304.

[Bei19] Paul Beirne, *On the 2-head of the colored Jones polynomial for pretzel knots*, arXiv:1902.07061 (2019).

[BO17] Paul Beirne and Robert Osburn, *q-series and tails of colored Jones polynomials*, Indag. Math. **28** (2017), no. 1, 247–260, MR3597046.

[DL06] Oliver T Dasbach and Xiao-Song Lin, *On the head and the tail of the colored Jones polynomial*, Compos. Math. **142** (2006), no. 5, 1332–1342, MR2264669.

[EH17] Mohamed Elhamdadi and Mustafa Haiji, *Pretzel knots and q-series*, Osaka J. Math. **54** (2017), 363–381, MR3657236.

[Gar11] Stavros Garoufalidis, *The Jones slopes of a knot*, Quantum Topol. **2** (2011), no. 1, 43–69, MR2763086.

[GL15] Stavros Garoufalidis and Thang T. Q. Lê, *Nahm sums, stability, and the colored Jones polynomial*, Res. Math. Sci. **2** (2015), Art. 1, 55 pp.

[GM13] Stavros Garoufalidis and Hugh Morton, *The $sl3$ colored Jones polynomial of the trefoil.*, Proc. Amer. Math. Soc. **141** (2013), no. 6, 2209–2220, MR3034446.

[GV17] Stavros Garoufalidis and Thao Vuong, *A stability conjecture for the colored jones polynomial*, Topology Proc. **49** (2017), 211–249, MR3570390.

[Haj14] Mustafa Hajij, *The bubble skein element and applications*, J. Knot Theory Ramifications **23** (2014), no. 14, 1450076, 30 pp, MR3312619.

[Jon85] Vaughan F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), no. 1, 103–111, MR0766964.

[Kas97] R. M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), no. 3, 269–275, MR1434238.

[Kau87] Louis H. Kauffman, *State models, and the Jones polynomial,*, Topology **26** (1987), no. 3, 395–407, MR0899057.

[Kaw] Kotaro Kawasoe, *The one-row colored* $\mathfrak{sl}_3$ *Jones polynomials for pretzel links*, J. Knot Theory Ramifications, doi:10.1142/S021821652250105X.

[Kim06] Dongseok Kim, *Trihedron coefficients for* $u_q(sl(3,\mathbb{C}))$, J. Knot Theory Ramifications **15** (2006), no. 4, 453–72, MR 2221529.

[Kim07] ———, *Jones-Wenzl idempotents for rank 2 simple Lie algebras*, Osaka J. Math. **44** (2007), no. 3, 691–722, MR2360947.

[KO16] Adam Keilthy and Robert Osburn, *Rogers-Ramanujan type identities for alternating knots*, J. Number Theory **161** (2016), 255–280, MR3435728.

[Kup96] Greg Kuperberg, *Spiders for rank* 2 *Lie algebras*, Comm. Math. Phys. **180** (1996), no. 1, 109–151, MR1403861.

[Law03] Ruth Lawrence, *The* $PSU(3)$ *invariant of the poincaré homology sphere*, Topology Appl **127** (2003), no. 1-2, 153–168, MR1953324.

[Le00] Thang T. Q. Le, *Integrality and symmetry of quantum link invariant*, Duke Math. J. **102** (2000), no. 2, 273–306, MR1749439.

[Lic97] W. B. Raymond Lickorish, *An introduction to knot theory*, vol. 175, Springer-Verlag, New York, 1997, MR1472978.

[OY97] Tomotada Ohtsuki and Shuji Yamada, *Quantum* $SU(3)$ *invariant of* 3-*manifolds via linear skein theory*, J. Knot Theory Ramifications **6** (1997), no. 3, 373–404.

[RJ93] Marc Rosso and Vauhjan F. R. Jones, *On the invariants of torus knots derived from quantum groups*, J.Knot Theory Ramificatications **2** (1993), no. 1, 97–112, MR1209320.

[Wen87] Hans Wenzel, *On sequences of projections*, C. R. Math. Rep. Acad. Sci. Canada 9 (1987), no. 1, 5–9, MR 873400.

[Yua17] Wataru Yuasa, *The* $\mathfrak{sl}_3$ *colored Jones polynomials for* 2*-bridge links*, J. Knot Theory Ramifications **26** (2017), no. 7, MR 3660093.

[Yua18a] ———, a_2 *colored polynomials of rigid vertex graphs*, New York J. Math **24** (2018), 355–374, MR3829741.

[Yua18b] ———, *A q-series identity via the* $\mathfrak{sl}_3$ *colored Jones polynomials for the* $(2,2m)$*-torus link*, Proc. Amer. Math. Sec **46** (2018), no. 7, 3153–3166, MR 3787374.

[Yua20] ———, *The zero stability for the one-row colored* $\mathfrak{sl}_3$ *Jones polynomial*, arXiv:2007.15621 (2020).

[Yua21] ———, *Twist formulas for one-row colored* a_2 *webs and* $\mathfrak{sl}_3$ *tails of* $(2,2m)$*-torus links*, Acta Math. Vietnam **46** (2021), no. 2, 369–387, MR4264242.