Spatial homogenization by perturbation on reaction-diffusion systems

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Spatial homogenization by perturbation on reaction-diffusion systems

反応拡散系の外乱による空間一様化

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Abstract.

Ginzburg-Landau equation has two types of behavior: one is spatio-temporal chaos remained inside the limit cycle on the phase plane, the other is a spatially homogeneous periodic solution on the limit cycle. If we perturb the solution behaving spatio-temporal chaos to the outside of a limit cycle, it is numerically observed that the perturbed solution converges to a spatially homogeneous periodic oscillation. This is the transition from chaos to regular motions based on a spatial homogenization by the perturbation. By constructing the invariant sets and using the asymptotic stability of the limit cycle, we prove analytically that the solution starting from an initial condition far from the limit cycle converges to the limit cycle oscillation.

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Chapter 1

Introduction

Reaction-diffusion systems or reaction-diffusion equations are widely used as mathematical models to describe many natural phenomena that are affected by two kinds of processes, diffusion and interaction in space. Reaction-diffusion systems have a wide range of applications, such as population dynamics, changes in population numbers due to interactions between phytoplankton and zooplankton, changes in the membrane potential of the heart and nerve cells, and body surface patterns in fish and mammals, chemical reaction systems such as the Belousov-Zhavodzinski reaction, Rayleigh-Benard convection, combustion theory, and phase transitions. The dynamics arising from the equations are also diverse, such as simple spatially uniform stationary solutions, spatially uniform periodic solutions on limit cycles, traveling wave solutions, stripe patterns, spot patterns, target patterns, spiral patterns, and even more. Complex phenomena such as spatio-temporal chaos and dissipative soliton solutions exist. Two or more spatial dimensions are often required to form remarkable spatial patterns such as stripe patterns and spot patterns on the surface of organisms, and spiral and target patterns appearing in BZ reactions.

The main purpose of this paper is to study spatial homogenization when the systems possess the complicated dynamics such as chaotic dynamics. Transitions between regular motions and chaotic dynamics have been often observed in many fields, such as fluid dynamics, motions of many particles, chemical reactions, biological systems and so on (e.g., see [10, 19, 26, 7, 8, 6, 23, 20]). There are various mathematical equations which possess both the chaotic dynamics and the regular periodic motions. Examples of equations in which such phenomena occur include the FitzHugh-Nagumo equation and the prey-predator model. FitzHugh-Nagumo equation describes the neuronal potential of the heart, and the prey-predator model, which describes population density due to interactions between zooplankton and phytoplankton. Prey and predator coexisting in the same space, mutually influencing each other and gradually expanding their habitat, it is a universal phenomenon widely seen in the natural world. Phytoplankton reproduces through photosynthesis and self-replication, and zooplankton feeds on phytoplankton and survives. The former serves as prey and the latter as predator. In this way, it can be said that relationships with other living things are essential for the existence of living things. The ecosystem created by such a predator and prey consists of the random movement(diffusion) and the interaction (reactions) in which each other influences itself through predation, self-proliferation, natural death, etc. It is generated by simultaneous diffusion. If we add the saturation effect and the diffusion term of the natural reproduction and predator predation following the logistic equation of the prey to the prey-predator model, then various behavior appear like the periodic oscillation on the limit cycle, the spiral pattern and spatiotemporal chaos. It shows complicated behavior.

Heartbeat is roughly classified into two types, namely regular periodic oscillations and irregular non-periodic oscillations. Examples of the latter case are arrhythmia and ventricular fibrillation. Periodic oscillations are represented by spatially uniform periodic solutions on limit cycles, and irregular non-periodic oscillations are represented by spiral patterns and spatio-temporal chaotic behavior. In the medical field, electric shocks such as pacemakers and defibrillators are widely used as treatments for restoring irregular vibrations to normal one, and their effectiveness is widely recognized. In a mathematical model, the phenomenon of returning irregular vibrations to regular vibrations by defibrillation corresponds to giving disturbances to spatio-temporal chaos to return it to a space-uniform periodic solution on the limit cycle. Periodic oscillations and chaotic behaviors are often studied individually, and there are few studies on their transient phenomena.

In this thesis, we aim to mathematically prove that defibrillation is effective in treating ventricular fibrillation and arrhythmia by showing analytically that the chaotic behavior converges to a regular oscillation when a disturbance is added. However, the FitzHugh-Nagumo equation is very difficult from the point of view of mathematical analysis. Specifically, the periodic solution cannot be obtained analytically, and it is difficult to prove the asymptotic behavior to the periodic solution. Therefore, it seems that there is no way to make the periodic behavior of this equation other than obtaining an approximate solution by numerical simulation. In order to facilitate the analysis, it is necessary to reduce the FitzHugh-Nagumo equation to a more manageable form. As a method of contraction, there is a diminishing contraction method. This method is used to construct a general reaction-diffusion equation that eliminates the effects of nonlinear terms of order 4 or higher by considering only the most important nonlinear terms with respect to deviations from the vicinity of the Hopf bifurcation. By this method, general reaction-diffusion systems with limit cycles are reduced to complex Ginzburg-Landau equations. The complex Ginzburg-Landau equation is a reaction-diffusion equation with a complex path function with nonlinear terms up to third order. Since the vector field is spherically symmetric and the conversion to polar coordinates simplifies the calculation, it is easy to obtain an exact solution approaching the limit cycle. From the existence of this exact solution, it becomes possible to discuss the convergence of the solution.

To tell the truth, it would be ideal if we could directly prove the FitzHugh-Nagumo equation, but various difficulties arise, thus we had no choice but to deal with the complex Ginzburg-Landau equation. In the process of proof, calculations that are difficult to perform without the complex Ginzburg-Landau equation are performed, thus the method in this paper cannot be used as it is.

This time, the proof was limited to abstract concepts such as chaos and limit cycles on the mathematical model corresponding to the beating of the heart, but with the development of theory in the future, proof will be made on the mathematical model that is closer to the actual dynamics of nerve cells. This remains for the future work.

This thesis is organized as follows. In Chapter 1.1, we define functional spaces (e.g. Banach space, L^p space, Sobolev space), a sectorial operator, inequalities, and Sobolev's embedding theorem to prove the theorem about the existence of solution of differential equations and to analyze them. In Chapter 2, we give definitions and examples of reaction-diffusion equations, the proof about the existence of the reaction-diffusion system on $\mathbb{R}^n$ and its Schauder estimation. Additionally, we explain the method of constructing invariant sets and the principle of the maximum value used in the subsequent proofs. Chapter 3 presents some lemmas on sectorial operators and characteristic multipliers in preparation for the book by Henry [9] on useful theorems on the asymptotic stability of limit cycles. Then we prove that a positive invariant set containing initial conditions moved far outside the limit cycle is sufficiently close to the limit cycle. This proves that the solution converges to a spatially uniform periodic solution due to the asymptotic stability of the limit cycle.

1.1 Preliminaries

1.1.1 The Gronwall inequality

Set $w(t)$ be the continuous function in class $C^0(I)$ which satisfies

$$w(t) \leq \alpha + \beta \int_0^t w(s)ds,$$

where $\alpha \geq 0$, $\beta > 0$ and $I = (0, T) \subset \mathbb{R}$. By setting $W(t) = \int_0^t w(s)ds$, we get

$$\frac{dW(t)}{dt} = w(t) \leq \alpha + \beta W(t).$$

By integrating the above inequality, we have

$$W(t) \leq \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right),$$

which implies

$$w(t) \leq \alpha + \beta W(t) \leq \alpha e^{\beta t}$$

is called *the Gronwall inequality*, and we often use this to prove the uniqueness of the many kinds of differential equations.

1.1.2 Poincaré-Bendixson's theorem

Consider the two-dimensional autonomous system (plane dynamical system) as

$$\begin{aligned}\frac{du}{dt} &= f(u, v)\\ \frac{dv}{dt} &= g(u, v).\end{aligned}$$

Then, there is the following theorem about the attractor of the plane dynamical system.

Theorem 1.1.1 (Poincaré-Bendixson's theorem, [32, 24, 31]). *The attractor* (ω*-limit set*) *of the bounded plane dynamical system is limited to the following three:*

(a): *the equilibrium point.*

(b): *the closed trajectory connecting finite numbers of equilibrium points.*

(c): *the periodic closed trajectory(the limit cycle).*

Specifically, the ω-limit set of the plane dynamical system without the equilibrium point is limited to the periodic closed trajectory. In other words, the trajectory confined in the closed bounded domain (invariant set, see Section 2.5 for the details) on the phase plane converges to the limit cycle oscillation.

1.1.3 Banach space

Banach space is one of the most representative functional spaces that is used in functional analysis.

Definition 1.1.2 (completeness)**.** *Consider a Cauchy sequence* $\{u_n\}_{n\geq 0}$ *in some vector space* X *with norm* $\|\cdot\|$. *Cauchy sequence is the sequence that satisfies*

$$\lim_{m,n\to\infty} \|u_n - u_m\| = 0.$$

If an arbitrary Cauchy sequence $\{u_n\}_{n\geq 0}$ *converges to* $\boldsymbol{u} \in X$, *then vector space* X *is complete.*

Definition 1.1.3 (Banach space)**.** *A complete vector space with the norm* $\|\cdot\|$ *that satisfies norm's axiom is Banach space. The norm must be satisfied follwing axioms:*

(semi-positivity) $\forall x \in X, \|x\| \geq 0. \quad \|x\| = 0 \Leftrightarrow x = 0.$

(homogeneity) $\forall x \in X, \forall \alpha \in \mathbb{R}, \|\alpha x\| = |\alpha|\|x\|.$

(triangle inequality) $\forall x, y \in X, \|x + y\| \leq \|x\| + \|y\|.$

Next we give the following examples of the Banach space.

Theorem 1.1.4. *Let* X *be a set of all bounded and uniformly continuous functions from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$, *then* X *is the Banach space with the norm*

$$\|\boldsymbol{u}\| := \sup_{\boldsymbol{x}\in\mathbb{R}^n} |\boldsymbol{u}(x)|.$$

Proof. Obviously the norm $\|\cdot\|$ satisfies the norm's axiom, thus it suffices to prove X is complete. Let $\{\boldsymbol{u}_j\}$ be a Cauchy sequence in X. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|\boldsymbol{u}_i - \boldsymbol{u}_j\| < \epsilon, \quad \text{if } i, j \geq N.$$

For each $\boldsymbol{x} \in \mathbb{R}^n$, $\{\boldsymbol{u}_i(\boldsymbol{x})\}_{i\geq 0}$ is a Cauchy sequence in a Banach space $\mathbb{R}^m$, then there exists $\boldsymbol{u}_\infty(\boldsymbol{x}) \in \mathbb{R}^m$ that satisfies

$$\lim_{i\to\infty} \boldsymbol{u}_i(\boldsymbol{x}) = \boldsymbol{u}_\infty(\boldsymbol{x}).$$

For any $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that

$$\|\boldsymbol{u}_i - \boldsymbol{u}_j\| < \frac{\epsilon}{4}, \quad \text{if } i, j \geq M.$$

Then

$$|\boldsymbol{u}_i(\boldsymbol{x}) - \boldsymbol{u}_j(\boldsymbol{x})| < \frac{\epsilon}{4}, \quad \text{if } i, j \geq M, \boldsymbol{x} \in \mathbb{R}^m.$$

When $j = M$ and $i \to \infty$, we have

$$|\boldsymbol{u}_\infty(\boldsymbol{x}) - \boldsymbol{u}_M(\boldsymbol{x})| \leq \frac{\epsilon}{4}, \quad \boldsymbol{x} \in \mathbb{R}^n,$$

since we get

$$|\boldsymbol{u}_\infty(\boldsymbol{x})| \leq |\boldsymbol{u}_M(\boldsymbol{x})| + \frac{\epsilon}{4} \leq \|\boldsymbol{u}_M\| + \frac{\epsilon}{4}.$$

$\boldsymbol{u}_\infty$ is a bounded function on $\mathbb{R}^n$. Since $\boldsymbol{u}_M \in X$, for any $\boldsymbol{x} \in \mathbb{R}^m$, there exists $\delta(\epsilon, \boldsymbol{x}) > 0$ such that

$$|\boldsymbol{u}_M(\boldsymbol{x}) - \boldsymbol{u}_M(\boldsymbol{y})| < \frac{\epsilon}{2}$$

if $|\boldsymbol{x} - \boldsymbol{y}| < \delta$. Now we have

$$\begin{aligned} |\boldsymbol{u}_\infty(\boldsymbol{x}) - \boldsymbol{u}_\infty(\boldsymbol{y})| <& |\boldsymbol{u}_\infty(\boldsymbol{x}) - \boldsymbol{u}_M(\boldsymbol{x})| + |\boldsymbol{u}_M(\boldsymbol{x}) - \boldsymbol{u}_M(\boldsymbol{y})| \\ &+ |\boldsymbol{u}_M(\boldsymbol{y}) - \boldsymbol{u}_\infty(\boldsymbol{y})| \\ \leq& \epsilon \end{aligned}$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, |\boldsymbol{x} - \boldsymbol{y}| < \delta$. Thus $\boldsymbol{u}_\infty(\boldsymbol{x})$ is uniformly continuous function from $\mathbb{R}^n$ to $\mathbb{R}^m$ and belongs to X. Finally we have

$$\lim_{j \to \infty} \|\boldsymbol{u}_i - \boldsymbol{u}_j\| = \|\boldsymbol{u}_i - \boldsymbol{u}_\infty\| < \epsilon,$$

and this implies the Cauchy sequence $\{\boldsymbol{u}_i\}$ converges to $\boldsymbol{u}_\infty \in X$ and X is complete. Therefore, X is the Banach space. □

1.1.4 Banach's fixed point theorem

To prove the existence of the solution of the differential equation, we often use the following fixed point's theorem about the contraction mapping.

Theorem 1.1.5. *Let X be the complete distance space (Cauchy space) with appropriate distance* $\mathrm{dist}(x, y)$ *for all $x, y \in X$ and $f : X \to X$ be the contraction mapping that satisfies*

$$\mathrm{dist}(f(x), f(y)) \leq \theta \, \mathrm{dist}(x, y)$$

for $\theta \in (0, 1)$. Then, f has the unique fixed point $x^* \in X$, that is, x^* satisfies $f(x^*) = x^*$.

Proof. Consider the sequence $\{x_n\}_{n\geq 0}$ that satisfies the following recurrence formula

$$x_{n+1} = f(x_n).$$

Using repeatedly

$$\mathrm{dist}(x_{n+1}, x_n) = \mathrm{dist}(f(x_n), f(x_{n-1})) \leq \theta \mathrm{dist}(x_n, x_{n-1}),$$

we have

$$\mathrm{dist}(x_{n+1}, x_n) \leq \theta^n \mathrm{dist}(x_1, x_0).$$

Next, set $m, n \in \mathbb{N}, m > n$. Then,

$$\begin{aligned}
\mathrm{dist}(x_m, x_n) &\leq \mathrm{dist}(x_m, x_{m-1}) + \mathrm{dist}(x_{m-1}, x_{m-2}) + \cdots + \mathrm{dist}(x_{n+1}, x_n) \\
&\leq \theta^{m-1}\mathrm{dist}(x_1, x_0) + \theta^{m-2}\mathrm{dist}(x_1, x_0) + \cdots + \theta^n \mathrm{dist}(x_1, x_0) \\
&= \theta^n \mathrm{dist}(x_1, x_0) \sum_{k=0}^{m-n-1} \theta^k \\
&\leq \frac{\theta^n \mathrm{dist}(x_1, x_0)}{1-\theta}.
\end{aligned}$$

For all $\epsilon > 0$, there exists the large $N > 0$ that satisfies

$$\frac{\theta^N \mathrm{dist}(x_1, x_0)}{1-\theta} < \epsilon.$$

Thus, for $m, n \geq N$, we have

$$\mathrm{dist}(x_m, x_n) \leq \frac{\theta^n \mathrm{dist}(x_1, x_0)}{1-\theta} < \epsilon.$$

Therefore, $\{x_n\}$ is the Cauchy sequence. Because X is complete, there exist x^* that satisfies

$$\lim_{n\to\infty} x_n = x^*.$$

Since contraction mapping f is obviously continuous, we have

$$\begin{aligned}
&\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} \\
&\Leftrightarrow f(x^*) = x^*,
\end{aligned}$$

thus $x^* \in X$ is a fixed point. Finally, we show the uniqueness of fixed point. If f has two fixed points x^*, y, these satisfy

$$0 \le \mathrm{dist}(x^*, y) = \mathrm{dist}(f(x^*), f(y)) \le \theta \mathrm{dist}(x^*, y),$$

thus we have

$$0 \le (1-\theta)\mathrm{dist}(x^*, y) \le 0, \quad \theta \in (0,1).$$

Thus $\mathrm{dist}(x^*, y) = 0$ and we get $x^* = y$. Eventually There exists the unique fixed point x^*. □

Note that an arbitrary Banach space X is the complete distance space because we can define the distance by any norm $\|\cdot\|_X$. For all $x, y \in X$, set $\mathrm{dist}(x, y) = \|x - y\|_X$, then this definition satisfies the axiom of the distance. We can easily check that the norm satisfies the following distance axiom.

(i) $\forall x, y \in X, \quad \mathrm{dist}(x, y) = \mathrm{dist}(y, x)$

(ii) $\forall x, y \in X, \quad \mathrm{dist}(x, y) = 0 \Rightarrow x = y.$

(iii) $\forall x, y, z \in X, \quad \mathrm{dist}(x, z) \le \mathrm{dist}(x, y) + \mathrm{dist}(y, z).$

Therefore, we can use Banach's fixed point theorem even if we replace the part of 'complete distance space' with 'Banach space'.

1.1.5 Existence of the solution of ODE

Using Banach's fixed point theorem, we can prove the existence of the solution of the following the differential equation.

Theorem 1.1.6. *Consider the following differential equation*

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{f}(\boldsymbol{u}), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{1.1.1}$$

here $\boldsymbol{u} : \mathbb{R} \to \mathbb{R}^m$ is continous bounded function that satisfies

$$|\boldsymbol{u}(t) - \boldsymbol{u}_0| < M$$

for $t \in (0, \tau)$. Note that M depends only on τ. Assume that $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^m$ is the Lipschitz-continuous and bounded function that satisfies

$$|\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v})| \le L|\boldsymbol{u} - \boldsymbol{v}|, \sup_{|\boldsymbol{u}-\boldsymbol{u}_0| \le M} |\boldsymbol{f}(\boldsymbol{u})| \le R.$$

with $L > 0, R > 0$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$. Then, for $t_0 > 0$, there exists the unique local solution that solve (1.1.1) on $t \in (0, t_0)$.

The Lipschitz continuity means that the solution is constrained by a linear function with some degree, and expresses a function that the rate of change cannot be infinite. Under this assumption, we can guarantee the uniqueness of the solution. Note that extending the existence of solutions locally in time, we can show global existence as long as they do not diverge.

Proof. Take the small $t_0 > 0$ such that

$$t_0 = \min\left\{\tau, \frac{M}{R}, \frac{1}{2L}\right\}.$$

Let X be the functional space consisted of all bounded continuous functions, namely,

$$X := \{\boldsymbol{u} \in C([0, t_0], \mathbb{R}^m) \mid \sup_{t \in (0, t_0)} |\boldsymbol{u}(t) - \boldsymbol{u}_0| \leq M\}.$$

X is the Banach space with the norm $\|\cdot\| = \sup_{t \in (0,t_0)} |\cdot|$.

Define the operator Φ such that

$$\Phi\boldsymbol{u} = \boldsymbol{u}_0 + \int_0^t \boldsymbol{f}(\boldsymbol{u}(s))ds.$$

At first, we show $\Phi : X \to X$. For $\boldsymbol{u} \in X$, by the definition of t_0, we have

$$|\Phi\boldsymbol{u} - \boldsymbol{u}_0| \leq \int_0^t |\boldsymbol{f}(\boldsymbol{u}(s))|ds \leq t_0 R \leq M.$$

Thus

$$\|\Phi\boldsymbol{u} - \boldsymbol{u}_0\| \leq M$$

and $\Phi\boldsymbol{u} \in X$ for $\boldsymbol{u} \in X$. Second, we show Φ is the contraction mapping. For any $\boldsymbol{u}, \boldsymbol{v} \in X$, we have

$$|\Phi\boldsymbol{u} - \Phi\boldsymbol{v}| \leq \int_0^t |\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v})| \leq t_0 L|u - v| \leq \frac{1}{2}|u - v|$$

for any $t \in (t, t_0)$, thus we have

$$\|\Phi\boldsymbol{u} - \Phi\boldsymbol{v}\| \leq \frac{1}{2}\|u - v\|.$$

Therefore, $\Phi : X \to X$ is the contraction mapping and we can apply the Banach's fixed point theorem. There exists the unique function $\boldsymbol{u}^* \in X$ that satisfies $\Phi\boldsymbol{u}^* = \boldsymbol{u}^*$ and

$$\boldsymbol{u}^* = \boldsymbol{u}_0 + \int_0^t \boldsymbol{f}(\boldsymbol{u}^*(s))ds.$$

This is nothing but the integral form of (1.1.1). □

1.1.6 L^p space

Consider the functional space of the function $u(x)$ in the spatial domain $\Omega \subset \mathbb{R}^n$ such that

$$\int_\Omega |u|^p d\boldsymbol{x} < \infty,$$

for $1 \le p < \infty$. Then, this space is called L^p space with norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p d\boldsymbol{x} \right)^{\frac{1}{p}}.$$

In case of $p = \infty$, the space of essential bounded function that satisfies

$$\text{ess.sup}_{\boldsymbol{x}\in\Omega} |u(\boldsymbol{x})| < \infty \tag{1.1.2}$$

is defined as L^∞ space with norm

$$\|u\|_{L^\infty} := \text{ess.sup}_{\boldsymbol{x}\in\Omega} |u(\boldsymbol{x})|.$$

For $1 \le p \le \infty, L^p$ is the Banach space with norm $\| \cdot \|_{L^p(\Omega)}$. Note that $\|u\|_{L^p(\Omega)} = 0$ doesn't mean $u \equiv 0$, but $u = 0$ except zero measure sets. It is often expressed as *almost everywhere*(a.e). Specifically, in case of $p = 2$, L^2 space is a kind of Hilbert space that is the complete functional space with the appropriate inner product. For any $f, g \in L^2(\Omega)$, we can define the inner product

$$(f, g)_{L^2} := \int_\Omega f g^* d\boldsymbol{x},$$

this is bounded by using the Cauchy Schwarz inequality and boundedness of $\|f\|_{L^2}, \|g\|_{L^2}$.

1.1.7 Sobolev space

At first, we define the weak derivative. Let u be the function in $L^p(\Omega)$ and the test function ϕ that satisfies $\phi = 0$ on $\partial\Omega$. Then, when $v \in L^p(\Omega)$ satisfies

$$\int_\Omega u \partial \phi d\boldsymbol{x} = - \int_\Omega v \phi d\boldsymbol{x},$$

v is the weak derivative of u and we write as $v := \partial u = \frac{\partial u}{\partial x_i} (i = 1, 2, \cdots, n)$. This is the definition by the integration by parts. In the same way, the higher derivative can be defined as the function v that satisfies

$$\int_\Omega u \partial^\alpha \phi d\boldsymbol{x} = (-1)^{|\alpha|} \int_\Omega v \phi d\boldsymbol{x},$$

here α is the multi-index

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), \quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

and ∂^α means

$$\partial^\alpha := \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Then, for $m \in \mathbb{N}, 1 \leq p < \infty$ we can define the following functional space $W^{m,p}$:

$$W^{m,p} = \{u \in L^p(\Omega) | \partial^\alpha u \in L^p(\Omega), \quad |\alpha| \leq m\}.$$

$W^{m,p}$ is the Banach space with norm

$$\|u\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

and it is called the Sobolev space. Specifically, $W^{m,2}$ is often written as H^m and is the Hilbert space with the following inner product

$$(f, g)_{H^m(\Omega)} := \sum_{|\alpha| \leq m} (\partial^\alpha f, \partial^\alpha g)_{L^2(\Omega)}$$

for any $f, g \in H^m(\Omega)$. Additionally, define $W_0^{m,p}(\Omega)$ as

$$W_0^{m,p}(\Omega) = \{u \in W^{m,p}(\Omega) | \overline{\{x \in \Omega | u(x) \neq 0\}} \subset \Omega\}.$$

$W_0^{m,p}(\Omega)$ is Sobolev space to which functions satisfying Dirihclet boundery condition belong.

1.1.8 Sobolev's embedding theorem

At first, we define the following Hölder space $C^{k,\sigma}(\mathbb{R}^n)$ for $k \in \mathbb{Z}, \sigma \in (0, 1]$: The function $u \in C^k(\mathbb{R}^n)$ and the $k-$th order derivative $\partial^\alpha u|_{|\alpha|=k}$ is uniformly σ-th order Hölder continuous that is

$$\exists L > 0, \forall x, y \in \mathbb{R}^n, \quad s.t. \quad |\partial^\alpha u(x) - \partial^\alpha u(y)| \leq L|x - y|^\sigma.$$

$C^{k,\sigma}(\mathbb{R}^n)$ is the Banach space with norm

$$\|u\|_{C^{k,\sigma}(\mathbb{R}^n)} := \sum_{|\alpha| \leq k} |\partial^\alpha u(x)| + \sum_{|\alpha|=k} \sup_{x \neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x - y|^\sigma}.$$

Then, there is the following theorem about the relation between Sobolev space $W^{m,p}$ and Hölder space $C^{k,\sigma}(\mathbb{R}^n)$ and their embedding.

Definition 1.1.7 (Embedding)**.** *Let X, Y be functional spaces such that $X \subset Y$ with norm $\|\cdot\|_X, \|\cdot\|_Y$. For any function $u \in X$, if*

$$\|u\|_Y \leq C\|u\|_X$$

is satisfied for some constant $C > 0$ that depends only on Ω, then we say X is embedding to Y and write as $X \hookrightarrow Y$.

Theorem 1.1.8 (Sobolev's embedding theorem for $\mathbb{R}^n$, [29])**.** *For $W^{m,p}(\mathbb{R}^n)(m \in \mathbb{N}, 1 \leq p < \infty)$, we have following statements:*

(i) *In case of $m - n/p < 0$: Set $p^* \in (p, \infty)$ be $1/p^* = 1/p - m/n$. Then, for all $q \in [p, p^*]$, we have $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$.*

(ii) *In case of $m - n/p = 0$: For all $q \in [p, \infty)$, we have $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$.*

(iii) *In case of $m - n/p > 0$ and $m - n/p$ is not an integer: Then we can write as $m - n/p > 0 = k + \sigma$ for $k \in \mathbb{N}, \sigma \in (0, 1)$, and we have $W^{m,p}(\mathbb{R}^n) \hookrightarrow C^{k,\sigma}(\mathbb{R}^n)$.*

(iV) *In case of $m - n/p > 0$ and $m - n/p$ is an integer: Set $k \in \mathbb{N}$ as $m - n/p = k$. Then, for any $\sigma \in (0, 1)$, we have $W^{m,p}(\mathbb{R}^n) \hookrightarrow C^{k-1,\sigma}(\mathbb{R}^n)$.*

Example 1.1.9. *Consider the $m = 2, p = 1, n = 3$, that is $m = n/p = 2 - 3/1 < 0$ (case(i) of Theorem 1.1.8), then for any $q \in [2, 6]$, we have $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$.*

Next, to apply the embedding theorem for a bounded domain $\Omega \subset \mathbb{R}^n$, we define the following expansion operator:

Definition 1.1.10 (expansion operator)**.** *For $m \in \mathbb{N}, p \in [1, \infty]$, set the bounded linear operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$ that satisfies*

$$\forall u \in W^{m,p}(\Omega), Eu|_\Omega = u,$$

here $u|_\Omega : \Omega \to \mathbb{R}; x \mapsto u(x)$. Then, E is called the expansion operator.

The expansion operator is applied as follows:

Theorem 1.1.11 (Sobolev's embedding theorem for Ω, [29])**.** *If there is the expansion operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$ for $\Omega \subset \mathbb{R}^n$, then we can replace parts of $\mathbb{R}^n$ to Ω in Sobolev's embedding theorem 1.1.8. That is, for $W^{m,p}(\Omega) \quad (m \in \mathbb{N}, 1 \leq p < \infty)$, we have following statements:*

(i) *In case of $m - n/p < 0$: Set $p^* \in (p, \infty)$ be $1/p^* = 1/p - m/n$. Then, for all $q \in [p, p^*]$, we have $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$.*

(ii) *In case of* $m - n/p = 0$: *For all* $q \in [p, \infty)$, *we have* $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$.

(iii) *In case of* $m-n/p > 0$ *and* $m-n/p$ *is not an integer: Then we can write as* $m-n/p > 0 = k+\sigma$ *for* $k \in \mathbb{N}, \sigma \in (0,1)$, *and we have* $W^{m,p}(\Omega) \hookrightarrow C^{k,\sigma}(\Omega)$.

(iv) *In case of* $m-n/p > 0$ *and* $m-n/p$ *is an integer: Set* $k \in \mathbb{N}$ *be* $l = m-n/p$. *Then, for any* $\sigma \in (0,1)$, *we have* $W^{m,p}(\Omega) \hookrightarrow C^{k-1,\sigma}(\Omega)$.

Proof. It might be sufficient to prove the case (i). Other cases are proved in the same way. From the assumption, $Eu \in W^{m,p}(\mathbb{R}^n)$ for $u \in W^{m,p}(\Omega), m - n/p < 0$. Additionally, because E is a bounded linear operator, for some $C > 0$, we have

$$\|Eu\|_{W^{m,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{m,p}(\Omega)}.$$

By case (i) of Theorem 1.1.8, for $q \in [p, p*]$, we have

$$\|Eu\|_{L^q(\mathbb{R}^n)} \leq C'\|Eu\|_{W^{m,p}(\mathbb{R}^n)}$$

for some constant C' .Therefore, we have

$$\begin{aligned} \|u\|_{L^q(\Omega)} &= \|Eu\|_{L^q(\Omega)} \\ &\leq \|Eu\|_{L^q(\mathbb{R}^n)} \\ &\leq C'\|Eu\|_{W^{m,p}(\mathbb{R}^n)} \\ &\leq CC'\|u\|_{W^{m,p}(\Omega)}, \end{aligned}$$

and $W^{m,p}(\Omega) \hookrightarrow L^q_\Omega$. □

Finally, we mention the condition of the existence of the expansion operator.

Theorem 1.1.12 ([29])**.** *If the spatial domain* Ω *has the* C^1 *smooth boundary, or satisfies the cone condition, there exists the expansion operator* E *on* Ω.

Definition 1.1.13 (cone condition, [29])**.** *Let* $\Omega \subset \mathbb{R}^n$ *be the open set with the bounded boundary that is not empty. Then,* Ω *satisfies the cone condition if the following conditions are satisfied: There are* $\{O_i\}_{\{i \geq 0\}}$, *the sequence of finite opened cover of* $\partial\Omega$ *and the family of conic domain* $\{C_i\}$, *and if they satisfies* $x + C_i \subset \Omega$ *for any* $i \in \mathbb{N}, x \in \Omega \cap O_i$, Ω *then satisfies the cone condition.*

For example, polygonal domains, cubic domains, or the domains with Lipschitz boundery satisfie the cone condition.

1.1.9 The Poincaré inequality

Theorem 1.1.14 ([29, 32]). *Let p be a real number satisfying $p \in [1, \infty)$ and Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundery. Consider $\boldsymbol{u} \in W_0^{1,p}(\Omega)$, then there exists a positive constant $\boldsymbol{C}$ depending only on Ω, p such that $\boldsymbol{u}$ satisfies*

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

When $\boldsymbol{u}$ satisfies the Neumann boundary condition $\nabla \boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ where $\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$, the above inequality does not hold because $\nabla \boldsymbol{u} = 0$ for any constant $\boldsymbol{u}$. The alternate following inequality holds:

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

where

$$\bar{u} = \frac{1}{|\Omega|} \int_\Omega u(x)dx,$$

and $|\Omega|$ is the measure of Ω.

1.2 Sectorial operator

Let X be a Banach space, the norm in the Banach space be given by $\|\cdot\|$, and A be the closed operator which has a dense domain in X. Consider the partial differential equation of $\boldsymbol{u}(\boldsymbol{x}, t)$ as follows:

$$\frac{du}{dt} + Au = 0.$$

For example, in case of $A = -\Delta$, this is the heat equation. Even if A is an operator on the Banach space, can we express the solution of (1.2.3) as $u(t) = e^{-At}u(0)$? The answer is "yes" if A is a *sectorial operator*. Hence, we will explain about the sectorial operator.

Definition 1.2.1. *Assume that the resolvent set $\rho(A)$ of A contains the open set on the complex plain with some real numbers $l, \theta \in (0, \pi/2)$ as follows:*

$$\rho(A) \subset \Sigma_{l,\theta} = \{\lambda \in \mathbb{C} \mid \theta \leq |\arg(\lambda - l)| \leq \pi, \lambda \neq l\},$$

and satisfies

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - l|},$$

for any $\lambda \in \Sigma_{l,\theta}$ with some $M \geq 1$. Then, A is called a sectorial operator.

Lemma 1.2.2. *Let* $-\Delta$ *be the Laplacian with the Neumann boundary condition on* $\Omega \subset \mathbb{R}^n$ *and the domain* $D(A) = \{u \in H^1(\Omega) | \nabla u \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega\}$. *For* $u, v \in D(A)$, *assume that* $-\Delta$ *satisfies the weak form*

$$(-\Delta u(x), v) = \int_\Omega \nabla u \cdot \nabla v d\boldsymbol{x},$$

with inner product

$$(f, g) = \int_\Omega f g d\boldsymbol{x}, \quad \text{for} \quad f, g \in H^1(\Omega)$$

Then, $-\Delta$ *is the sectorial operator on* $L^2(\Omega)$.

Proof. For any eigenfunction $\phi \in D(A)$, consider the eigenvalue problem

$$-\Delta\phi = \lambda\phi$$

with the Neumann boundary condition. Then, we have

$$\begin{aligned} \lambda \|\phi\|^2_{L^2(\Omega)} &= (\lambda\phi, \phi) \\ &= (-\Delta\phi, \phi) \\ &= \int_\Omega |\nabla\phi\|^2 d\boldsymbol{x} \\ &= \|\nabla\phi\|^2_{L^2(\Omega)} \\ &\geq C\|\phi - \overline{\phi}\|^2_{L^2(\Omega)} \end{aligned}$$

for some positive constant $C > 0$. Here we use the Poincaré inequality and $\overline{\phi}$ is an average of ϕ on Ω. Thus for $\phi \neq 0$, $\lambda \geq C > 0$. Obviously $\lambda = 0$ in case of $u = 0$, finally we get $\lambda \geq 0$. All eigenvalue of $-\Delta$ lies in $R_+ \cap \{0\}$. Therefore $-\Delta$ is the sectorial operator. □

Example 1.2.3 ([9, 27])**.** *Due to the definition, we can give the following examples:*

(1) *If* A *is a bounded linear operator on a Banach space, then* A *is sectorial.*

(2) *If* A *is a self-adjoint densely defined operator in a Hilbert space, and if* A *is bounded below, then* A *is sectorial.*

(3) *If* A *is sectorial in* X, B *is sectorial in the another Banach space* Y, *then* $A \times B$ *is also sectorial in* $X \times Y$, *where* $(A \times B)(x, y) = (Ax, By)$ *for* $x \in D(A), y \in D(B)$.

(4) *If* $Au(x) = -\Delta u(x)$, *when* $u \in C_0^2(\Omega)$ $(\Omega \subset \mathbb{R}^m)$,*and* A *is the closure in* $L_p(\Omega)$ *of* $-\Delta|_{C_0^2(\Omega)}(1 \leq p < \infty)$, *then* A *is sectorial.*

(5) *The elliptic operator* L *defined as*

$$Lu := \sum_{1 \leq i,j \leq n} \frac{\partial}{\partial x_i} a_i(x) \frac{\partial u}{\partial x_j} + c(x)u$$

with $u \in L^2(\Omega)$ *under Neumann or Dirichlet boundary condition is sectorial.*

(6) *If* A *is sectorial in* X *and* B *is a linear operator with* $D(B) \in D(A)$ *and for all* $x \in D(A), \|Bx\| \leq \epsilon\|Ax\| + K(\epsilon)\|x\|$ *(for sufficiently small* $\epsilon > 0$*), then* $A + B$ *is sectorial.*

To prove some properties of the sectorial operator, we define *analytic semigroup* as follows:

Definition 1.2.4. *An analytic semigroup on a Banach space* X *is a family of continuous linear operators on* X*, a function* $\{T(t)\}_{t \geq 0}$ *satisfying*

(i) $T(0) = I, T(t)T(s) = T(t+s)$ *for* $t, s \geq 0$.

(ii) $T(t)x \to x$ *as* $t \to 0+$, *for each* $x \in X$

(iii) $t \to T(t)x$ *is real analytic on* $0 < t < \infty$ *for each* $x \in X$.

The infinitesimal generator L *of this semigroup is defined by*

$$Lx = \lim_{t \to 0+} \frac{1}{t}(T(t)x - x),$$

with the domain $D(L)$ *consisting of all* $x \in X$ *for which this limit (in* X*) exists. The expression* $T(t) = e^{Lt}$ *is often used.*

Lemma 1.2.5 ([9, Theorem 1.3.4])**.** *Let set the Banach space* X, Y *and* $A : X \to Y$*. Assume that the domain* $D(A) \in X$ *is dense in* X*. Then,* $-A$ *is the infinitesimal generator of an analytical semigroup* $\{e^{-tA}\}_{t>0}$ *defined by*

$$e^{-tA} = \frac{1}{2\pi i} \int_\Gamma (\lambda + A)^{-1} e^{\lambda t} d\lambda, \tag{1.2.3}$$

where Γ *is a contour in* $\rho(A)$ *with* $\arg\lambda \to \pm\theta$ *as* $|\lambda| \to \infty$ *for some* θ *in* $(\frac{\pi}{2}, \pi)$*. Further* e^{-tA} *can be continued analytically into* $\{t \neq 0 \mid |\arg t| < \varepsilon\}$*. If* $\operatorname{Re}\lambda > a$ *for all* $\lambda \in \sigma(A)$*, then*

$$\|e^{-tA}\| \leq Ce^{-at}, \|Ae^{-tA}\| \leq \frac{C}{t}e^{-at} \tag{1.2.4}$$

for some constant C *and* $t > 0$*. Finally,*

$$\frac{d}{dt}e^{-tA} = -Ae^{-tA}$$

is satisfied.

Proof. Without loss of the generality, assume $a = 0$ and $\|(\lambda+A)^{-1}\| \leq M/|\lambda|+\delta$ for $|\pi - \arg\lambda| \geq \phi$ for some constants $\delta > 0, M > 0$ and $\phi \in (0, \pi/2)$, otherwise replace A by $A - \lambda I$. Choose $\theta \in (\pi/2, \pi - \phi)$. Define e^{-tA} by (1.2.3). Note that the integral in (1.2.3) converges absolutely when $t > 0$. From Cauchy's integral theorem, the integral is unchanged when the contour Γ is shifted to the right a small distance. We denote the shifted contour by Γ' Then, for $t > 0, s > 0$

$$\begin{aligned} e^{-tA}e^{-sA} &= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} e^{\lambda t}(\lambda I + A)^{-1}e^{\mu s}(\mu I + A)^{-1}d\mu d\lambda \\ &= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} e^{\lambda t \mu s}\frac{(\lambda I + A)^{-1} - (\mu I + A)^{-1}}{(\mu - \lambda)}d\mu d\lambda, \end{aligned}$$

using the resolvent identity. Additionally, for $\lambda \in \Gamma, \mu \in \Gamma'$,

$$\int_\Gamma e^{\lambda t}(\mu - \lambda)^{-1}d\lambda = 0,$$
$$\int_{\Gamma'} e^{\mu s}(\mu - \lambda)^{-1}d\mu = 2\pi i e^{\lambda s},$$

thus we have

$$e^{-tA}e^{-sA} = \int_\Gamma \frac{1}{2\pi i}e^{\lambda(t+s)}(\lambda I + A)^{-1}d\lambda = e^{-A(t+s)}.$$

Thus, $\{e^{-tA}\}_{t\geq 0}$ is the semigroup. For $0 < \epsilon < \theta - \pi/2$, the integral converges uniformly in any compact set of $\{|\arg t| < \epsilon\}$, thus the semigroup is analytic. For $t > 0, z = \lambda t$ in the integral

$$\|e^{-tA}\| = \left\|\frac{1}{2\pi i}\int_\Gamma e^z(\frac{z}{t} + A)^{-1}\frac{dz}{t}\right\| \leq \frac{M}{2\pi}\int_\Gamma |e^z|\frac{|dz|}{|z|}$$

and

$$\|Ae^{-tA}\| \leq \frac{M}{2\pi\delta}\int_\Gamma |e^z|\frac{|dz|}{|z|}\cdot\frac{1}{t}.$$

These inequalities immediately imply (1.2.4).

Next, we prove $e^{-tA}x - x$ as $t \to 0+$ for each $x \in X$. By the assumption, it is sufficient to prove this for $x \in D(A)$ For $x \in D(A), t > 0$, since $\|e^{-tA}\| \leq C$ for all $t \geq 0$,

$$e^{-tA}x - x = \int_\Gamma e^{\lambda t}((\lambda I + A)^{-1} - \lambda^{-1})x d\lambda$$
$$= -\frac{1}{2\pi i}\int_\Gamma \lambda^{-1}e^{\lambda t}A(\lambda I + A)^{-1}x d\lambda,$$

thus $\|e^{-tA}x - x\| \leq Constant \cdot \|Ax\|t$. Thus $\{|\arg t| < \epsilon\}$ is a strongly continuous semigroup and can extends to analytic semigroup in $|\arg t| < \epsilon$. In the same way, we have

$$\frac{d}{dt}e^{-tA}x + Ae^{-tA}x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A)(\lambda + A)^{-1}d\lambda = 0.$$

Namely,

$$\frac{d}{dt}e^{-tA}x = -Ae^{-tA}x$$

for $x \in D(A)$ as $t > 0$. □

We can define the fractional powers of A when $\operatorname{Re}\sigma(A) > 0$.

Definition 1.2.6. *Suppose A is a sectorial operator and $\operatorname{Re}\sigma(A) > 0$, then $A^{-\alpha}$ is defined as*

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-tA}dt.$$

for any $\alpha > 0$.

Example 1.2.7 ([9, 27])**.** *Due to the definition, we can give the following examples:*

(1) *If A is a positive scalar ($X = \mathbb{R}$), then $A^{-\alpha}$ is the usual $(-\alpha)$ power of A.*

(2) *If A is a self-adjoint definite, self-adjoint operator in a Hilbert space with spectral representation $A = \int_0^\infty \lambda dE(\lambda)$, then $A^{-\alpha} = \int_0^\infty \lambda^{-\alpha}dE(\lambda)$*

(3) *If $\alpha = 1$, A^{-1} is the inverse of A.*

We can calculate $A^{-\alpha}$ in a following way:

Lemma 1.2.8 ([9, Theorem 1.4.2]). *If A is a sectorial operator in Banach space X with $\operatorname{Re}\sigma(A) > 0$, then for any $\alpha > 0, A^{-\alpha}$ is a bounded linear operator on X that satisfies*

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1} d\lambda$$

Note that $A^{-\alpha}$ is the inverse of $A^\alpha(\alpha > 0)$, $D(A^\alpha) = R(A^{-\alpha})$, A^0 is the identity on X

Proof. By the assumption, there is a positive constant δ such that $\operatorname{Re}\sigma(A) > \delta$. Then, Lemma 1.2.5 implies that $\|e^{-tA}\| \leq Ce^{-\delta t}$ for $t > 0$. Thus for $x \in X$,

$$\|A^{-\alpha}x\| \leq \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} Ce^{-\delta t} dt \|x\|,$$

and $A^{-\alpha}$ is bounded when α with $\alpha > 0$. Also for $\alpha, \beta > 0$

$$\begin{aligned}
A^{-\alpha}A^{-\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-A(t+s)} ds dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty e^{-uA} du \int_0^u t^{\alpha-1}(u-t)^{\beta-1} dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty u^{\alpha+\beta-1}e^{-uA} du \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} dz \\
&= A^{-(\alpha+\beta)},
\end{aligned}$$

where we used the beta function $B(\alpha, \beta)$ as

$$B(\alpha, \beta) = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} dz = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}.$$

Finally, $(\lambda + A)^{-1} = \int_0^\infty e^{-tA}e^{-\lambda t} dt$ for $\lambda \geq 0$, so

$$\begin{aligned}
\int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1} d\lambda &= \int_0^\infty e^{-tA}\left(\int_0^\infty e^{-\lambda t}\lambda^{-\alpha} d\lambda\right) dt \\
&= \int_0^\infty e^{-tA}t^{\alpha-1}\Gamma(1-\alpha) dt \\
&= \int_0^\infty A^{-\alpha}\Gamma(\alpha)\Gamma(1-\alpha) \\
&= A^{-\alpha}\frac{\pi}{\sin \pi\alpha},
\end{aligned}$$

using the reflective formula

$$\frac{\pi}{\sin \pi\alpha} = \Gamma(\alpha)\Gamma(1-\alpha).$$

□

Next, we should show the evaluation about $A^{-\alpha}$ as follows:

Lemma 1.2.9 ([9, Theorem 1.4.3]). *Suppose A is a sectorial and* $\mathrm{Re}\,\sigma(A) > \delta > 0$. *There exists a positive constant C_α depending on α such that*

$$\|A^\alpha e^{-tA}\| \leq C_\alpha t^{-\alpha} e^{-\delta t}$$

for $t > 0$.

Proof. From Lemma 1.2.5,

$$\|e^{-tA}\| \leq Ce^{-\delta t}, \|Ae^{-tA}\| \leq \frac{Ce^{-\delta t}}{t}$$

for $t > 0$. thus for $m \in \mathbb{Z}$,we have

$$\|A^m e^{-tA}\| = \|(Ae^{-(t/m)A})^m\| \leq \frac{(Cm)^m e^{-\delta t}}{t^m}$$

If $0 < \alpha < 1, t > 0$,

$$\begin{aligned}
\|A^\alpha e^{-tA}\| &= \|Ae^{-tA} \cdot A^{-(1-\alpha)}\| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} \|Ae^{-(t+s)A}\| ds \\
&\leq Ct^{-\alpha} e^{-\delta t} \Gamma(\alpha).
\end{aligned}$$

Additionally, we have

$$\|A^{\alpha+\beta} e^{-tA}\| \leq \|A^\alpha e^{-tA/2}\| \|A^\beta e^{-tA/2}\| \leq C_\alpha C_\beta 2^{\alpha+\beta} t^{-(\alpha+\beta)} e^{-\delta t}.$$

□

Lemma 1.2.10 ([9, Theorem 1.4.4]). *For $\beta \in [0, 1], x \in D(A)$,*

$$\|A^\beta\| \leq C\|Ax\|^{1-\beta} \|x\|^\beta.$$

Proof. Let $0 < \beta < 1, \epsilon > 0$, thus we have

$$\begin{aligned}
\|\Gamma(\beta) A^{-\beta} x\| &= \left\| \left(\int_0^\epsilon + \int_\epsilon^\infty \right) t^{\beta-1} e^{-tA} x dt \right\| \\
&\leq C\|x\| \frac{\epsilon^\beta}{\beta} + \left\| \epsilon^{\beta-1} e^{-\epsilon A} A^{-1} x + (\beta - 1) \int_\epsilon^\infty t^{\beta-2} e^{-tA} A^{-1} x dt \right\| \\
&\leq C\|x\| \frac{\epsilon^\beta}{\beta} + 2C\|A^{-1}x\| \epsilon^{\beta-1}.
\end{aligned}$$

The right hand side can be minimized as

$$\|\Gamma(\beta)A^{-\beta}x\| \le \frac{2(2(1-\beta))^{\beta-1}}{\Gamma(1+\beta)} C\|x\|^{1-\beta}\|A^{-1}x\|^{\beta}.$$

□

Lemma 1.2.11. *Suppose A, B are sectorial in X with $D(A) = D(B)$,* $\operatorname{Re}\sigma(A) > 0$*, and for some $\alpha \in (0,1)$, $(A-B)A^{-\alpha}$ is bounded on X. Then for any $\beta \in (0,1)$, $A^{-\beta}B^{-\beta}$ and $B^{-\beta}A^{-\beta}$ are bounded in X.*

Proof. From Lemma 1.2.8, $\|A^{\beta}(\lambda+A)^{-1}\| \le C|\lambda|^{\beta-1}$ for $0 \le \beta \le 1$, $|\pi - \arg\lambda| \ge \phi$ for some constants C and $\phi < \pi/2$. For $0 < \beta < 1$,

$$B^{-\beta}A^{-\beta} = \frac{\sin\pi\beta}{\pi}\int_0^{\infty} \lambda^{-\beta}(\lambda+B)^{-1}(A-B)(\lambda+A)^{-1}d\lambda$$

is bounded. In the same way, $A^{-\beta}B^{-\beta}$ is also bounded. □

We can define the Banach space X^{α} with a $\|\cdot\|_{\alpha}$-norm as follows. The space X_{α} provide a basic topology to prove Theorem 1.3.1.

Definition 1.2.12. *Let A be a sectorial operator in a Banach space X with norm $\|\cdot\|$. Define for each $\alpha \ge 0$*

$$X^{\alpha} = D(A_1^{\alpha})$$

with the graph norm $\|x\|_{\alpha} = \|A_1 x\|$, $x \in X^{\alpha}$, where $A_1 = A + aI$ with a chosen as $\operatorname{Re}\sigma(A_1) > 0$*. A different choice of a gives equivalent norms on X^{α}.*

Theorem 1.2.13 ([9, Theorem 1.4.8])**.** *Define X^{α} as Definition 1.2.12. for $\alpha \ge \beta \ge 0$, $X^0 = X$, X^{α} is a dense subspace of X^{β} with the continuous inclusion. If A has compact resolvent, the inclusion $X^{\alpha} \subset X^{\beta}$ is compact. Let A_1, A_2 be sectorial operators in X with same domain and* $\operatorname{Re}\sigma(A_j) > 0$ *for $j = 1, 2$. If $(A_1 - A_2)A_1^{-\alpha}$ is a bounded operator for some $\alpha < 1$, then $X_1^{\beta} = X_2^{\beta}$ with $X_j^{\beta} = D(A_j^{\beta})$ and equivalent norms for $0 \le \beta \le 1$.*

1.3 Asymptotic stability of periodic solutions of complex Ginzburg-Landau equation

There is the following theorem about conversion to spatial homogeneous periodic solution on a kind of PDE. Consider a solution of

$$\boldsymbol{u}_t + A\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{u}), \tag{1.3.5}$$

where A is the sectorial operator on the Banach space X with norm $\|\cdot\|$ and $\boldsymbol{f}$: $X^\alpha \to X$ is differentiable and Hölder continuous, here X^α is also the Banach space with norm $\|\cdot\|_\alpha$.

Theorem 1.3.1 ([9]). *Assume that this equation (1.3.5) has non-constant spatial homogeneous periodic solution $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{p}(t)$ with a period T, and $\Gamma = \{\boldsymbol{p}(t); 0 < t < T\} \subset X^\alpha$ with some $\alpha > 0$, and the characteristic multiplier 1 is an isolated simple eigenvalue, others satisfy $|\mu| < e^{-\beta T}$ for some $\beta > 0$. Then, the trajectory Γ is asymptotically stable. Namely, there exist positive constants ρ and M such that if*

$$\mathrm{dist}_{X^\alpha}(\boldsymbol{u}(\boldsymbol{x},0),\Gamma) = \min_t \|\boldsymbol{u}(\boldsymbol{x},0) - \boldsymbol{p}(t)\|_\alpha < \frac{\rho}{2M},$$

then the solution $\boldsymbol{u}(t)$ exists for all $t > 0$ and satisfies

$$\|\boldsymbol{u}(t) - \boldsymbol{p}(t-\boldsymbol{\theta})\|_\alpha < 2\rho e^{-\beta t}$$

for all $t > 0$ with some $\theta = \theta(\boldsymbol{u}(\boldsymbol{x},0))$.

1.4 Characteristic multiplier

Let $B(t) \in \mathbb{R}^{n\times n}$ be the square matrix with a time period T,and Y be the unknown vector sequence as $Y = [\boldsymbol{y_1}(t), \boldsymbol{y_2}(t), \cdots, \boldsymbol{y_n}(t)]$. We consider

$$\frac{dY}{dt} = BY, \quad Y(0) = C.$$

$Y(t), Y(t+T)$ is written as

$$Y(t) = \Phi(t)C, \quad Y(t+T) = \Phi(t)D, \quad \Phi(0) = I$$

by $\Phi(t)$. Then, $F = C^{-1}D$ is called fundamental matrix of $Y(t)$, $\Phi(T)$ is the fundamental matrix of $\Phi(t)$. We can calculate as

$$Y(t+T) = \Phi(t+T)C = \Phi(t)\Phi(T)C = Y(t)C^{-1}\Phi(T)C,$$

thus we have

$$F = C^{-1}\Phi(T)C.$$

This implies $\Phi(T)$ and F have identical eigenvalues. thus eigenvalues of $\Phi(T)$, that is solutions of $\det(\mu I - \Phi(T))$ are representatively called characteristic multiplier. moreover,

$$\nu = \frac{1}{T}\log\mu$$

are called Lyapunov exponent. Lyapunov exponent expresses the intensivity how two nearby solution trajectories separate as time development. A positive Lyapunov exponent means the solution of the dynamic system has the initial sensitivity, negative Lyapunov exponent means that nearby orbits collapse and converge, and zero Lyapunov exponent means solution has a periodicity.

About the stability of the periodic solution $\boldsymbol{p}(t)$, we show following theorem

Theorem 1.4.1. *Let $\boldsymbol{p}(t) \in \mathbb{R}^2$ be the periodic solution of plane dynamic system*

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{f}(\boldsymbol{u}), \quad \boldsymbol{u} \in \mathbb{R}^2$$

and its linearized equation of $\boldsymbol{\xi} = \boldsymbol{u} - \boldsymbol{p}(t)$ near the $\boldsymbol{p}(t)$ can be written as

$$\frac{d\boldsymbol{\xi}}{dt} = J(\boldsymbol{p}(t))\boldsymbol{\xi}.$$

If

$$\int_0^T \mathrm{Tr}(J)dt < 0,$$

then $\boldsymbol{p}(t)$ is stable except for a phase shift.

Proof. $\boldsymbol{p}(t)$ is the periodic solution and satisfies

$$\frac{d\boldsymbol{p}(t)}{dt} = \boldsymbol{f}(\boldsymbol{p}(t)).$$

Differentiate by t, then we have

$$\frac{d^2\boldsymbol{p}(t)}{dt^2} = J(\boldsymbol{p}(t))\frac{d\boldsymbol{p}(t)}{dt}.$$

This implies that $\boldsymbol{\xi} = \frac{d\boldsymbol{p}(t)}{dt}$ is the solution of linearlized equation and we can written as

$$\frac{d\boldsymbol{p}(T)}{dt} = \Phi(T)\frac{d\boldsymbol{p}(0)}{dt} = \frac{d\boldsymbol{p}(0)}{dt}.$$

thus 1 is a characteristic multiplier and $\frac{d\boldsymbol{p}(0)}{dt}$ is the one of eigenvector of $\Phi(T)$. $\frac{d\boldsymbol{p}(0)}{dt}$ is tangent vector of periodic orbit at $t = 0$. By Liouville's theorem, we have

$$\nu_1\nu_2 = \det\Phi(T) = e^{\int_0^T \mathrm{Tr}(J)dt}.$$

From the assumption, we have two different characteristic multipliers are $1, e^{\int_0^T \mathrm{Tr}(J)dt}(< 1)$. If all characteristic multipliers satisfie $|\mu| < 1$,

$$|\boldsymbol{\xi}(nT)| = |\boldsymbol{\xi}(0)||\Phi(T)^n| \to 0.$$

This implies 1 characteristic multiplier means the trajectory doesn't variant along the corresponding eigenvector. thus the periodic solution is stable except for the phase shift. □

1.4.1 Calculation of Lyapunov exponent and characteristic multiplier

To calculate Lyapunov exponent and characteristic multiplier of $d\boldsymbol{u}(t)dt = \boldsymbol{f}(\boldsymbol{u})$, there is the following algorithm.

(1) Calculate numerical solution until sufficiently long time t_1

(2) Discretize as

$$\boldsymbol{u}_{n+1} = (I + \Delta t J(\boldsymbol{u}_n))\boldsymbol{u}_n = A_n \boldsymbol{u}_n,$$

here J is the Jacobian of $\boldsymbol{f}$ and Δt is time step, and get $A_n = A_n(\boldsymbol{u}_n)$ for $t \geq t_1$.

(3) Let $Q_0 = I$, then numerically calculate QR-decomposition of $A_n Q_{n-1} = Q_n R_n$ for each time step, $n \geq 0$. Q_n, R_n are an orthogonal matrix and a lower triangle matrix respectively. QR-decomposition is also called Gram-Schmidt decomposition. We can get QR-decomposition of square matrix A by using the Gram-Schmidt's orthogonalization with regarding A as n-dimensional linear independent vectors.

(4) Calculate Lyapunov exponents ν_i as

$$\nu_i = \frac{1}{N\Delta t} \sum_n^N \log |(R_n)_{ii}|$$

for sufficiently large N.

(5) If the solution has time-periodicity, we can calculate characteristic multipliers as

$$\mu_i = e^{\nu_i T}$$

for the time period T.

Chapter 2

Fundamental properties of reaction-diffusion systems

2.1 Reaction-diffusion systems

Reaction-diffusion systems are used as a mathematical model to describe many phenomena interacting reactions and diffusion in space such as population dynamics, changes in population numbers due to interactions between phytoplankton and zooplankton, changes in the membrane potential of the heart and nerve cells, and body surface patterns in fish and mammals. generation, chemical reaction systems such as the Belousov-Zhavodzinski reaction, Rayleigh-Benard convection, combustion theory, and phase transitions. A reaction-diffusion system is a kind of partial differential equation consisting of two kinds of elements, a reaction term and a diffusion term. Set Ω be a bounded region in $\mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$ be unknown m-compact functions that depend on position $\boldsymbol{x} = (x_1, \cdots, x_n)$ and time t, and a diagonal matrix D like as $\mathrm{diag}(d_1, d_2, \cdots, d_m)$ be diffusion coefficient. Then, consider the following differential equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} = D\Delta \boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}) \tag{2.1.1}$$

where Δ means n-dimensional Laplacian that expresses

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

This equation is called a *reaction diffusion system*, and the first and second term on the right hand side are called diffusion term and reaction term respectively. This equation is in the form of a heat equation with a nonlinear term. Additionally, consider the case that $d_1 = d_2 = \cdots = d_m = d$ (all diffusion coefficients are same value). Reaction diffusion systems, especially in the case of equal diffusion coefficients, are related to ODE like

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{f}(\boldsymbol{u}),$$

thus we can apply an effective theorem for analyze the solution. As we will explain later, bounded convex positive invariant sets can be constructed for reaction-diffusion systems when the diffusion coefficients are equal. Specifically, if we can construct a positive invariant set that becomes a bounded convex region of the reaction equation(ODE) by considering the confinement of the vector field, then this region is the positive invariant sets of the reaction-diffusion system when the diffusion coefficients are equal. In this paper, we mainly treat this case.

2.2 Existence of solutions and the Schauder estimates on R^n

Set $\boldsymbol{u} \in \mathbb{R}^m$ be unknown m-compact functions that depend on position $\boldsymbol{x} = (x_1, \cdots, x_n)$ and time t, and a diagonal matrix D like as $\mathrm{diag}(d_1, d_2, \cdots, d_m)$ be diffusion coefficients. Then, consider the following equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} = D\Delta \boldsymbol{u}, \qquad \boldsymbol{x} \in \mathbb{R}^n, t > 0. \tag{2.2.2}$$

This equation is called a *heat equation*. Let X be a set of all bounded and uniformly continuous functions, then X is a Banach space with the norm

$$\|\boldsymbol{u}\| := \sup_{\boldsymbol{x} \in \mathbb{R}^n} |\boldsymbol{u}(\boldsymbol{x})|.$$

The heat equation has fundamental solution

$$K_j(\boldsymbol{x}, t) = \frac{1}{(4\pi d_j t)^{\frac{n}{2}}} e^{-\frac{|\boldsymbol{x}|^2}{4 d_j t}},$$

that satisfies

$$\frac{\partial K_j}{\partial t} = d_j \Delta K_j, \quad K_j(x, 0) = \delta(x) \quad (j = 1, 2, \cdots, m),$$

here $\delta(x)$ is the Dirac's delta function.$K_j(\boldsymbol{x}, t)$ is a heat kernel. By using the heat kernel we can solve heat equation as

$$u_j(\boldsymbol{x}, t) = \int_{\mathbb{R}^n} K_j(\boldsymbol{x} - \boldsymbol{y}, t) u_j(\boldsymbol{y}, 0) d\boldsymbol{y}.$$

Define the diagonal matrix $\boldsymbol{K}(\boldsymbol{x},t) := \mathrm{diag}(K_1, K_2, \cdots, K_m)$, then we have the solution

$$\boldsymbol{u}(\boldsymbol{x},t) = \int_{\mathbb{R}^n} \boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y},t)\boldsymbol{u}(\boldsymbol{y},0)d\boldsymbol{y}.$$

For any $\boldsymbol{u}_0 \in X$ and any $t \geq 0$, we define the operator $T(t)$ as

$$T(t)\boldsymbol{u}_0 := \int_{\mathbb{R}^n} \boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y},t)\boldsymbol{u}_0(\boldsymbol{x})d\boldsymbol{y}.$$

Then $\{T(t)\}_{t\geq 0}$ is the analytic semigroup defined in Definition 1.2.4, which satisfies following conditions:

(1) $T(0) = I$ (the identity mapping on X).

(2) $T(t)T(s) = T(t+s)$ for $t, s \geq 0$ (the associated law).

(3) $\lim_{t\to 0+} \|T(t)u_0 - u_0\| = 0$ for each $u_0 \in X$.

(4) $\|T(t)\|_{L(X)} \leq 1$ for all $t \geq 0$.

Here

$$\|T(t)\|_{L(X)} := \sup_{\boldsymbol{v}\in X, \boldsymbol{v}\neq 0} \frac{\|T(t)\boldsymbol{v}\|}{\|\boldsymbol{v}\|}$$

for each $t \geq 0$. According to Lemma 1.2.5, we have

$$\frac{d}{dt}T(t)\boldsymbol{v} = -AT(t)\boldsymbol{v}, \quad T(t)\boldsymbol{v}|_{t=0} = \boldsymbol{v},$$

where $A = D\Delta$. This implies that we can can define $e^{-tA} := T(t)$ and can solve it as $\boldsymbol{v} = e^{-tA}\boldsymbol{v}_0$. Then, consider the following reaction-diffusion system

$$\begin{cases} \dfrac{d\boldsymbol{u}}{dt} = -A\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}), & \boldsymbol{x} \in \mathbb{R}^n, t > 0 \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^n, \end{cases} \tag{2.2.3}$$

where $\boldsymbol{u}_0 \in X$ is an initial data, A is a sectorial operator in X (see Section1.2) and $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^m$ is a given nonlinear continuous function of class $\boldsymbol{C}^1(\mathbb{R}^m)$. Then, we can solve (2.2.3) as

$$\boldsymbol{u}(\boldsymbol{x},t) = T(t)u_0 + \int_{\mathbb{R}^n} T(t-s)\boldsymbol{f}(\boldsymbol{u}(\boldsymbol{x},s))ds \tag{2.2.4}$$

for $t > 0$. Additionally, we show the following theorem about the existence of solution of (2.2.3).

Theorem 2.2.1 ([13]). *There exists* $t_0 \in (0, \infty)$ *and* (2.2.4) *is the unique solution of* (2.2.3) *in* $C([0, t_0]; X)$.

Proof. By the variation of constants, we have

$$\boldsymbol{u}(\boldsymbol{x}, t) = e^{-tA}\boldsymbol{u}_0 + \int_0^t e^{-(t-s)A}\boldsymbol{f}(\boldsymbol{u}(\boldsymbol{x}, s))ds.$$

Thus, we see that a solution of (2.2.3) satisfies (2.2.4) with $T(t) = e^{-tA}$. Next, we show the local existence of solutions. Let $t_0 > 0$ be a enough small constant such that

$$\max_{0 \le t \le t_0} \|T(t)\boldsymbol{u}_0 - \boldsymbol{u}_0\| \le \frac{1}{2},$$
$$t_0 \max\{\max\{|\boldsymbol{f}(\boldsymbol{u})|, |J(\boldsymbol{u})|\}; |\boldsymbol{u}| \le 1 + \|\boldsymbol{u}_0\|\} \le \frac{1}{2},$$

where $J(\boldsymbol{u})$ is a Jacobian. Let Banach space Y as

$$Y = \{\boldsymbol{u} \in C([0, t_0]; X| \max_{t \in [0, t_0]} \|u(\boldsymbol{x}, t) - \boldsymbol{u}_0\|)\} \le 1,$$

with norm

$$\| \cdot \|_Y := \max_{t \in [0, t_0]} \| \cdot \|.$$

For any $u \in Y$, define the operator Φ as

$$\Phi\boldsymbol{u} = T(t)\boldsymbol{u}_0 + \int_0^t T(t-s)\boldsymbol{f}(\boldsymbol{u}(\boldsymbol{x}, s))ds.$$

First, we show $\Phi\boldsymbol{u} \in Y$. Due to the assumption, we have

$$\|\boldsymbol{u}(\boldsymbol{x}, t)\|_Y \le 1 + \|\boldsymbol{u}_0\|$$

and

$$\Phi\boldsymbol{u} - \boldsymbol{u}_0 = T(t)\boldsymbol{u}_0 - \boldsymbol{u}_0 + \int_0^t T(t-s)\boldsymbol{f}(\boldsymbol{u}(\boldsymbol{x}, s))ds.$$

Then,

$$\begin{aligned} \|\Phi\boldsymbol{u} - \boldsymbol{u}_0\| &\le \|T(t)\boldsymbol{u}_0 - \boldsymbol{u}_0\| + \int_0^t \|T(t-s)\boldsymbol{f}(\boldsymbol{u}(\boldsymbol{x}, s))\|ds \\ &\le \frac{1}{2} + t_0 \max\{|\boldsymbol{f}(\boldsymbol{u})|; \|\boldsymbol{u}\|_Y = 1 + \|\boldsymbol{u}_0\|\} \\ &\le 1, \end{aligned}$$

thus we get $\Phi \boldsymbol{u} \in Y$. Assume that $\boldsymbol{u}_1, \boldsymbol{u}_2$ are two solutions of (2.2.3). Using

$$\Phi \boldsymbol{u}_1 - \Phi \boldsymbol{u}_2 = \int_0^t T(t-s)(\boldsymbol{f}(\boldsymbol{u}_1(\boldsymbol{x},s)) - \boldsymbol{f}(\boldsymbol{u}_2(\boldsymbol{x},s)))ds,$$

we have

$$\|\Phi \boldsymbol{u}_1 - \Phi \boldsymbol{u}_2\| \leq \int_0^t \|T(t-s)(\boldsymbol{f}(\boldsymbol{u}_1(\boldsymbol{x},s)) - \boldsymbol{f}(\boldsymbol{u}_2(\boldsymbol{x},s)))\|ds.$$

Combining this inequality and using the mean-value theorem

$$\boldsymbol{f}(\boldsymbol{u}_1(\boldsymbol{x},s)) - \boldsymbol{f}(\boldsymbol{u}_2(\boldsymbol{x},s)) = \int_0^1 J(\theta \boldsymbol{u}_1 + (1-\theta)\boldsymbol{u}_2)d\theta(\boldsymbol{u}_1 - \boldsymbol{u}_2),$$

finally we get

$$\begin{aligned} \|\Phi \boldsymbol{u}_1 - \Phi \boldsymbol{u}_2\|_Y &\leq t_0 \max\{|J(\boldsymbol{u})|; |\boldsymbol{u}| \leq 1 + \|u_0\|\} \cdot \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_Y \\ &\leq \frac{1}{2}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_Y, \end{aligned}$$

for $\boldsymbol{u}_1, \boldsymbol{u}_2 \in Y$. Then, using Banach's fixed point theorem, there exists the fixed point $\boldsymbol{u}^*$ that satisfies

$$\Phi \boldsymbol{u}^*(\boldsymbol{x},t) = \boldsymbol{u}^*(\boldsymbol{x},t) = T(t)\boldsymbol{u}_0 + \int_0^t T(t-s)\boldsymbol{f}(\boldsymbol{u}^*(\boldsymbol{x},s))ds.$$

Note that $\boldsymbol{u}^*$ is nothing but the solution of (2.2.3).

Next, we show the uniqueness of solution. Since $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ satisfies

$$\boldsymbol{w} = \int_0^t T(t-s)(\boldsymbol{f}(\boldsymbol{u}_1(\boldsymbol{x},s)) - \boldsymbol{f}(\boldsymbol{u}_2(\boldsymbol{x},s)))ds,$$

we have

$$\|\boldsymbol{w}(t)\| \leq k \int_0^t \|\boldsymbol{w}(s)\|ds$$

for $0 \leq t \leq t_0$ where

$$k = \max\{|J(\boldsymbol{u})|; |bu| \leq 1 + \|u_0\|\}.$$

By the Gronwall inequality,

$$\|\boldsymbol{w}(t)\| \leq \|\boldsymbol{w}(0)\|e^{kt_0}.$$

Since $\boldsymbol{w}(0) = 0$, $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2 \equiv 0$. Therefore, we get the uniqueness of solutions of (2.2.3). □

Next, we will get the following Schauder's type estimate for

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} = D\Delta \boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}), \quad \boldsymbol{x} \in \mathbb{R}^n, t > 0 \\ \boldsymbol{u}(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n. \end{cases} \tag{2.2.5}$$

This equation is a case of $A = -D\Delta$ in (2.2.3).

Theorem 2.2.2 ((Schauder's estimate) [13]). *Let* $\boldsymbol{u} = (u_1, u_2, \cdots, u_m)$ *be the solution of* (2.2.3)*, and let* $T > 0$*. Now we define*

$$\|u_i\| := \sup_{\boldsymbol{x} \in \mathbb{R}^n} |u_i|.$$

Then, for $1 \le j \le n, 1 \le i \le m$*, we have*

$$\left|\frac{\partial}{\partial x_j} u_i(\boldsymbol{x}, t)\right| \le \frac{1}{\sqrt{\pi t}} \|u_i(\boldsymbol{x}, 0)\| + \frac{2\sqrt{t}}{\sqrt{\pi}} \sup_{\boldsymbol{y} \in \mathbb{R}^n, 0 < \tau < t} |f_i(\boldsymbol{u}(\boldsymbol{y}, \tau))|$$

for all $(\boldsymbol{x}, t) \in \mathbb{R}^n \times (0, T)$*.*

Proof. From (2.2.4), $\boldsymbol{u}$ can be written as

$$\begin{aligned} u_i(\boldsymbol{x}, t) &= \int_{\mathbb{R}^n} K_i(\boldsymbol{x} - \boldsymbol{y}, t) u_0 + \int_0^t \Big(\int_0^t K_i(\boldsymbol{x} - \boldsymbol{y}, t - s) \boldsymbol{f}(\boldsymbol{u}(\boldsymbol{y}, s)) d\boldsymbol{y} \Big) ds \\ &= I_1 + I_2. \end{aligned}$$

Now

$$K_i(\boldsymbol{x}, t) = \frac{1}{(4\pi d_i t)^{\frac{n}{2}}} e^{-\frac{|\boldsymbol{x}|^2}{4 d_i t}}$$

satisfies

$$(D_t - d_i \Delta) K_i(\boldsymbol{x}, t) = 0,$$

$$\frac{\partial}{\partial x_j} K_i(\boldsymbol{x}, t) = -\frac{x_j}{2t} K_i(\boldsymbol{x}, t),$$

$$\int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_j} K_i(\boldsymbol{x}, t) \right| d\boldsymbol{x} = \frac{1}{\sqrt{\pi t}},$$

$$\int_0^t \Big(\int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_j} K_i(\boldsymbol{x}, t - s) \right| d\boldsymbol{x} \Big) ds = 2\sqrt{\frac{t}{\pi}}.$$

A part of I_1 can be calculated as

$$\begin{aligned} \left| \frac{\partial}{\partial x_j} I_1 \right| &\le \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_j} K_i(\boldsymbol{x} - \boldsymbol{y}, t) | u_i(\boldsymbol{y}, 0) \right| d\boldsymbol{y} \\ &\le \frac{1}{\pi t} \|u_i(\boldsymbol{x}, 0)\|. \end{aligned}$$

Similarly, we get

$$
\begin{aligned}
\left|\frac{\partial}{\partial x_j} I_2\right| &\leq \int_0^t \Big(\int_{\mathbb{R}^n} \left|\frac{\partial}{\partial x_j} K_i(\boldsymbol{x}-\boldsymbol{y}, t-s)\right| |f_i(\boldsymbol{y}, s)| d\boldsymbol{y}\Big) ds \\
&\leq \int_0^t \Big(\int_{\mathbb{R}^n} \left|\frac{\partial}{\partial x_j} K_i(\boldsymbol{x}-\boldsymbol{y}, t-s)\right| d\boldsymbol{y}\Big) ds \cdot \sup_{\boldsymbol{y}\in\mathbb{R}^n, 0<\tau<t} |f_i(\boldsymbol{u}(\boldsymbol{y}, \tau))| \\
&= 2\sqrt{\frac{t}{\pi}} \sup_{\boldsymbol{y}\in\mathbb{R}^n, 0<\tau<t} |f_i(\boldsymbol{u}(\boldsymbol{y}, \tau))|.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
\left|\frac{\partial}{\partial x_j} u_i(\boldsymbol{x}, t)\right| &\leq \left|\frac{\partial}{\partial x_j} I_1\right| + \left|\frac{\partial}{\partial x_j} I_2\right| \\
&\leq \frac{1}{\sqrt{\pi t}} \|u_i(\boldsymbol{x}, 0)\| + 2\sqrt{\frac{t}{\pi}} \sup_{\boldsymbol{y}\in\mathbb{R}^n, 0<\tau<t} |f_i(\boldsymbol{u}(\boldsymbol{y}, \tau))|.
\end{aligned}
$$

□

2.3 Existence of solutions on a bounded domain Ω

Even in case of $\Omega \subset \mathbb{R}^n$, we can prove the existence of the reaction-diffusion system as the case of $\mathbb{R}^n$.

Theorem 2.3.1 (Theorem 3.3.3 [9]). *Let X be a Banach space with norm $\|\cdot\|$. Consider the following equations:*

$$\boldsymbol{u}_t - D\Delta\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{u}), \quad \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0$$

with Neumann boundary condition, where $-D\Delta$ is a sectorial operator that the fractional powered of $A_1 \equiv A + a\boldsymbol{I}$ are well defined, and the space $X^\alpha = D(A_1^\alpha)$ with the norm $\|u\|_\alpha = \|A_1 u\|$ can be defined for $\alpha \geq 0$. Assume that $\boldsymbol{f} : U \to X$ where $U \in X^\alpha$ for some $0 \leq \alpha < 1$ and locally Lipschitz continuous. More precisely, for any $\boldsymbol{u}_1 \in U$, there exists a neighborhood $V \in U$ of $\boldsymbol{u}_1$ that satisfies

$$\|\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v})\| \leq L\|\boldsymbol{u} - \boldsymbol{v}\|_\alpha$$

for $\boldsymbol{u}, \boldsymbol{v} \in V, L \geq 0$. Then, for any $\boldsymbol{u}_0 \in U$, (2.3.6) *has a unique solution $\boldsymbol{u}$ on $(\boldsymbol{x}, t) \in \Omega \times (0, t_1)$ with initial value $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0$ and the time $t_1 = t_1(\boldsymbol{u}_0)$.*

Proof. We can define the time evolute operator $T(t) = e^{-tA}$ for $A = -D\Delta$, thus we evaluate the solution and can write the solution of (2.3.6) as

$$\boldsymbol{u}(\boldsymbol{x},t) = T(t)\boldsymbol{u}_0 + \int_0^t T(t-s)\boldsymbol{f}(\boldsymbol{u}(\boldsymbol{x},s))ds$$

like the case of $\mathbb{R}^n$. Choose $\delta > 0, \tau > 0$, such that the set

$$V = \{\boldsymbol{u} \in U \,|\, t \in [0,\tau], \|\boldsymbol{u} - \boldsymbol{u}_0\|_\alpha \leq \delta\},$$

and

$$\|\boldsymbol{f}(\boldsymbol{u}_1) - \boldsymbol{f}(\boldsymbol{u}_2)\| \leq L\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_\alpha$$

for $\boldsymbol{u}_1, \boldsymbol{u}_2 \in V$. Let $C = \|\boldsymbol{f}(\boldsymbol{u}_0)\| < \infty$ and choose $t_1 \in (0,\tau)$ and

$$\|(T(h) - I)\boldsymbol{u}_0\| \leq \frac{\delta}{2}, \quad \text{for} \quad 0 \leq h \leq t_1,$$

$$M(C + L\delta)\int_0^{t_1} u^{-\alpha}e^{au}du \leq \frac{\delta}{2}$$

where

$$\|A_1^\alpha e^{-tA}\| \leq Mt^{-\alpha}e^{at}$$

for $t > 0$ by Lemma 1.2.9. Set a functional space S as

$$S = \{\boldsymbol{u} \in [0,t_1] \longrightarrow X^\alpha \mid \|\boldsymbol{u} - \boldsymbol{u}_0\|_\alpha \leq \delta\}.$$

S is the Banach space with norm

$$\|\boldsymbol{u}\|_S = \sup_{t\in[0,t_1]} \|\boldsymbol{u}\|_\alpha.$$

For $\boldsymbol{u} \in S$, we can define the following operator Φ:

$$\Phi\boldsymbol{u}(\boldsymbol{x},t) = T(t)\boldsymbol{u}_0 + \int_0^t T(t-s)\boldsymbol{f}(\boldsymbol{u}(\boldsymbol{x},s))ds.$$

We show that $\Phi : S \longrightarrow S$ is the contraction mapping. Note that

$$\begin{aligned}
\|\Phi\boldsymbol{u} - \boldsymbol{u}_0\|_\alpha &\leq \|T(t)\boldsymbol{u}_0 - \boldsymbol{u}_0\|_\alpha + \int_0^t \|T(t-s)\boldsymbol{f}(\boldsymbol{u})\|ds \\
&\leq \frac{\delta}{2} + \int_0^t \|A_1^\alpha e^{-(t-s)A}\|(\|\boldsymbol{f}(\boldsymbol{u}_0)\| + \|\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{u}_0)\|)ds \\
&\leq \frac{\delta}{2} + \int_0^t \|A_1^\alpha e^{-(t-s)A}\|(C + L\delta)ds \\
&\leq \frac{\delta}{2} + M(C + L\delta)\int_0^t (t-s)^{-\alpha}e^{a(t-s)}ds \\
&\leq \delta
\end{aligned}$$

for $0 \le t \le t_1$. Thus $\Phi\boldsymbol{u} \in Y$ for any $\boldsymbol{u} \in Y$. From the assumption

$$ML\delta \int_0^t u^{-\alpha} e^{au} du \le M(C + L\delta) \int_0^t u^{-\alpha} e^{au} du \le \frac{\delta}{2},$$

we have

$$ML \int_0^t u^{-\alpha} e^{au} du \le \frac{1}{2}.$$

Then if $\boldsymbol{u}, \boldsymbol{v} \in Y$, we have

$$\begin{aligned}
\|\Phi\boldsymbol{u} - \Phi\boldsymbol{v}\|_\alpha &\le \int_0^t \|A_1^\alpha e^{-(t-s)A}\| \|(\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v}))\| ds \\
&\le ML\delta \int_0^t (t-s)^{-\alpha} e^{a(t-s)} ds \cdot \|\boldsymbol{u} - \boldsymbol{v}\|_\alpha \\
&\le \frac{1}{2} \|\boldsymbol{u} - \boldsymbol{v}\|_\alpha.
\end{aligned}$$

Finally we get

$$\|\Phi\boldsymbol{u} - \Phi\boldsymbol{v}\|_Y \le \frac{1}{2} \|\boldsymbol{u} - \boldsymbol{v}\|_Y.$$

Therefore Φ is a contraction mapping and we can apply the Banach's fixed point theorem. There exists the unique fixed point $\boldsymbol{u}^* \in Y$ that satisfies $\Phi\boldsymbol{u}^* = \boldsymbol{u}^*$ and

$$\boldsymbol{u}^*(\boldsymbol{x}, t) = T(t)\boldsymbol{u}_0 + \int_0^t T(t-s)\boldsymbol{f}(\boldsymbol{u}^*(\boldsymbol{x}, s)) ds.$$

Hence, u^* is nothing but the unique solution of (2.3.6). □

2.4 Maximum principle of parabolic equations

Consider now bounded region $\Omega \in \mathbb{R}^n$, $\overline{Q_T} = \Omega \times (0, T)$ and the parabolic boundary $\Gamma = \Omega \times t = 0 \subset \partial\Omega \times [0, T]$ for any time $T > 0$. Suppose that the smooth function $u : Q_T \to \mathbb{R}$ is $C^{2,1}$ on $\overline{Q_T}$ and $C^{1,0}$ on Q_T. Here, $C^{2,1}$ means C^2 for $\boldsymbol{x}$ and C^1 on t and $C^{1,0}$ means C^1 for $\boldsymbol{x}$ and continuous on t. Then, u satisfies the following equation:

$$\frac{\partial u}{\partial t} = D\Delta u + f(\boldsymbol{x}, t)u \qquad (\boldsymbol{x} \in \Omega) \tag{2.4.6}$$

with Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad (\boldsymbol{x} \in \partial\Omega) \tag{2.4.7}$$

that means no flux from the boundary, where $f(\boldsymbol{x}, t)$ is a bounded continuous function on $\overline{Q_T}$ and $D \in \mathbb{R}^{n\times n}$ is the diagonal matrix defined as $D := \mathrm{diag}(d_1, d_2, \cdots, d_n)$. Then, we have the following theorem.

Theorem 2.4.1 (Theorem 5.12 [30, 21])**.** *The solution of* (2.4.6) *and* (2.4.7) *satisfies the following statements.*

(a) *If $u(x, 0) \leq 0$ on Ω, then $u(\boldsymbol{x}, t) \leq 0$ for any $t > 0$. Especially, if the solution is not constant function $u \equiv 0$, then $u(\boldsymbol{x}, t) < 0$.*

(b) *Due to* (a)*, if $u(\boldsymbol{x}, 0) = 0$ on Ω, then $u(\boldsymbol{x}, t) = 0$ for any $t > 0$. This implies the uniqueness of solution of the single reaction diffusion system*

$$u_t = D\Delta u + f(u),$$

with Neumann boundary condition.

To prove this theorem, we should show the following lemmas.

Lemma 2.4.2 ([30, Theorem 5.7])**.** *The function $u \in C^{2,1}(Q_T) \cap C^0(\overline{Q_T})$ satisfies*

$$u_t - D\Delta u + f(\boldsymbol{x}, t)u \leq 0.$$

Then, if $u \leq 0$ on Γ, $u \leq 0$ on $\overline{Q_T}$.

Proof. At first, assume $f(\boldsymbol{x}, t) \equiv 0$, and

$$u_t - D\Delta u < 0.$$

Then, for $(\boldsymbol{x}_0, t_0) \in \Omega \times (0, T] = Q_T$ that satisfies

$$\sup_{(\boldsymbol{x},t)\in Q_T} u(\boldsymbol{x}, t) = (\boldsymbol{x}_0, t_0),$$

thus we have

$$u_t(\boldsymbol{x}_0, t_0) = 0, \quad \nabla u(\boldsymbol{x}_0, t_0) = 0, \quad \sum_{i=1}^{n} d_i \frac{\partial^2}{\partial x_i^2} u(\boldsymbol{x}_0, t_0) \leq 0.$$

Thus

$$u_t - D\Delta u \geq 0,$$

and this is contradictory to the assumption. Therefore we have

$$\max_{(\boldsymbol{x},t)\in\overline{Q_T}} u(\boldsymbol{x},t) = \max_{(\boldsymbol{x},t)\in\Gamma} u.$$

Next, consider the case of

$$u_t - D\Delta u \leq 0.$$

Set $v = \epsilon e^{-t} + u$ for $\epsilon > 0$ and v satisfies

$$v_t - D\Delta v = u_t - D\Delta u - \epsilon e^{-t} < 0,$$

we have

$$\max_{(\boldsymbol{x},t)\in\overline{Q_T}} u(\boldsymbol{x},t) \leq \max_{(\boldsymbol{x},t)\in\overline{Q_T}} (u(\boldsymbol{x},t) + \epsilon e^{-t}) \leq \max_{(\boldsymbol{x},t)\in\Gamma} (u(\boldsymbol{x},t) + \epsilon e^{-t}).$$

Thus we get

$$\max_{(\boldsymbol{x},t)\in\overline{Q_T}} u(\boldsymbol{x},t) \leq \max_{(\boldsymbol{x},t)\in\Gamma} u(\boldsymbol{x},t)$$

as $\epsilon \to 0$. Finally, consider the case of a general function $f(\boldsymbol{x},t)$. Take $v = ue^{-Mt}$ for $M = \max_{(\boldsymbol{x},t)\in\overline{Q_T}} |f(\boldsymbol{x},t)|$. Then, we have

$$v_t - D\Delta v + (f(\boldsymbol{x},t) + M)v = (u_t - D\Delta u + f(\boldsymbol{x},t))e^{-Mt} \leq 0.$$

If v has the maximum value $v_{max} > 0$, then we have

$$(v_{max})_t - D\Delta v_{max} \leq -(f(\boldsymbol{x},t) + M)v_{max} \leq 0,$$

thus it is sufficient to consider the case of $f(\boldsymbol{x},t) \equiv 0$. By the assumption $u \leq 0$ on Γ, we have

$$\max_{(\boldsymbol{x},t)\in\overline{Q_T}} v \leq 0,$$

but it is contradictory to $v_{max} > 0$. Therefore, if $u \leq 0$ on Γ , then $u \leq 0$ on $\overline{Q_T}$. □

Lemma 2.4.3 (north or south pole, [30, Lemma 5.9]). *Let* $B_R(\boldsymbol{x})$ *be the* n*-th dimensional ball with the origin* $\boldsymbol{x} \in \mathbb{R}^n$. *The function* $u \in C^{2,1}(Q_T) \cap C^0(\overline{Q_T})$ *satisfies*

$$u_t - D\Delta u + f(\boldsymbol{x}, t)u \leq 0,$$

and $u \leq 0$ *on* Γ. *Assume that* $u < 0$ *on* $B_R(\boldsymbol{x}_0, t_0)$ *for* $(\boldsymbol{x}_0, t_0) \in Q_T$ *and* $u(\boldsymbol{x}_1, t_1) = 0$ *on* $B_R(\boldsymbol{x}_1, t_1)$ *for* $(\boldsymbol{x}_1, t_1) \in Q_T$. *Then,* $(\boldsymbol{x}_1, t_1)$ *is the north or south pole of the* $B_R(\boldsymbol{x}_1, t_1)$. *That is,* $(\boldsymbol{x}_1, t_1)$ *is limited to* $(x_0, t_0 + R)$ *or* $(x_0, t_0 - R)$.

Proof. Consider $v = e^{-Mt}$ instead of u like 2.4.2, we can assume $f(\boldsymbol{x}, t) \geq 0$ and $u \leq 0$ on $u \leq 0$ on Q_T. Moreover, we can assume $u = 0$ on only $(\boldsymbol{x}_1, t_1) \in B_R(\boldsymbol{x}_0, t_0)$ since we can set $B_R(\boldsymbol{x}_0, t_0)$ that is tangent to only $(\boldsymbol{x}_1, t_1)$. Set w as

$$w := e^{-\gamma(|\boldsymbol{x}-\boldsymbol{x}_0|^2+(t-t_0)^2)} - e^{-\gamma R^2}$$

for $\gamma > 0$, then $w > 0$ inside $B_R(\boldsymbol{x}_0, t_0)$, $w = 0$ on $\partial B_R(\boldsymbol{x}_0, t_0)$, $w < 0$ outside $B_R(\boldsymbol{x}_0, t_0)$. Assume $\boldsymbol{x}_1 \neq \boldsymbol{x}_0$ and show the contradiction. Consider the sufficiently small $\delta \in (0, |\boldsymbol{x}_1 - \boldsymbol{x}_0|)$ and $B_\delta(\boldsymbol{x}_1, t_1) \subset Q_T$. Then we have

$$\begin{aligned}
&w_t - D\Delta w + f(\boldsymbol{x}, t)w \\
&= \left\{-2\gamma(t - t_0) - \sum_{i=1}^{n} 4d_i\gamma^2|\boldsymbol{x} - \boldsymbol{x}_0|^2 + \sum_{i=1}^{n} 2d_i\gamma\right\} e^{-\gamma(|\boldsymbol{x}-\boldsymbol{x}_0|^2+(t-t_0)^2)} \\
&\quad + f(\boldsymbol{x}, t)w \\
&\leq \{-2\gamma(t - t_0) - 4n\gamma^2 d_{min}|\boldsymbol{x} - \boldsymbol{x}_0|^2 + 2nd_{max}\gamma + C\}e^{-\gamma(|\boldsymbol{x}-\boldsymbol{x}_0|^2+(t-t_0)^2)} \\
&\quad - Ce^{-\gamma R^2}
\end{aligned}$$

with some positive constant C. By taking sufficiently large $\gamma > 0$, we have that $w_t - D\Delta w + f(\boldsymbol{x}, t)w < 0$ in $\overline{B_\delta(\boldsymbol{x}_1, t_1)}$. Set $v := u + \epsilon w < 0$ for $\epsilon > 0$. Since $u \leq 0$ for $(\boldsymbol{x}, t) \in \partial B_\delta(\boldsymbol{x}_1, t_1) \cap \overline{B_R(x_0, t_0)}$, we can assume $v < 0$ for sufficiently small ϵ. Similarly since $u < 0, w < 0$ for $(\boldsymbol{x}, t) \in \partial B_\delta(\boldsymbol{x}_1, t_1) \backslash \overline{B_R(x_0, t_0)}$, we can assume $v < 0$. Then, $v < 0$ on $\partial B_\delta(\boldsymbol{x}_1, t_1)$. On the other hand,

$$\begin{aligned}
&v_t - D\Delta v + f(\boldsymbol{x}, t)v \\
&= u_t - D\Delta u + f(\boldsymbol{x}, t)u + \epsilon(w_t - D\Delta w + f(\boldsymbol{x}, t)w) < 0
\end{aligned}$$

on $B_\delta(\boldsymbol{x}_1, t_1)$. By applying Lemma 2.4.2, we get

$$\max_{(\boldsymbol{x},t)\in\overline{B_\delta(\boldsymbol{x}_1,t_1)}} v = \max_{(\boldsymbol{x},t)\in\partial B_\delta(\boldsymbol{x}_1,t_1)} v < 0,$$

but it is contradictory to $v = 0$ at a point $(\boldsymbol{x}_1, t_1)$. Therefore, $\boldsymbol{x}_1 = \boldsymbol{x}_0$ and this also implies $t_1 = (t_0 \pm R)$. □

Lemma 2.4.4 ([30, Lemma 5.10]). *The function* $u \in C^{2,1}(Q_T) \cap C^0(\overline{Q_T})$ *satisfies*

$$u_t - D\Delta u + f(\boldsymbol{x}, t)u \leq 0,$$

$u \leq 0$ *on* Γ. *If there is* $(\boldsymbol{x}_2, t_2) \in Q_T$ *that satisfies* $u(\boldsymbol{x}_2, t_2) < 0$, *then* $u(\boldsymbol{x}, t_2) < 0$ *for all* $\boldsymbol{x} \in \Omega$.

Proof. Assume that $\boldsymbol{x}_3 \in \Omega$ that satisfies $u(\boldsymbol{x}_3, t_2) = 0$. Set E and $d(\boldsymbol{x})$ as

$$E = \{(\boldsymbol{y}, s) \in Q_T; u(\boldsymbol{y}, s) = 0\}, \quad d(\boldsymbol{x}) = \inf_{(\boldsymbol{y},s)\in E} \sqrt{(|\boldsymbol{x} - \boldsymbol{y}|^2 + (t_2 - s)^2)},$$

then $d(\boldsymbol{x}) \leq |\boldsymbol{x} - \boldsymbol{x}_3|$. By Lemma 2.4.3, we have $u(\boldsymbol{x}_2, t_2 + d(\boldsymbol{x}_2)) = 0$ or $u(\boldsymbol{x}_2, t_2 - d(\boldsymbol{x}_2)) = 0$ if $d(\boldsymbol{x}_2) > 0$, that is $(\boldsymbol{x}_2, t_2 + d(\boldsymbol{x}_2)) \in E$ or $(\boldsymbol{x}_2, t_2 - d(\boldsymbol{x}_2)) \in E$. For any $\delta > 0$ and $\boldsymbol{c}$ such that $|\boldsymbol{c}| = 1$, the distance from $(\boldsymbol{x}_2 + \delta\boldsymbol{c}, t_2)$ to $(\boldsymbol{x}_2, t_2 \pm d(\boldsymbol{x}_2))$ is $\sqrt{\delta^2 + d(\boldsymbol{x}_2)^2}$ and satisfies

$$d(\boldsymbol{x}_2 + \delta\boldsymbol{c}) \leq \sqrt{\delta^2 + d(\boldsymbol{x}_2)^2} \leq d(\boldsymbol{x}_2) + \frac{\delta^2}{2d(\boldsymbol{x}_2)}.$$

This implies d is non-increasing and monotonous for δ. By $d(\boldsymbol{x}_3) = 0.$,we have $d \equiv 0$ and $u(\boldsymbol{x}, t_2) = 0$. But it is contradictory to Lemma 2.4.3. Therefore, there doesn't exist $(\boldsymbol{x}, t_2) \in Q_T$ that satisfies $u(\boldsymbol{x}, t_2) = 0$. By the assumption $u \leq 0$ on Γ, we have $u(\boldsymbol{x}, t_2) < 0$ on Q_T. □

Lemma 2.4.5 (Strong maximum principle of the parabolic equation, [30, Theorem 5.11]). *The function* $u \in C^{2,1}(Q_T) \cap C^0(\overline{Q_T})$ *satisfies*

$$u_t - D\Delta u + f(\boldsymbol{x}, t)u \leq 0,$$

$u \leq 0$ *on* Γ. *Then,* $u < 0$ *or* $u \equiv 0$ *on* Q_T.

Proof. In the same way to Lemma 2.4.2, we can regard as $f(\boldsymbol{x}, t) \geq 0$. Assume that u attains the maximum value 0 at $(\boldsymbol{x}_0, t_0) \in Q_T$. We can retake t_0 as

$$t_0 = \inf_{u(\boldsymbol{x},t)=0;(\boldsymbol{x},t)\in Q_T} t.$$

set v as

$$v := e^{-\gamma|\boldsymbol{x}-\boldsymbol{x}_0|^2 - \beta(t-t_0)} - 1.$$

Then, we have

$$\begin{aligned} v_t - &D\Delta v + f(\boldsymbol{x}, t)v \\ &= \left(-\beta - \sum_{i=1}^{n} 4d_i\gamma^2|\boldsymbol{x} - \boldsymbol{x}_0|^2 + \sum_{i=1}^{n} 2d_i n\gamma\right) e^{-\gamma(|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (t-t_0)^2)} + f(\boldsymbol{x}, t)v \\ &\leq \left(-\beta - 4nd_{min}\gamma^2|\boldsymbol{x} - \boldsymbol{x}_0|^2 + 2nd_max\gamma\right) e^{-\gamma(|\boldsymbol{x}-\boldsymbol{x}_0|^2 + (t-t_0)^2)} + C_1 v, \end{aligned}$$

for some constant $C_1 > 0$. By taking sufficiently large β, $v_t - D\Delta v + f(\boldsymbol{x}, t)v < 0$. Set $\delta > 0$ such that satisfies $K_\delta \subset Q_T$ where

$$K_\delta = \{(\boldsymbol{x}, t) \in B_\delta(\boldsymbol{x}_0, t_0) \mid \gamma|\boldsymbol{x} - \boldsymbol{x}_0|^2 + \beta(t - t_0) \leq 0\}.$$

Then, $w = u + \epsilon v < 0$ on $\partial K_\delta \cap \partial B_\delta(\boldsymbol{x}_0, t_0)$ for a sufficiently small $\epsilon > 0$, because $v = 0$ on $\gamma|\boldsymbol{x} - \boldsymbol{x}_0|^2 + \beta(t - t_0) = 0$. On the other hand, since

$$w_t - D\Delta w + f(\boldsymbol{x}, t)w < 0,$$

we can use Lemma 2.4.2 and $w \leq 0$ on ∂K_δ. Since $w = 0$ on $(\boldsymbol{x}_0, t_0)$, w attains the maximum value 0 at $(\boldsymbol{x}_0, t_0)$. This implies that $w_t(x_0, t_0) \geq 0$. Then,

$$u_t(\boldsymbol{x}_0, t_0) \geq -\epsilon v_t(\boldsymbol{x}_0, t_0) = \epsilon\beta > 0,$$

and it contradicts the choice of $(\boldsymbol{x}_0, t_0)$. □

Finally, we show the maximum principle for the parabolic equations.

Proof of Theorem 2.4.1.

(a): Assume that the non-negative maximum value M on $\overline{\Omega} \times (0, T]$. Then we can take minimum t_0 that is $u(\boldsymbol{x}, t_0) = 0$, and there exists $\boldsymbol{x}_0 \in \overline{\Omega}$ that satisfies $u(\boldsymbol{x}_0, t_0) = 0$ and $u(\boldsymbol{x}, t) < 0$ on $(\boldsymbol{x}, t) \in \Omega \times [0, t_0)$. If $(\boldsymbol{x}_0, t)$ is an inner point of Q_T, we have $u \equiv 0$ by the strong maximum principle(Lemma 2.4.5).Thus $\boldsymbol{x}_0$ is a point on the boundary, and $u(\boldsymbol{x}, t_0) < 0$ for $x \in \Omega$. Set the ball $B_R(x_1, t_0) \in Q_T$ that is tangent to $\partial\Omega$ on x_0. Consider v, w defined as

$$\begin{aligned} v &= e^{-\gamma(|\boldsymbol{x}-\boldsymbol{x}_1|^2+(t-t_0))} - e^{-\gamma R^2}, \\ w &= u + \epsilon v, \\ K_\delta &= \{(\boldsymbol{x}, t) \in Q_T | (\boldsymbol{x}, t) \in B_\delta(\boldsymbol{x}_0, t_0) \cap B_R(\boldsymbol{x}_1, t_0), t \in (0, t_0)\} \end{aligned}$$

for sufficiently small $\epsilon, \delta > 0$. In the same way to 2.4.3, we can calculate as

$$w_t - D\Delta w + f(\boldsymbol{x}, t)w < 0$$

and $w < 0$ on K_δ except $(\boldsymbol{x}_0, t_0)$. Then we have

$$\frac{\partial w}{\partial \nu}(\boldsymbol{x}_0, t_0) \geq 0,$$

namely,

$$\begin{aligned} \frac{\partial u}{\partial \nu}(\boldsymbol{x}_0, t_0) &\geq -\epsilon \frac{\partial v}{\partial \nu}(\boldsymbol{x}_0, t_0) \\ &= 2\epsilon\gamma(\boldsymbol{x}_0 - \boldsymbol{x}_1) \cdot \boldsymbol{n} e^{-\gamma(|\boldsymbol{x}_0-\boldsymbol{x}_1|^2+(t-t_0))} \\ &> 0, \end{aligned}$$

here $\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$. But it is contradictory to the assumption $\frac{\partial u}{\partial \nu}(\boldsymbol{x}_0, t_0) \le 0$. Therefore, (2.4.6) has the negative maximum on $\overline{\Omega} \times (0, T]$,that is, $u(x,t) < 0$ for any $t > 0$.

(b): Let u, v be two solutions of the single reaction diffusion system $u_t = D\Delta u + f(u)$ with the same initial data. Then, $w = u - v$ satisfies

$$\frac{\partial w}{\partial t} = D\Delta w + h(\boldsymbol{x}, t)w, \quad \frac{\partial w}{\partial \nu} = 0, w(\boldsymbol{x}, 0) = 0,$$

here $h(\boldsymbol{x}, t)$ is defined as

$$h(\boldsymbol{x}, t) = \begin{cases} \dfrac{\partial f}{\partial u}(\boldsymbol{x}, t), & (u \equiv v) \\ \dfrac{f(u) - f(v)}{u - v} & (\text{otherwise}) \end{cases}$$

We can regard the equation of w as

$$\frac{\partial w}{\partial t} \le D\Delta w + h(\boldsymbol{x}, t)w, \quad \frac{\partial w}{\partial \nu} \le 0, \quad w(\boldsymbol{x}, 0) \le 0$$
$$\frac{\partial(-w)}{\partial t} \le D\Delta(-w) + h(\boldsymbol{x}, t)(-w), \quad \frac{\partial(-w)}{\partial \nu} \le 0, \quad -w(\boldsymbol{x}, 0) \le 0.$$

Now we can apply theorem 2.4.1 to these inequalities, thus we have $w(\boldsymbol{x}, t) \le 0$ and $-w(\boldsymbol{x}, t) \le 0$. Therefore, we get $w(\boldsymbol{x}, t) \equiv 0$ for any $t > 0, \boldsymbol{x} \in \overline{\Omega}$. □

2.5 Invariant regions

In this section, we review the theory of invariant sets. Consider the m-component ordinary differential equations

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{f}(\boldsymbol{u}) \tag{2.5.8}$$

where m is a positive integer and $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{f}$ is a smooth function from $\mathbb{R}^m$ to $\mathbb{R}^m$. Denote a solution of (2.5.8) with $\boldsymbol{u}(0) = \boldsymbol{u}_0$ by $\boldsymbol{u}(t; \boldsymbol{u}_0)$.

First we define the invariant set as follows.

Definition 2.5.1. *A family of closed bounded sets $\Sigma(t)$ in $\mathbb{R}^m$ is called an invariant set of (2.5.8) if $\boldsymbol{u}(t; \boldsymbol{u}_0) \in \Sigma(t)$ for any $\boldsymbol{u}_0 \in \Sigma(0)$ and any $t > 0$.*

Next consider the m-component reaction–diffusion system

$$\frac{\partial \boldsymbol{w}}{\partial t} = \Delta \boldsymbol{w} + \boldsymbol{f}(\boldsymbol{w}), \quad \boldsymbol{x} \in \Omega, \ t > 0, \tag{2.5.9}$$

with the Neumann boundary condition. We can also define an invariant set for (2.5.9). Define a set $\mathrm{Im}(\boldsymbol{w}(\cdot)) := \{\boldsymbol{w}(\boldsymbol{x}) \mid \boldsymbol{x} \in \Omega\}$. We also denote the solution of (2.5.9) satisfying $\boldsymbol{w}(x,0) = \boldsymbol{w}_0(x)$ for $x \in \Omega$ by $\boldsymbol{w}(x,t;\boldsymbol{w}_0)$.

Definition 2.5.2. *A family of closed bounded sets $\Sigma(t)$ in $\mathbb{R}^m$ is called an invariant set of* (2.5.9) *if* $\mathrm{Im}(\boldsymbol{w}(\cdot,t;\boldsymbol{w}_0)) \in \Sigma(t)$ *for any $t > 0$ and any $\boldsymbol{w}_0 \in C^0(\Omega)$ satisfying* $\mathrm{Im}(\boldsymbol{w}_0(\cdot)) \in \Sigma(0)$.

This definition implies that a solution with the initial data inside the invariant set remains at the same set for all time. If we can obtain $\Sigma(t)$ for a given partial differential equation, we can know the rough behavior of the solution without knowing the exact solution of the equation. In addition, if we can construct an invariant set that shrinks over time, it would be a powerful tool to specify the destination of solutions to equations and to show that solutions can only exist in a limited range after a sufficient amount of time. We have the following property for the invariant sets.

Theorem 2.5.3 ([25, 32]). *Let $\Sigma(t) := \bigcap_{j=1}^{k} \Sigma_j(t)$ be a closed bounded set defined by*

$$\Sigma_j(t) := \{\boldsymbol{w} \in \mathbb{R}^m \mid H_j(\boldsymbol{w},t) \leq 0\} \qquad (j = 1,2,\dots k),$$

where k is a positive integer and $H_1,\cdots,H_k$ are smooth functions satisfying

$$\frac{\partial H_j}{\partial t}(\boldsymbol{w},t) + \nabla_{\boldsymbol{w}} H_j(\boldsymbol{w},t) \cdot \boldsymbol{f}(\boldsymbol{w}) \leq 0 \tag{2.5.10}$$
$$\text{on } \boldsymbol{w} \in \partial\Sigma_j(t) \cap \partial\Sigma(t) \quad (j = 1,2,\dots,k,\ t > 0).$$

Then, a family of $\Sigma(t)$ is the invariant set of (2.5.8)*. Moreover, it is the invariant set of* (2.5.9)*, if $\Sigma(t)$ is convex.*

Proof. We can assume that

$$\frac{\partial^2 H_j}{\partial w_i^2} \geq 0, \qquad (i = 1,\cdots,k)$$

near the $\partial\Sigma_j(t) \cap \partial\Sigma(t) \quad (j = 1,2,\dots,k)$ because of the convexity of $\Sigma(t)$. For example, set $H_j(\boldsymbol{w},t)$ as the signed distance $U(\boldsymbol{x},t)$ near the $\partial\Sigma_j(t)$ from $\partial\Sigma(t)$. The signed distance is the shortest distance from $\boldsymbol{w}$ and $\partial\Sigma_j(t)$ with the negative sign inside $\Sigma(t)$, or the positive sign outside. Then,

$$\begin{aligned} U_t &= \nabla_{\boldsymbol{w}} H_j(\boldsymbol{w},t) \cdot \boldsymbol{w}_t + \frac{\partial}{\partial t} H_j(\boldsymbol{w},t), \\ \nabla U &= \sum_{i=1}^{m} \frac{\partial H_j}{\partial w_i} \nabla w_i, \\ \Delta U &= \sum_{i=1}^{m} \frac{\partial^2 H_j}{\partial w_i^2} |\nabla w_i|^2 + \nabla_{\boldsymbol{w}} H_j \cdot \Delta \boldsymbol{w}. \end{aligned}$$

Note that $\nabla_{\boldsymbol{w}}$ is the gradient on the phase space of $\boldsymbol{w}$. From this, we can get

$$\begin{aligned}U_t - \Delta U =&\nabla_{\boldsymbol{w}} H_j(\boldsymbol{w},t) \cdot \boldsymbol{w}_t + \frac{\partial}{\partial t} H_j(\boldsymbol{w},t)\\ &- \sum_{i=1}^{m} \frac{\partial^2 H_j}{\partial w_i^2} |\nabla w_i|^2 - \nabla_{\boldsymbol{w}} H_j(\boldsymbol{w},t) \cdot \Delta \boldsymbol{w}\\ =&\nabla_{\boldsymbol{w}} H_j(\boldsymbol{w},t) \cdot (\boldsymbol{w}_t - \Delta \boldsymbol{w}) + \frac{\partial}{\partial t} H_j(\boldsymbol{w},t)\\ &- \sum_{i=1}^{m} \frac{\partial^2 H_j}{\partial w_i^2} |\nabla w_i|^2\\ =&\nabla_{\boldsymbol{w}} H_j(\boldsymbol{w},t) \cdot \boldsymbol{F}(\boldsymbol{w}) + \frac{\partial}{\partial t} H_j(\boldsymbol{w},t) - \sum_{i=1}^{m} \frac{\partial^2 H_j}{\partial w_i^2} |\nabla w_i|^2.\end{aligned}$$

Because the initial condition is in $\Sigma(0)$, $U(x,0) \leq 0$. Moreover, using the assumption of this theorem $\frac{\partial^2 H_j}{\partial w_i^2}$, we have

$$\nabla_{\boldsymbol{w}} H_j(\boldsymbol{w},t) \cdot \boldsymbol{F}(\boldsymbol{w}) + \frac{\partial}{\partial t} H_j(\boldsymbol{w},t) \leq 0, \tag{2.5.11}$$

thus $U_t - \Delta U \leq 0$. Therefore, we can apply Theorem 2.4.1 and get $U(\boldsymbol{x},t) \leq 0$ that implies that $\Sigma(t)$ is the invariant set of (2.5.9). □

Consider diffusion coefficients cannot be expressed as $\boldsymbol{D} = d\boldsymbol{I}$ where $\boldsymbol{I}$ is identity matrix. In this case, Theorem 2.5.3 is not applicable. Instead of Theorem 2.5.3, we have the following theorems.

Theorem 2.5.4. *Let $\boldsymbol{u}$ be a solution of (2.1.1) with the Neumann boundary condition with $\boldsymbol{D} = \operatorname{diag}(d_1, \cdots, d_m)$ where $d_j > 0$. Denote the j-component of $\boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{f}(\boldsymbol{u}(\boldsymbol{x},t))$ by $u_j(\boldsymbol{x},t), F_j(\boldsymbol{u}(\boldsymbol{x},t))$ respectively. Suppose that a function $a(t)$ satisfies following the inequality*

$$\frac{da(t)}{dt} \leq F_j(u_1, u_2, \cdots, u_{j-1}, a(t), u_{j+1}, \cdots, u_m).$$

Then, if $\boldsymbol{u}(\boldsymbol{x},0) \leq a(0)$, we have $\boldsymbol{u}(\boldsymbol{x},t) \leq a(t)$ for any $t > 0$.

Proof. Since u_j satisfies

$$\frac{\partial u_j}{\partial t} = d_j \Delta u_j + F_j(u_1, u_2, \cdots u_j, \cdots u_m),$$

we have

$$\frac{\partial(u_j - a)}{\partial t} = d_j\Delta(u_j - a) + F_j(u_1, u_2, \cdots u_j, \cdots u_m) \\ -F_j(u_1, u_2, \cdots a, \cdots u_m).$$

Define the continuous function $c(\boldsymbol{x}, t)$ by

$$c(\boldsymbol{x}, t) = \begin{cases} \dfrac{\partial F_j}{\partial u_j}, & (u_j(\boldsymbol{x}, t) = a(t)) \\ \dfrac{F_j(u_1, \cdots, u_j, \cdots u_m) - F_j(u_1, \cdots, u_{j-1}, a_j(t), u_{j+1}, \cdots, u_m)}{u_j - a}, & \text{(otherwise)} \end{cases}$$

Then, by setting $U := u_j - a$, we have

$$\frac{\partial U}{\partial t} = d_j\Delta U + c(\boldsymbol{x}, t)U.$$

Due to the assumption, $U(\boldsymbol{x}, 0) = u_j(\boldsymbol{x}, 0) - a(0) \leq 0$. Therefore, we can apply the maximum principle (Theorem 2.4.1) to the above equation and we get $U(\boldsymbol{x}, t) = u_j(\boldsymbol{x}, t) - a(t) \leq 0$. □

From Theorem 2.5.4, in case of $D \neq dI$, we can obviously construct an axis-parallel rectangular invariant set $\Sigma(t) := \Pi_{j=1}^{m}[a_j(t), b_j(t)]$. Therefore, we have the following theorem about a axis-parallel invariant set.

Theorem 2.5.5. *Assume that* $(a_j(t), b_j(t)) \quad (j = 1, 2, \cdot, m)$ *satisfies*

$$\frac{da_j(t)}{dt} \geq F_j(u_1, u_2, \cdots, u_{j-1}, a_j(t), u_{j+1}, \cdots, u_m)$$
$$\frac{db_j(t)}{dt} \leq F_j(u_1, u_2, \cdots, u_{j-1}, b_j(t), u_{j+1}, \cdots, u_m).$$

Then, the following rectangular set

$$\Sigma(t) := \{\boldsymbol{u} \mid a_j(t) \leq u_j(\boldsymbol{x}, t) \leq b_j(t) \text{ for } \boldsymbol{x} \in \Omega,\ j = 1, \cdots, m\}$$

is invarinat for (2.1.1).

This theorem immediately follows from Theorem 2.5.4.

2.6 Linear stability of ODE

For $\boldsymbol{u} = (u, v) \in \mathbb{R}^m$, consider the following autonomous system

$$\frac{du}{dt} = \boldsymbol{f}(\boldsymbol{u}), \tag{2.6.12}$$

with a equilibrium point $\boldsymbol{u} = \boldsymbol{u}_{eq}$ that satisfies $\boldsymbol{f}(\boldsymbol{u}_{eq}) = 0$. We aim to investigate how the solution of the ordinary differential equation behaves near this equilibrium point. The Taylor expansion of F near the equilibrium point $\boldsymbol{u} = \boldsymbol{u}_{eq}$ is as follows:

$$F(\boldsymbol{u}) = J(\boldsymbol{u} - \boldsymbol{u}_{eq}) + \text{second order or higher nonlinearity of} \quad (\boldsymbol{u} - \boldsymbol{u}_{eq}),$$

where J is a Jacobian on $\boldsymbol{u}_{eq}$, which is given by $\partial \boldsymbol{f}(\boldsymbol{u}_{eq})/\partial \boldsymbol{u}$. Now we consider the solution near the equilibrium point,thus the infection of second-order or higher nonlinearity can be ignored and residue $\boldsymbol{\xi} = \boldsymbol{u} - \boldsymbol{u}_{eq}$ is sufficiently small. Accordingly we linearize the solution in the vicinity of the equilibrium point $\boldsymbol{u} = \boldsymbol{u}_{eq}$ and obtain the equation of $\boldsymbol{\xi}$ as

$$\frac{d\boldsymbol{\xi}}{dt} = J\boldsymbol{\xi}. \tag{2.6.13}$$

Any autonomous system can be linearized in this form. Assume that the solution (2.6.13) can be written as $\boldsymbol{\xi} = \boldsymbol{p}e^{\lambda t}$ for some constant λ and some vector $\boldsymbol{p}$ and substitute it for (2.6.13). Then, we get the equation

$$(J - \lambda I)\boldsymbol{p} = 0.$$

This is the eigenvalue problem with the eigenvalue λ and the eigenvector $\boldsymbol{p}$. Since it is $\boldsymbol{\xi} \in \mathbb{R}^m$, there exist m eigenvalues $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ and the corresponding eigenvectors $(\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_m)$ if multiple roots of the characteristic equation are not considered. In this case, since eigenvectors belonging to different eigenvalues are linearly independent each other, m independent solutions

$$\boldsymbol{p}_1 e^{\lambda_1 t}, \boldsymbol{p}_2 e^{\lambda_2 t}, \cdots, \boldsymbol{p}_m e^{\lambda_m t}$$

are obtained. By the principle of superposition, the linear combination of these independent solutions is also the solution of the linearization equation, thus we get

$$\boldsymbol{\xi} = \boldsymbol{p}_1 e^{\lambda_1 t} + \boldsymbol{p}_2 e^{\lambda_2 t} + \cdots + \boldsymbol{p}_m e^{\lambda_m t}.$$

If at least one of eigenvalues has a positive real part, $|\boldsymbol{\xi}|$ increases exponentially along the eigenvector direction. In other words, $\boldsymbol{u}$ near the equilibrium point

$\boldsymbol{u}_{eq}$ leaves exponentially and never converges to $\boldsymbol{u}_{eq}$ (unstable). If the real parts of eigenvalues are all negative, $|\boldsymbol{\xi}|$ converges to 0 as $t \to \infty$, and $\boldsymbol{u}$ in the vicinity of $\boldsymbol{u}_{eq}$ finally converges to the equilibrium point (stable). If there is an eigenvalue with a real part 0, the solution trajectory will neither move toward nor move away from the corresponding eigenvector direction.

The *repeller* (repulsion point) is the equilibrium point with only positive real part of eigenvalues, the *attractor* is the equilibrium point with only negative real part of eigenvalues, and the *saddle* is the equilibrium with the positive and negative real parts of eigenvalues.

2.7 Prey-Predator model

A typical example of reaction-diffusion systems is a Prey-Predator model as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} &= d\Delta u + u(1-u) - \frac{uv}{u+h}, \\ \frac{\partial v}{\partial t} &= d\Delta v + \frac{ruv}{u+h} - mv \end{aligned} \tag{2.7.14}$$

with Neumann boundary condition, where u, v describe population densities of prey (e.g. phytoplankton) and predator (e.g. zooplankton) respectively. A positive parameter r means the intensity of predation, m means the death rate of predator, and h means the agility of prey to evade from predation. In addition, u, v always take positive values or zero for consistency with the actual biological environment. The second term of the right hand side of (2.7.14) represents the increase or decrease in concentration like the logistic equation and it shows that a prey self-propagates until $u = 1$. The part of the reaction term $u/(u+h)$ is called a Holling II type response function, which approaches 1 when u is sufficiently large compared to h and never goes above 1. This means that even if there are a large number of prey, predators will saturate their prey intake. Also, the higher h, the more prey population is needed to saturate. The term $ruv/(u+h)$ represents the growth due to predation of predators, and the term $-mv$ of the following equation represents the natural death of predators. The values of u and v variable due to such interactions working at the same time.

2.7.1 The stability of equilibrium points

Here, we discuss about the stability of the prey-predator model on ODE, that is written as

$$\begin{aligned} \frac{du}{dt} &= u(1-u) - \frac{uv}{u+h} = f(u,v) \\ \frac{dv}{dt} &= r\frac{uv}{u+h} - mv = g(u,v). \end{aligned} \tag{2.7.15}$$

Then, the curve $f(u,v) = 0, g(u,v) = 0$ are respectively called u-*nullcline*, v-*nullcline*. The intersection of the two nullclines is the equilibrium point. According to this definition, nullclines are given by

$$\begin{aligned} u + u(1-u) - \frac{uv}{u+h} &= 0 \\ r\frac{uv}{u+h} - mv &= 0. \end{aligned}$$

Solving nullclines, we obtain three equilibrium points

$$(u,v) = (0,0), (1,0), (u_{eq}, v_{eq}),$$

where

$$u_{eq} := \frac{m}{r-m}h > 0, \qquad v_{eq} := (1-u_{eq})(u_{eq}+h).$$

Here u_{eq}, v_{eq} are positive and parameters satisfy

$$r > m > 0, \qquad \frac{r}{1+h} - m > 0$$

Then, under this condition, three equilibrium points respectively exist on the origin, u-axis, and the first quadrant as Figure 2.1.

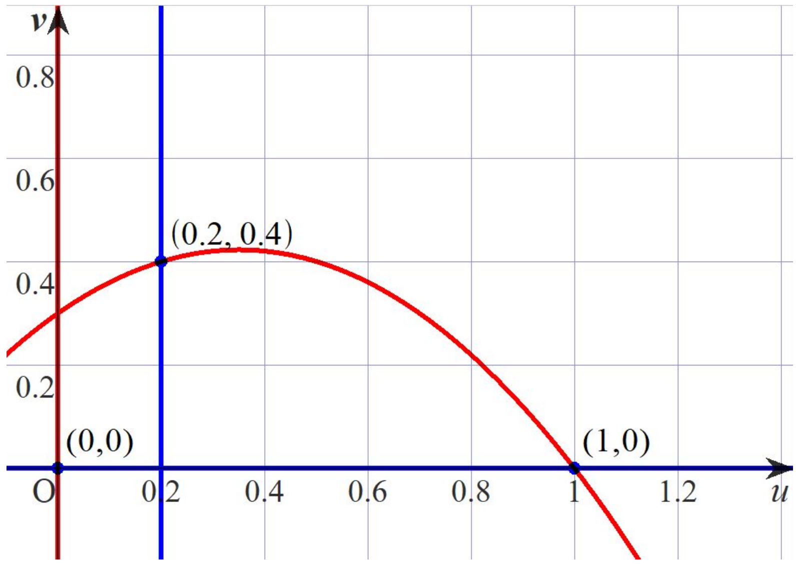


Figure 2.1: Distribution of three equilibrium points of prey-predator model when $r = 2.0,\ m = 0.8,\ h = 0.3$ with $u-$nullcline(red line) and $v-$nullcline(blue line).

The linearized equation on the equilibrium point (u_{eq}, v_{eq}) can be written as follows:

$$\frac{d\boldsymbol{\xi}}{dt} = \begin{pmatrix} 1 - 2u_{eq} - \frac{h}{(u_{eq}+h)^2} v_{eq} & -\frac{u_{eq}}{u_{eq}+h} \\ r\frac{h}{(u_{eq}+h)^2} v_{eq} & r\frac{h}{u_{eq}+h-m} \end{pmatrix} \boldsymbol{\xi}.$$

Since this prey-predator model has three equilibrium points, we need to study three cases:

case(i) :$(u, v) = (0, 0)$, both prey and predator become extinct. The linearized equation is

$$\frac{d\boldsymbol{\xi}}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & -m \end{pmatrix} \boldsymbol{\xi}.$$

Obviously eigenvalues are $\lambda = 1, -m$. Since there is the positive eigenvalue, (0,0) is always unstable(saddle). Thus, it can be seen that orbits near $(0, 0)$ on the phase plane are attracted to (0,0) along the eigenvector $(0, 1)$ and move away along the eigenvector $(1, 0)$ direction.

case(ii) : $(u, v) = (1, 0)$, prey survived and predator becomes extinct. The linearized equation is

$$\frac{d\boldsymbol{\xi}}{dt} = \begin{pmatrix} -1 & -\frac{1}{1+h} \\ 0 & \frac{r}{1+h} - m \end{pmatrix} \boldsymbol{\xi}.$$

Obviously, eigenvalues are $\lambda = -1, \frac{r}{1+h} - m$ and the corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 + \frac{r}{1+h} - m \end{pmatrix}$$

respectively. In case of $\frac{r}{1+h} - m < 0$, (1,0) is stable (attractor), and case of $\frac{r}{1+h} - m > 0$, (1,0) is unstable (saddle). Since conditions $u_{eq}, v_{eq} > 0$ contain $r - m(1+h) > 0$, $(1, 0)$ is always saddle if (u_{eq}, v_{eq}) lies in the first quadrant.

case(iii) : $(u, v) = (u_{eq}, v_{eq})$, both coexist.
Set

$$H(u) = \frac{u}{u+h}, \quad k = \frac{m}{r}.$$

The Holling type-II response function $H(u)$ is the monotonous increasing function with $u \geq 0$, thus there exists inverse function $H^{-1}(u), \quad (0 \leq u \leq 1)$ that satisfies

$$\begin{aligned} u_{eq} &= H^{-1}(k) \\ v_{eq} &= \frac{u_{eq}(1 - u_{eq})}{k} = \frac{h}{1-k}\left(1 - \frac{k}{1-k}h\right), \end{aligned}$$

here $k < 1, H(u_{eq}) = k$. Moreover, using the derivative of inverse function, we have

$$H'(u_{eq}) = \frac{1}{dH^{-1}(k)/dk} = \frac{(1-k)^2}{h}.$$

Thus the linearized equation on (u_{eq}, v_{eq}) is written as

$$\frac{d\boldsymbol{\xi}}{dt} = \begin{pmatrix} k\frac{(1-k)-(1+k)h}{1-k} & -k \\ r(1-k-kh) & 0 \end{pmatrix} \boldsymbol{\xi},$$

and trace and determine can be calculated as

$$\begin{aligned} \mathrm{Tr}J &= k\frac{(1-k)-(1+k)h}{1-k} \\ \det J &= rk(1-k-kh). \end{aligned}$$

We obtain the characteristic equation $\lambda^2 - \mathrm{Tr}J\lambda + \det J = 0$. It is very difficult to solve this equation and determine whether the eigenvalues are real or imaginary. For example, we can calculate the eigenvalue λ in case of $r = 2.0, k = 0.4$ as follows:

$$\lambda = \frac{3 - 7h \pm \sqrt{49h^2 + 30h - 99}}{15}$$

When $49h^2 + 30h - 99 < 0, h > 0$, namely

$$0 < h < \frac{-15 + 6\sqrt{141}}{49} \simeq 1.1479\cdots,$$

eigenvalues are imaginary. When $h > 1.1479$, eigenvalues are real. Additionally, if $h > 1.5$, then

$$\frac{3 - 7h + \sqrt{49h^2 + 30h - 99}}{15} > 0,$$

(u_{eq}, v_{eq}) is the saddle. In this case $(1, 0)$ becomes the attractor. Moreover, if $1.1479 < h < 1.5$, then (u_{eq}, v_{eq}) is the stable attractor. When the value of trace is positive, namely

$$h < \frac{1-k}{1+k} = \frac{3}{7} \simeq 0.4286\cdots,$$

(u_{eq}, v_{eq}) is the repeller (rotating), and when $0.4286 < h < 1.1479$, (u_{eq}, v_{eq}) is attractor (rotating). For example, in case of $h = 0.3$,

$$\lambda = 0.06 \pm 0.616765i.$$

Note that near the equilibrium point with imaginary eigenvalue, the rotating solution appears. The solution of the linearized equation with imaginary eigenvalue $\lambda = a + bi$ $(a, b \in \mathbb{R}, b \neq 0)$ can be calculated as

$$\boldsymbol{\xi} = \boldsymbol{p}_1 e^{(a+bi)t} + \boldsymbol{p}_2 e^{(a-bi)t}.$$

Using the Euler formula, we have

$$\begin{aligned}\boldsymbol{\xi} &= \boldsymbol{p}_1 e^{at}(\cos bt + i \sin bt) + \boldsymbol{p}_2 e^{at}(\cos bt - i \sin bt) \\ &= e^{at}(\boldsymbol{A} \cos bt + \boldsymbol{B} \sin bt),\end{aligned}$$

here $\boldsymbol{A} = \boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{B} = i(\boldsymbol{p}_1 - \boldsymbol{p}_2)$ are the constant vector. When $b \neq 0$, this solution implies the rotation around the equilibrium on the phase space appears by the part of $(\cos bt + i \sin bt)$.

2.7.2 Numerical simulation of ODE

We calculate the numerical solution of (2.7.15) by using the Runge-Kutta method with the 4th-order accuracy. In this section, we use the same initial data as

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and the time step $\Delta t = 0.001$, the terminal time $T = 1000$. We get the numerical solution as Figure 2.2(a) under these conditions. When $r = 2.0, m = 0.8, h = 0.3$, by the previous section, the stability of equilibrium points is as follows:

$$\begin{aligned} (0,0) &: \text{unstable (saddle) by } m > 0 \\ (1,0) &: \text{unstable (saddle) by } \frac{r}{1+h} - m = \frac{2}{1.3} - 0.8 > 0 \\ (u_{eq}, v_{eq}) &: \text{unstable (rotation) by } h = \frac{1-k}{1+k} < \frac{3}{7}. \end{aligned}$$

Since there is no stable equilibrium point, the solution is attracted to the periodic solution on the limit cycle. Next when $r = 2.0, m = 0.8, h = 0.45$, the stability of equilibrium points is as follows:

$$\begin{aligned} (0,0) &: \text{unstable (saddle) by } m > 0 \\ (1,0) &: \text{unstable (saddle) by } \frac{r}{1+h} - m = \frac{2}{1.3} - 0.8 > 0 \\ (u_{eq}, v_{eq}) &: \text{stable (rotation) by } h = \frac{1-k}{1+k} > \frac{3}{7}. \end{aligned}$$

As seen in the previous section, when $0.4286 < h < 1.1479$, the eigenvalue on (u_{eq}, v_{eq}) has the imaginary and negative real part, thus (u_{eq}, v_{eq}) is the stable attractor that appears the rotation in vicinity of the (u_{eq}, v_{eq}). Finally, we set $r = 2.0, m = 0.8, h = 1.4$, then the stability of equilibrium points is as follows:

$$\begin{aligned} (0,0) &: \text{unstable (saddle) by } m > 0 \\ (1,0) &: \text{unstable (saddle) by } \frac{r}{1+h} - m = \frac{2}{1.3} - 0.8 > 0 \\ (u_{eq}, v_{eq}) &: \text{stable attractor (without the rotation) by} \\ & h = \frac{1-k}{1+k} > \frac{-15 + 6\sqrt{141}}{49}. \end{aligned}$$

Under this condition, the solution converges to (u_{eq}, v_{eq}) without any rotation.

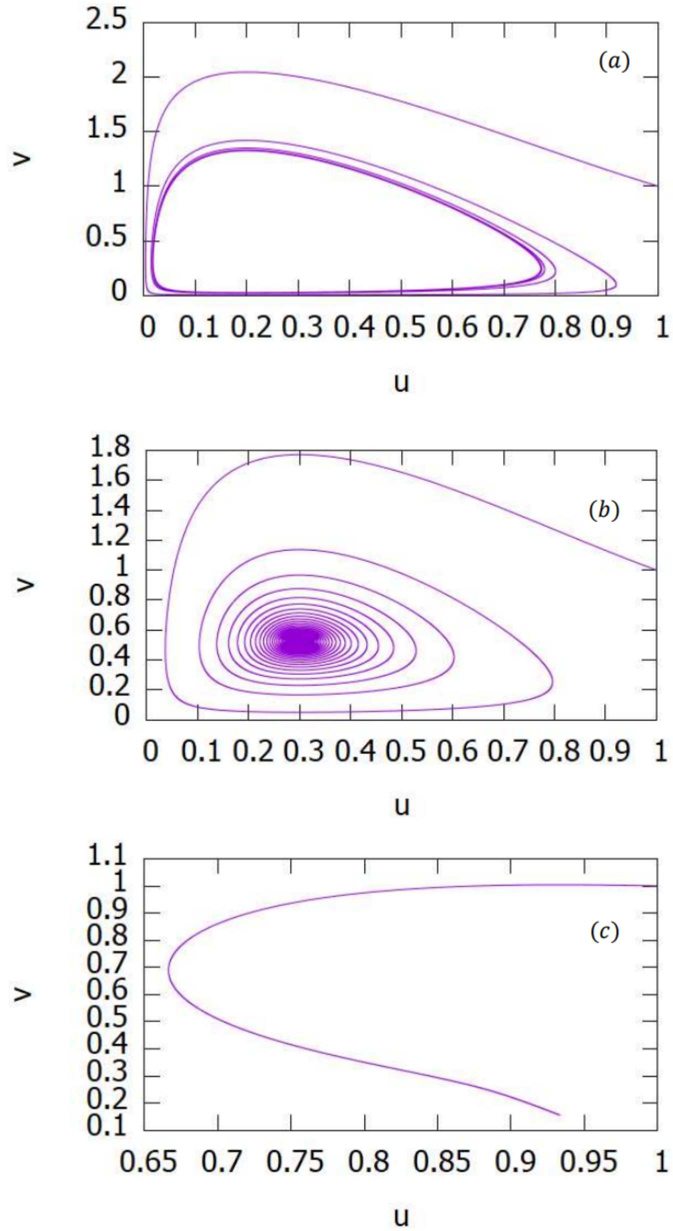


Figure 2.2: Numerical simulations of prey-predator model.

(a) :The case of $r = 2.0, m = 0.8, h = 0.3$

(b) :The case of $r = 2.0, m = 0.8, h = 0.45$

(c) :The case of $r = 2.0, m = 0.8, h = 1.4$

2.7.3 Numerical simulation of prey-predator model on PDE

The equation (2.7.14) has many kinds of numerical solutions, such as equilibrium point, the periodic solution on the limit cycle like ODE (2.7.15), additionally as specific solution on (2.7.14), spiral pattern and spatial temporal chaos. However, the *spatio-temporal chaotic* behavior does not have unite definition, we treat the following behaviors 'spatio-temporal chaos' in this paper:

(i) Select a point in space $\boldsymbol{\Omega}$, observe the solution trajectory at that point over and over time, and draw it in phase space. Then, we observe the aspect that is similar to the strange attractor in ordinary differential equations, in other word, the solution trajectory is bounded and non-periodic.

(ii) The spatial non-homogeneity of the solution is preserved over time, and its profile changes over time. In other words, the spatial structure is constantly fluctuating over time.

We can calculate the spatial non-homogeneity by the following function

$$V(t) := \frac{1}{2}\int_{\Omega} |\nabla \boldsymbol{u}|^2 d\boldsymbol{x} = \frac{1}{2}\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)}.$$

When the numerical solution satisfies (i) and (ii) and shows complicated behavior in time and space, it is called spatio-temporal chaos. Set the domain $\Omega = (0, L) \times (0, L) \in \mathbb{R}^2, L = 200,\ r = 2.0,\ m = 0.8,\ h = 0.3$ and the following initial conditions and boundary conditions:

$$\begin{aligned} u(x, y, 0) &= u_{eq}\left(1 - \cos\frac{\pi}{L}x\right) \\ v(x, y, 0) &= v_{eq}\left(1 - \cos\frac{\pi}{L}y\right) \\ u_{eq} &= \frac{r}{r - m} = 0.2 \\ v_{eq} &= (1 - u_{eq})(u_{eq} + h) = 0.4 \\ \nabla \boldsymbol{u} \cdot \boldsymbol{n} &= 0, \quad \text{on} \quad \partial\Omega. \end{aligned}$$

We use the Crank-Nicolson method to discretize the diffusion term and the Adams-Bashforth method with 3rd-order accuracy to discretize the reaction term and we apply the finite-elements-method (FEM) to solve discretized equation numerically. The time step is $\Delta t = 0.1$, and the number of space divisions is 40 per side.

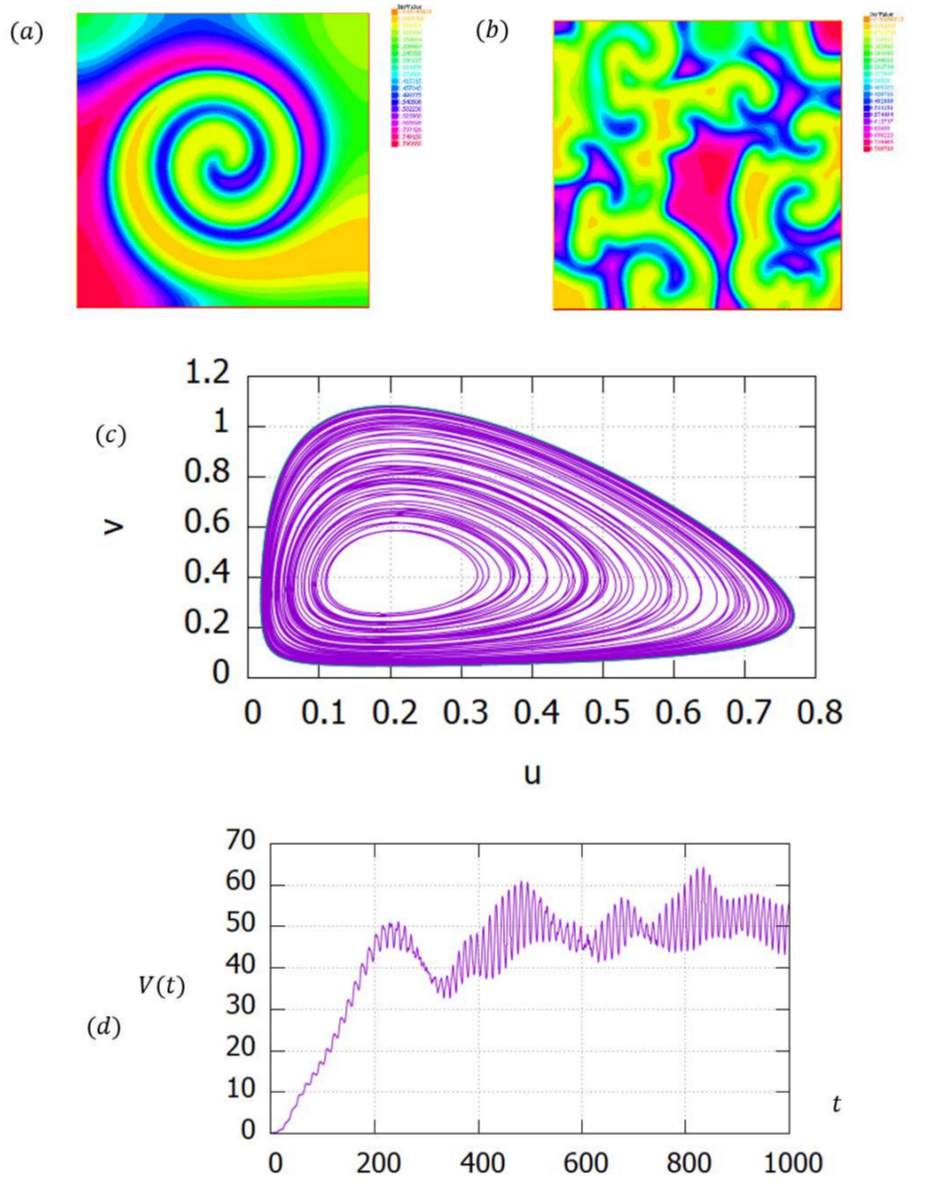


Figure 2.3: Numerical simulations of prey-predator model(2.7.14).

(a): The spatial pattern u at $t = 160$

(b): The spatial pattern u at $t = 1000$

(c): (u, v) plot on $(x, y) = (10, 10)$(purple line), the limit cycle(green line)

(d): Graph of $V(t)$

Then, we get the numerical solution of (2.7.14) under the following condition as Figure 2.3. Figure 2.3 (a) and (b) are respectively spatial patterns of the numerical solution u at $t = 160, 1000$, (c) is the (u, v) plot on $(x, y) = (10, 10)$ over the time, and (d) is the graph of $V(t)$ that is the intensity of the spatial non-homogeneity. By (a) and (b), the spiral pattern at $t = 160$ collapsed with time and became a chaotic spatial pattern at $t = 1000$. Also, the plot of (u, v) (purple line) on a settled position in the space domain (x, y) from (c), is the solution trajectory that is similar to the strange attractor in the ordinary differential equation, and (u, v) always lies in limit cycle(green line) of (2.7.15). The limit cycle is obviously the invariant set of ODE. If this limit cycle is convex, plot of (u, v) always lies inside the limit cycle by theorem 2.5.3. Since We cannot calculate analytically the trajectory of the limit cycle, it is difficult to know whether the limit cycle is convex or not, anyway the numerical solution (c) lies in. In addition, by (d), $V(0)$ takes a very small positive value in the initial state, but it gradually increases and keeps a much larger positive value than the initial state while constantly fluctuating until $t = 1000$. From this, it can be seen that spatio-temporal chaos occurs from the initial condition using the judgment criteria.

2.7.4 Invariant set of the prey-predator model

Theorem 2.7.1 ([28]). *(2.7.14) and (2.7.15) has the following invariant set* Σ *without depending to time as Fig 2.4:*

$$
\begin{aligned}
&H_1(u, v) = -u\\
&H_2(u, v) = -v\\
&H_3(u, v) = v - a(1 + h)(1 - u)\\
&H_4(u, v) = ru + v - r\frac{(1 + m)^2}{4m}\\
&H_5(u, v) = v - a(1 + h)(1 - u_e q)\\
&H_6(u, v) = v - r\left[\frac{(1 + m)^2}{4m} - u_e q\right]\\
&a = 1 - m + \frac{r}{1 + h}, \quad a > 1, r > m\\
&\Sigma_j = \{(u, v) | H_j(u, v) \leq 0\} \quad (j = 1, 2, \cdots, 6)\\
&\Sigma = \bigcap_{j=1}^{6} \Sigma_j .
\end{aligned}
$$

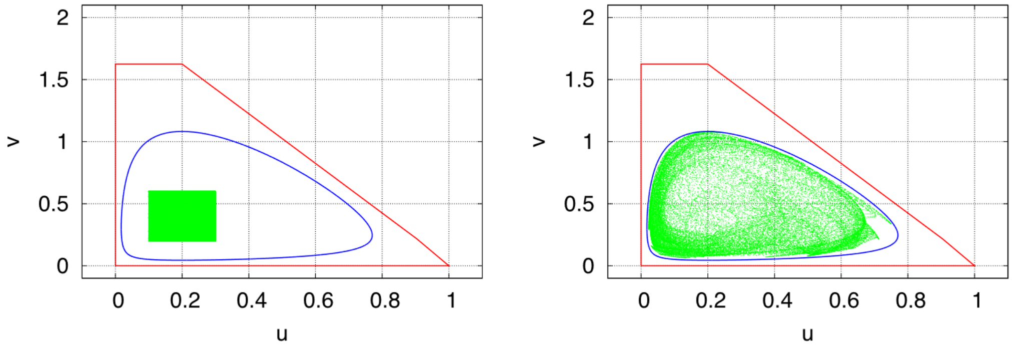


Figure 2.4: Comparison of the image of solution (green) and positive invariant set Σ (red) at each time ($r = 2.0, m = 0.8, h = 0.3$). Blue curves denote the numerically obtained limit cycles, (left)$t = 0$, (right) $t = 600$.

2.8 FitzHugh-Nagumo systems

In 1952, neurophysiologists Hodgkin and Huxley experimentally investigated how electrical signals are transmitted through the nerve fibers of spear squid, and devised the following four-component simultaneous partial differential equations (Hodgkin-Huxley equations):

$$\begin{cases} u_t &= D\Delta u + f(u, \boldsymbol{v}) \\ \boldsymbol{v}_t &= \boldsymbol{g}(u, \boldsymbol{v}), \end{cases}$$

where t is the time, x is the distance on the nerve, $u \in \mathbb{R}$ is the magnitude of the nerve potential, and $\boldsymbol{v} \in \mathbb{R}^3$ is a three-dimensional vector-valued function representing the condition of the nerve membrane. Also, $f, \boldsymbol{g}$ are non-linear functions that represent the function of the nerve membrane. The Hodgkin-Huxley equation is very difficult to handle mathematically because it is very difficult to precisely describe the actual behavior of neurons. Therefore, FitzHugh and Nagumo devised a simplified model of the Hodgkin-Huxley equation without losing its essence, and derived the following two-component reaction-diffusion system containing only one nonlinear term:

$$\begin{cases} u_t &= c(u - \dfrac{u^3}{3} - v + I), \\ v_t &= u - bv + a. \end{cases}$$

Here u, v are variables respectively representing membrane potential and inactivation, a, b, c, I are parameters. This equation is called the FitzHugh-Nagumo equation(ODE).

It is known that when I is smaller than a certain value determined by a, b, c, the solution converges to the equilibrium point (u_{eq}, v_{eq}). This behavior represents the neuronal excitation generated by the stimulation current, and the equilibrium point means the quiescent state of the nerve. When the initial value is near (u_{eq}, v_{eq}), the orbit converges to (u_{eq}, v_{eq}) while remaining near it, while on the right side of $(\sqrt{3}, 0)$ it makes a large turn on the phase plane. It will eventually converge to the equilibrium point. This is a mathematical model that expresses the characteristics of actual nerves in that they do not respond very well to small stimuli and are excited by stimuli that exceed the threshold. On the other hand, when I has a sufficiently large value, the solution of the FitzHugh-Nagumo equation exhibits a periodic orbit called a limit cycle. When a sufficiently strong external stimulus current is applied, neurons will cycle between excitation and rest-state. In this way, the FitzHugh-Nagumo equation has been used in various studies because it can qualitatively reproduce many behaviors of nerve cells even though it has a simple form. By considering the spatial non-uniformity and adding the diffusion term, we have

$$\begin{cases} u_t &= D\Delta u + c(u - \frac{u^3}{3} - v + I), \\ v_t &= (u - bv + a). \end{cases} \tag{2.8.16}$$

This is also called the FitzHugh-Nagumo equation, which is one of the reaction-diffusion systems. Such a reaction-diffusion system is used, for example, as a model of electrical signals that travel over the heart rather than simple linear nerves. The dynamics by (2.8.16) include complex behaviors such as the spatially uniform periodic solution on limit cycles corresponding to regular oscillations, spiral patterns and spatio-temporal chaos corresponding to arrhythmia or ventricular fibrillation as well as the prey-predator model.

We calculate the numerical solution of the FitzHugh-Nagumo equation(ODE) as Figure2.5. Figure 2.5 implies this equation has the periodic solution on the limit cycle, thus we expect the FitzHugh-Nagumo equation has two characteristic multipliers 1 and $\boldsymbol{u}(< 1)$. By the algorithm on 1.4.1, we can numerically calculate and get the Lyapunov exponent ν_1, ν_2 and characteristic multiplier μ_1, μ_2 from numerical solution as follows:

$$\nu_1 = 4.8308810469287955 \times 10^{-9} \simeq 0, \quad \nu_2 \simeq -6.9045454792035645$$
$$\mu_1 \simeq 1, \quad \mu_2 = e^{\nu_2 T} < 1,$$

here T is the time period of the limit cycle. thus we can get the approximated value of the Lyapunov exponent and characteristic multiplier, and this satisfies the assumption of 1.3.1.

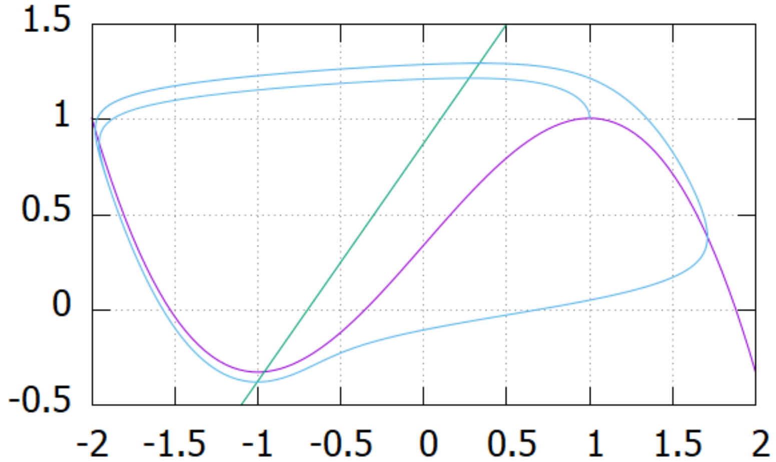


Figure 2.5: The numerical solution of the FitzHugh-Nagumo equation on $a = 0.7, b = 0.8, c = 10, I = 0.34, (u(0), v(0)) = (1, 1)$ (purple line), u-nullcline(green line),v-nullcline(purple line)

Chapter 3

Spatial homogenization of the complex Ginzburg-Landau equation

3.1 Introduction

Transitions from regular motions to chaotic dynamics have been observed in many fields, such as fluid dynamics, motions of many particles, chemical reactions, biological systems and so on (e.g., see [10, 19, 26, 7]). In the chaotic dynamics, complicated spatio-temporal regimes play an important role. Namely, the spatial inhomogeneity persists under the chaotic behavior. One of the examples of this transition from regular motions to chaos is ventricular fibrillation. If chaotic behaviors occur in the cardiac tissue of the ventricle, coordinated regular heartbeats are lost and it causes ventricular fibrillation (e.g., see [8, 6]).

The transition from chaos to regular motions is also important for controlling chaos. One example of the transition from chaos to the regular motion is electrical defibrillation. Electrical defibrillation of a strong electric shock to the heart is an effective therapy for the ventricular fibrillation (e.g., see [23, 20]). One way to regain regular motions is to reduce spatial inhomogeneity. Because chaotic systems can be characterized by the extreme sensitivity to small perturbations, large perturbations should be necessary to regain regular motions. We encounter the following natural question: *what happens when large perturbations are added ?* Since the system is sensitive to even small perturbations, the basins of attraction are complicated. Therefore, we may expect that it is difficult to control from the spatio-temporal complicated dynamics to the regular one. The purpose of this paper is to answer this question. We will show the spatial homogenization and the transition from chaos to the regular motions by adding large perturbations. We will also explain the mechanism by which large perturbations restore the regular motions.

There are various mathematical equations which possess both the chaotic dynamics and the regular periodic motions. The equations for the heartbeat such as the FitzHugh-Nagumo system [5, 17] and the Aliev-Panfilov model [1] also possess this property. However, it is difficult to analyze directly such equations because it is difficult to obtain the precise information of the periodic solutions of these systems and to examine the transient dynamics. To avoid these technical difficulties, we use a simpler reaction-diffusion system, the complex Ginzburg-Landau equation. The complex Ginzburg-Landau equation is one of the most-studied nonlinear equations in physics such as phase transitions, superconductivity, Bose-Einstein condensation and so on (see [2, 4] for example). In 1974, the reductive perturbation method of reaction diffusion was first introduced by [15] to understand nonlinear waves. If the reaction-diffusion systems possess the limit cycle oscillation with small amplitude near a Hopf bifurcation point, we can obtain the complex Ginzburg-Landau equation as the governing equation of such oscillations by using the reductive perturbation method (see [12] for the details). Therefore, the complex Ginzburg-Landau equation has been intensively studied in the nonlinear phenomena of synchronization. The complex Ginzburg-Landau equation possesses the essential structure for spatial homogenization and the transition from chaos to the regular motions, since this equation has both the spatio-temporal chaos and the limit cycle oscillation like FitzHugh-Nagumo system and the Aliev-Panfilov model. Because this equation is rotationally invariant, it is easier to analyze it mathematically than other reaction diffusion systems with limit cycles. Thus, we believe that the analysis of the complex Ginzburg-Landau equation gives a well-understanding of the underlying structure for more realistic model equations such as the FitzHugh-Nagumo system and the Aliev-Panfilov model. Hence, we treat the following complex Ginzburg-Landau equation:

$$\frac{\partial z}{\partial t} = d\Delta z + (1 + i\omega)z - (1 + ia\omega)|z|^2 z \qquad \text{in } \Omega \tag{3.1.1}$$

with the Neumann boundary condition

$$\boldsymbol{n} \cdot \nabla z = 0 \qquad \text{on } \partial\Omega,$$

where $z = u + iv \in \mathbb{C}$, Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary, $\boldsymbol{n}$ is the outer normal vector of the boundary $\partial\Omega$ and N is the spatial dimension. The diffusion coefficient d is a positive constant and a, ω are real numbers. The equation (3.1.1) is a special case of the complex Ginzburg-Landau equation because the diffusion coefficient is real. Using $z = u + iv$, we can rewrite (3.1.1) as

$$\begin{cases} \dfrac{\partial u}{\partial t} &= d\Delta u + u - \omega v - u^3 - uv^2 + a\omega(u^2+v^2)v, \\ \dfrac{\partial v}{\partial t} &= d\Delta v + v + \omega u - v^3 - u^2 v - a\omega(u^2+v^2)u. \end{cases} \tag{3.1.2}$$

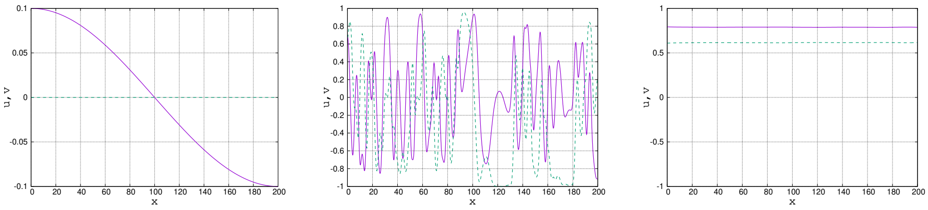
We also use the vector form:

$$\frac{\partial \boldsymbol{u}}{\partial t} = d\Delta \boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}), \tag{3.1.3}$$

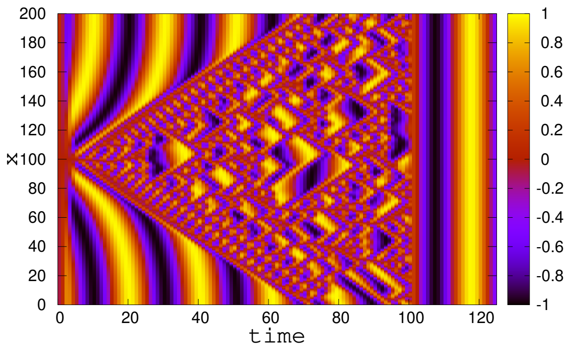
where $\boldsymbol{u} = (u, v)$ and

$$\boldsymbol{f}(\boldsymbol{u}) := \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix} = \begin{pmatrix} u - \omega v - u^3 - uv^2 + a\omega(u^2+v^2)v \\ v + \omega u - v^3 - u^2 v - a\omega(u^2+v^2)u \end{pmatrix}. \tag{3.1.4}$$

The existence of classical solutions of (3.1.1) follows from the standard theory of parabolic equations. For example, see [27] and [9, Chapter 3]. The existence of the invariant sets guarantees the existence of global solutions. The system (3.1.2) has obviously a constant unstable solution $(u, v) = (0, 0)$. Substituting $z = e^{ict}$ into (3.1.1), we obtain $c = (1 - a)\omega$. Therefore, the complex Ginzburg–Landau equation (3.1.2) possesses a homogeneous periodic solution $\boldsymbol{p}(t) := (\cos((1-a)\omega t + \theta_0), \sin((1-a)\omega t + \theta_0))$. It is often called a limit cycle. However, this system possesses spiral patterns and spatially inhomogeneous non-periodical solutions called spatio-temporal chaos in some parameter regions (see [12, 11, 18] for the details). Spatio-temporal chaos appears after collapsing the spiral pattern (e.g., see [22, 4, 2]). Actually, the spatio-temporal "chaotic" behavior of (3.1.1) in one-dimensional interval is observed numerically as in Figure 3.1 (b).



(a) $t = 0$ (b) $t = 100$ (c) $t = 120$



(d) space(vertical)-time(horizontal) plot of u

Figure 3.1: Numerical solution (u, v) of (3.1.2) in $\Omega = (0, 200)$ with $a = 0.9, \omega = 3.0$ and $(u(x, 0), v(x, 0)) = (0.1\cos(\pi x/200), 0)$. The solid curve (resp. the dotted curve) indicates the profile of u (resp. v) (a) $t = 0$ (initial condition), (b) $t = 100$ (spatio-temporal chaos before the perturbation), (c) $t = 120$ (after the perturbation). (d) color map of space(vertical)-time(horizontal) plot of u. Before the perturbation at $t = 100$, the spatio-temporal chaotic behavior is observed. After the perturbation, u suddenly becomes spatially homogeneous and oscillates only in time.

Next, we consider the way to retain the regular oscillation. As seen in Figure 3.1 (b), the "chaotic" solution is spatially inhomogeneous. Thus, the solution must be spatially homogenized to retain the regular oscillation. One way to homogenize solutions is to increase the diffusion coefficients. It is known that the solution becomes spatially homogenized (see [3]). Spatially homogeneous solutions satisfy the system of ordinary differential equations corresponding to (3.1.3):

$$\frac{d\boldsymbol{U}}{dt} = \boldsymbol{f}(\boldsymbol{U}), \tag{3.1.5}$$

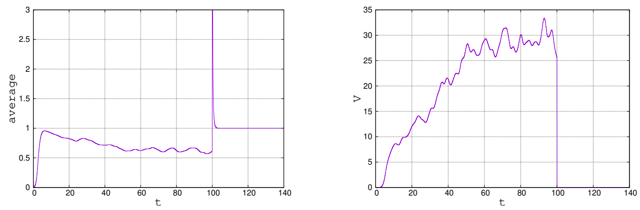
where $\boldsymbol{U} = (u, v)$ and $\boldsymbol{f}$ is given in (3.1.4). All solutions of (3.1.5) except for $(0, 0)$ converge to the limit cycle. By changing the diffusion coefficients, however, the dynamics also becomes simple and the chaotic behavior are no longer observed. Therefore, we fix the diffusion coefficient and we can assume that $d = 1$ in (3.1.1), (3.1.2) and(3.1.3) without loss of generality. In this paper, we put the perturbation on the external force to homogenize a solution. If we add the perturbation of the delta function such as $\gamma\delta(t - t_0)$ to the right-hand side of the equation (3.1.1), then the perturbed solution is regarded as the solution starting from the shifted initial data $z(\cdot, t_0) + \gamma$ at $t = t_0$. In the numerical simulation of Figure 3.1, we add the external force

$$P(t) := \begin{cases} \dfrac{\gamma}{\varepsilon} & (100 < t \leq 100 + \varepsilon), \\ 0 & \text{otherwise,} \end{cases}$$

to (3.1.1) instead of $\gamma\delta(t - 100)$ where $\gamma = 30$ and $\varepsilon = 0.1$. As seen in Figure 3.1 (c), the numerical solution becomes almost spatially homogeneous after the perturbation. The graphs of $\|z\|^2_{L_2(\Omega)}/|\Omega|$ and $\|\nabla z\|^2_{L_2(\Omega)}/2$ are shown in Figure 3.2 (a) and (b) respectively. Figure 3.2 (a) shows that the perturbation forces the solution to move to the outside of the limit cycle at $t = 100$. Because the average of the gradient of z indicates the spatial inhomogeneity, as shown in Figure 3.2 (b), the solution quickly homogenizes spatially after the perturbation (see also Figure 3.1 (d)). Thus, we expect that it converges to the limit cycle oscillation. We state this observation mathematically as in the following theorem.

Theorem 3.1.1 (Approach to the limit cycle by perturbation).
Assume that $a\omega > 0$. For any initial data $\boldsymbol{w}(\cdot) = (w_1(\cdot), w_2(\cdot)) \in C^0(\overline{\Omega})^2$, there is a positive constant K_1 depending only on $\|\boldsymbol{w}\|_{C^0(\Omega)}$ such that the solution $\boldsymbol{u}$ of (3.1.3) with $\boldsymbol{u}(\cdot, 0) = \boldsymbol{w}(\cdot) + (K_1, 0)$ converges to the homogenous periodic solution $\boldsymbol{p}(t - \eta_0)$ with some η_0 as $t \to \infty$.

Note that the perturbed moment is regarded as $t = 0$ in Theorem 3.1.1. For simplicity, we perturb the solution along the horizontal direction in the phase



(a) The average of $|z|^2$ (b) $V = \|\nabla z(\cdot,t)\|^2_{L_2(\Omega)}/2$

Figure 3.2: The time dependence of the average of $\|z(\cdot,t)\|^2_{L^2(\Omega)}/|\Omega|$ and $\|\nabla z(\cdot,t)\|^2_{L_2(\Omega)}/2$. Parameters are chosen as in Figure 3.1.

plane. Since (3.1.1) is invariant under the rotation of $z = u + iv \in \mathbb{C}$, Theorem 1.1 also holds even for any directions (K_1, K_2) if $|(K_1, K_2)|$ is sufficiently large.

3.2 Construction of invariant sets

3.2.1 Construction of invariant sets of the Complex Ginzburg-Landau equation

In this subsection, we construct the invariant set $\Sigma(t)$ of (3.1.2) by applying Theorem 2.5.3. To show (2.5.10), we first study the system (3.1.5) of the ordinary differential equations corresponding to (3.1.2). We also denote the map from $\boldsymbol{U}(0) = (u(0), v(0))$ to $\boldsymbol{U}(t) = (u(t), v(t))$ by $S(t)$, namely, $\boldsymbol{U}(t) = S(t)\boldsymbol{U}(0)$. By the polar coordinates as $(u, v) = R(\cos\Theta, \sin\Theta)$, (3.1.5) is rewritten to

$$\begin{cases} \dfrac{dR}{dt} = R - R^3, \\ \dfrac{d\Theta}{dt} = \omega(1 - aR^2). \end{cases} \tag{3.2.6}$$

It is easily seen that the solution of (3.2.6) with the initial condition $(R(0), \Theta(0)) = (R_0, \Theta_0)$ is given by

$$R(t) = \frac{R_0 e^t}{\sqrt{R_0^2(e^{2t} - 1) + 1}}, \tag{3.2.7}$$

$$\begin{aligned} \Theta(t) &= \Theta_0 + \omega t - \frac{1}{2} a\omega \log(R_0^2(e^{2t} - 1) + 1) \\ &= \Theta_0 + (1 - a)\omega t - \frac{1}{2} a\omega \log(R_0^2(1 - e^{-2t}) + e^{-2t}). \end{aligned} \tag{3.2.8}$$

It is denoted by $(R(t; R_0), \Theta(t; R_0, \Theta_0))$. Note that

$$R_0 = \frac{R(t)e^{-t}}{\sqrt{R(t)^2(e^{-2t} - 1) + 1}}.$$

Thus we set

$$R_0(t, r) := \frac{re^{-t}}{\sqrt{r^2(e^{-2t} - 1) + 1}}.$$

Remark 3.2.1. *It follows from* (3.2.7) *and* (3.2.8) *that any solution of* (3.2.6) *converges to the time-periodic solution* $(\cos((1-a)\omega t + \theta_0), \sin((1-a)\omega t + \theta_0))$ *with* $\theta_0 = \Theta_0 - a\omega \log R_0$.

We have the following properties.

Lemma 3.2.2. *Let* $(R_1, \Theta_1), (R_2, \Theta_2)$ *be two solutions of* (3.2.6).

(i) *If* $R_1(0) \leq R_2(0)$*, then* $R_1(t) \leq R_2(t)$ *for any* $t > 0$.

(ii) *If* $R_1(0) = R_2(0)$ *and* $\Theta_1(0) \geq \Theta_2(0)$*, then* $\Theta_1(t) \geq \Theta_2(t)$ *for any* $t > 0$.

Proof. The statement (i) is easily shown by (3.2.6). We will show (ii). By (3.2.6) with $R_1(0) = R_2(0)$, $R_1(t) = R_2(t)$ for any $t \geq 0$, which is denoted by $R(t)$. Then Θ_j satisfies the same equation

$$\frac{d\Theta_j}{dt} = \omega(1 - aR(t)^2), \quad \Theta_1(0) \geq \Theta_2(0).$$

Integrating the right-hand side of the above equation, we can conclude the second statement (ii). □

Next we will construct the invariant set of (3.1.1) by using three solutions of (3.2.6). Let $(R_j(t), \Theta_j(t)) := (R(t; R_j(0)), \Theta(t; R_j(0), \Theta_j(0)))$ $(j = 1, 2, 3)$ be three different solutions of (3.2.6) with initial conditions satisfying

$$\Theta_1(0) < \Theta_2(0) = \Theta_3(0), \quad 1 < R_3(0) < R_1(0) = R_2(0). \tag{3.2.9}$$

Set $\boldsymbol{U}_j(t) = (R_j(t)\cos\Theta_j(t), R_j(t)\sin\Theta_j(t))$ for $j = 1, 2, 3$. It immediately follows from Lemma 3.2.2 and (3.2.9) that

$$R_3(t) \leq R_1(t) = R_2(t), \quad \Theta_1(t) < \Theta_2(t).$$

Since $\Theta_3'(t) - \Theta_2'(t) = a\omega(R_2(t)^2 - R_3(t)^2) \geq 0$, we get that $\Theta_1(t) < \Theta_2(t) < \Theta_3(t)$ when $a\omega > 0$. Moreover, by (3.2.8), we have

$$\begin{aligned} \Theta_3(t) - \Theta_1(t) &= \Theta_3(0) - \Theta_1(0) + a\omega \log \frac{R_1(0)^2(e^{2t} - 1) + 1}{R_3(0)^2(e^{2t} - 1) + 1} \\ &\leq \Theta_3(0) - \Theta_1(0) + 2a\omega \log \frac{R_1(0)}{R_3(0)}. \end{aligned} \tag{3.2.10}$$

Under the assumption

$$\Theta_3(0) - \Theta_1(0) + 2a\omega \log \frac{R_1(0)}{R_3(0)} < \pi, \tag{3.2.11}$$

we have $0 < \Theta_3(t) - \Theta_1(t) < \pi$ for any $t > 0$.

Let $\Sigma(t)$ be a set enclosed by three curves:

(i) the curve transferred from the line segment connecting $\boldsymbol{U}_2(0)$ and $\boldsymbol{U}_3(0)$ by the map $S(t)$;

(ii) the line segment connecting $\boldsymbol{U}_1(t)$ and $\boldsymbol{U}_3(t)$;

(iii) the arc between $\boldsymbol{U}_1(t)$ and $\boldsymbol{U}_2(t)$ with radius $|\boldsymbol{U}_1(t)| = |\boldsymbol{U}_2(t)|$.

See Figure 3.3 for its illustrative profiles. To represent the above curves explicitly, we introduce the auxiliary functions

$$\begin{aligned} H_1(\boldsymbol{w}, t) &:= \arg \boldsymbol{w} - \Theta(t; R_0(t, |\boldsymbol{w}|), \Theta_2(0)), \\ H_2(\boldsymbol{w}, t) &:= \boldsymbol{n}(t) \cdot (\boldsymbol{U}_3(t) - \boldsymbol{w}), \\ H_3(\boldsymbol{w}, t) &:= |\boldsymbol{w}|^2 - R_1(t)^2, \end{aligned}$$

where

$$\boldsymbol{n}(t) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\boldsymbol{U}_1(t) - \boldsymbol{U}_3(t)).$$

Here $\arg \boldsymbol{w}$ is a multivalued function defined as $\arctan(w_2/w_1) + k\pi$ where k is integer specified later. Introduce three sets $\Sigma_j(t)$ $(j = 1, 2, 3)$ by

$$\Sigma_j(t) := \{\boldsymbol{w} \in \mathbb{R}^2 \mid H_j(\boldsymbol{w}, t) \leq 0\} \qquad (j = 1, 2, 3). \tag{3.2.12}$$

Then set

$$\Sigma(t) := \bigcap_{j=1}^{3} \Sigma_j(t).$$

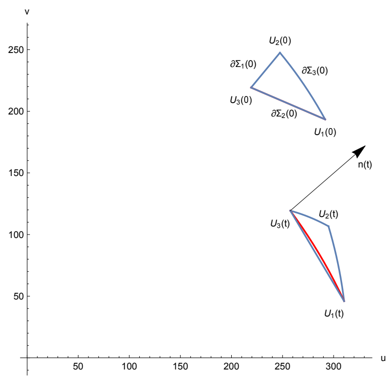


Figure 3.3: The invariant sets $\Sigma(0)$ and $\Sigma(t)$.

When $\boldsymbol{w} \in \Sigma(t)$, $\arg \boldsymbol{w}$ is chosen as $\min_{j=1,2,3} \Theta_j(t) \leq \arg \boldsymbol{w} \leq \max_{j=1,2,3} \Theta_j(t)$.

Consider the line segment $\partial\Sigma_1(0) \cap \partial\Sigma(0)$ described as

$$\{(r\cos\Theta_2(0), r\sin\Theta_2(0)) \mid R_3(0) \leq r \leq R_2(0)\}.$$

The semiflow defined by (3.1.5) maps the line segment to the curve

$$\{R(t;r)(\cos\Theta(t;r,\Theta_2(0)), \sin\Theta(t;r,\Theta_2(0))) \mid R_3(0) \leq r \leq R_2(0)\},$$

which coincides with the boundary $\partial\Sigma_1(t) \cap \partial\Sigma(t)$. Noting that $\boldsymbol{n}(t) \cdot (\boldsymbol{U}_3(t) - \boldsymbol{U}_1(t)) = 0$, we see that the boundary $\partial\Sigma_2(t) \cap \partial\Sigma(t)$ is the line segment between $\boldsymbol{U}_1(t)$ and $\boldsymbol{U}_3(t)$. Since $\Sigma_3(t) = \{\boldsymbol{w} \in \mathbb{R}^2 \mid |\boldsymbol{w}| \leq R_1(t)\}$, the boundary $\partial\Sigma_3(t) \cap \partial\Sigma(t)$ is the arc of a circle with radius $R_1(t)$ between $\boldsymbol{U}_1(t)$ and $\boldsymbol{U}_2(t)$.

Proposition 3.2.3. *Assume that $a\omega > 0$. Then, the family of $\Sigma(t)$ defined as above is the invariant set of (3.1.5) if*

$$\begin{cases} \Theta_2(0) = \Theta_3(0), \quad 0 < \Theta_2(0) - \Theta_1(0) \leq \varepsilon, \\ 1 < R_3(0) < R_1(0) = R_2(0) < (1+\delta)R_3(0) \end{cases} \tag{3.2.13}$$

for some positive constants ε and δ.

Proof. First we show $\Sigma(t)$ is a bounded closed set. It is obvious that $\Sigma(t)$ is closed. Since $\Sigma \subset \Sigma_1(t)$, it is bounded. The condition (3.2.11) is satisfied for small $\varepsilon > 0$ and $\delta > 0$. Next, by noting $\Sigma_j(t) \cap \partial\Sigma(t) = S(t)\{\Sigma_j(0) \cap \partial\Sigma(0)\}$ $(j = 1, 3)$, we can see that (2.5.10) holds when $j = 1, 3$, here we show it directly. Take any positive t and any $\boldsymbol{w} \in \partial\Sigma_1(t) \cap \partial\Sigma(t)$. First note that, for $\boldsymbol{w} = (w_1, w_2) = R(\cos\Theta, \sin\Theta)$,

$$\begin{aligned} RR_t &= \boldsymbol{w} \cdot \boldsymbol{f}(\boldsymbol{w}), \\ R^2\Theta_t &= \begin{pmatrix} -w_2 \\ w_1 \end{pmatrix} \cdot \boldsymbol{f}(\boldsymbol{w}). \end{aligned}$$

From (3.2.6), we have

$$\begin{aligned}
&\frac{d}{dt}\Theta(t;R_0(t,|\boldsymbol{w}|),\Theta_2(0)) \\
&= \omega(1-aR(t;R_0)^2)+\frac{\partial}{\partial R_0}\Theta(t;R_0(t,|\boldsymbol{w}|),\Theta_2(0))\frac{\partial}{\partial t}R_0(t,|\boldsymbol{w}|), \\
&(\nabla_{\boldsymbol{w}}\arg \boldsymbol{w})\cdot \boldsymbol{f}(\boldsymbol{w}) \\
&= \frac{1}{w_1^2+w_2^2}\begin{pmatrix}-w_2\\ w_1\end{pmatrix}\cdot \boldsymbol{f}(\boldsymbol{w}) = \omega(1-aR(t;R_0)^2), \\
&(\nabla_{\boldsymbol{w}}\Theta(t;R_0(t,|\boldsymbol{w}|),\Theta_2(0)))\cdot \boldsymbol{f}(\boldsymbol{w}) \\
&= \frac{\partial}{\partial R_0}\Theta(t;R_0(t,|\boldsymbol{w}|),\Theta_2(0))\nabla_{\boldsymbol{w}}R_0(t,|\boldsymbol{w}|)\cdot \boldsymbol{f}(\boldsymbol{w}).
\end{aligned}$$

Thus, these equalities imply that

$$\begin{aligned}
&\frac{\partial H_1}{\partial t}(\boldsymbol{w},t)+\nabla_{\boldsymbol{w}}H_1(\boldsymbol{w},t)\cdot \boldsymbol{f}(\boldsymbol{w}) \\
&= -\omega(1-aR(t;R_0)^2)-\frac{\partial\Theta}{\partial R_0}\frac{\partial}{\partial t}R_0(t,|\boldsymbol{w}|) \\
&\quad +\omega(1-aR(t;R_0)^2)-\frac{\partial\Theta}{\partial R_0}\nabla_{\boldsymbol{w}}R_0(t,|\boldsymbol{w}|)\cdot \boldsymbol{f}(\boldsymbol{w}) \\
&= -\frac{\partial\Theta}{\partial R_0}\Big(\frac{\partial}{\partial t}R_0(t,|\boldsymbol{w}|)+\boldsymbol{f}(\boldsymbol{w})\cdot\nabla_{\boldsymbol{w}}R_0(t,|\boldsymbol{w}|)\Big) \\
&= -\frac{\partial\Theta}{\partial R_0}\Big(\frac{\partial}{\partial t}R_0(t,|\boldsymbol{w}|)+\frac{1}{|\boldsymbol{w}|}\boldsymbol{w}\cdot \boldsymbol{f}(\boldsymbol{w})\frac{\partial R_0}{\partial R}\Big) \\
&= 0
\end{aligned}$$

for any $\boldsymbol{w}\in\partial\Sigma_1(t)\cap\partial\Sigma(t),\ t>0$.

Consider the case where $j=3$ and take $\boldsymbol{w}\in\partial\Sigma_3(t)\cap\partial\Sigma(t)$. Since the first equation of (3.2.6) is autonomous, $R_1(t)=R(t;R_{\boldsymbol{w}})$ where $R(t;R_{\boldsymbol{w}})=|\boldsymbol{w}|$ similarly to the above case. Therefore

$$\begin{aligned}
&\frac{\partial H_3}{\partial t}(\boldsymbol{w},t)+\nabla_{\boldsymbol{w}}H_3(\boldsymbol{w},t)\cdot \boldsymbol{f}(\boldsymbol{w}) \\
&= -2R_1(t)^2(1-R_1(t)^2)+2|\boldsymbol{w}|^2(1-|\boldsymbol{w}|^2)=0.
\end{aligned}$$

Thus (2.5.10) with $j=3$ holds for $\boldsymbol{w}\in\partial\Sigma_3(t)\cap\partial\Sigma(t)$ and $t>0$.

Finally we prove (2.5.10) when $j=2$. Set

$$\boldsymbol{u}^s(t) := s\boldsymbol{U}_1(t)+(1-s)\boldsymbol{U}_3(t).$$

Because $\boldsymbol{n}(t)\cdot(\boldsymbol{U}_3(t)-\boldsymbol{u}^s(t))=0$ for any $t>0$, we have

$$\frac{d}{dt}\Big[\boldsymbol{n}(t)\cdot(\boldsymbol{U}_3(t)-\boldsymbol{u}^s(t))\Big]=0.$$

Recall that $H_2(\boldsymbol{u},t) = \boldsymbol{n}(t)\cdot(\boldsymbol{U}_3(t)-\boldsymbol{u})$. Then

$$\frac{\partial H_2}{\partial t} + \nabla_u H_2 \cdot \boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{n}'(t)\cdot(\boldsymbol{U}_3(t)-\boldsymbol{u}) + \boldsymbol{n}(t)\cdot\boldsymbol{U}_3'(t) - \boldsymbol{n}(t)\cdot\boldsymbol{f}(u).$$

Substituting $\boldsymbol{u} = \boldsymbol{u}^s(t)$ yields

$$\begin{aligned}
&\left(\frac{\partial H_2}{\partial t} + \nabla_u H_2 \cdot \boldsymbol{f}(\boldsymbol{u})\right)\Big|_{\boldsymbol{u}=\boldsymbol{u}^s(t)} \\
&= \boldsymbol{n}'(t)\cdot(\boldsymbol{U}_3(t)-\boldsymbol{u}^s(t)) + \boldsymbol{n}(t)\cdot\boldsymbol{U}_3'(t) - \boldsymbol{n}(t)\cdot\boldsymbol{f}(\boldsymbol{u}^s(t)) \\
&= \frac{d}{dt}[\boldsymbol{n}(t)\cdot(\boldsymbol{U}_3(t)-\boldsymbol{u}^s(t))] + \boldsymbol{n}(t)\cdot\left(\frac{d\boldsymbol{u}^s(t)}{dt} - \boldsymbol{f}(\boldsymbol{u}^s(t))\right) \\
&= \boldsymbol{n}(t)\cdot\left(\frac{d\boldsymbol{u}^s(t)}{dt} - \boldsymbol{f}(\boldsymbol{u}^s(t))\right)
\end{aligned}$$

Set

$$\begin{aligned}
I(s) &:= \boldsymbol{n}(t)\cdot[(1-s)\boldsymbol{f}(\boldsymbol{U}_3(t)) + s\boldsymbol{f}(\boldsymbol{U}_1(t))] \\
&\quad -\boldsymbol{n}(t)\cdot\boldsymbol{f}((1-s)\boldsymbol{U}_3(t) + s\boldsymbol{U}_1(t)) \\
&= \boldsymbol{n}(t)\cdot\left(\frac{d\boldsymbol{u}^s(t)}{dt} - \boldsymbol{f}(\boldsymbol{u}^s(t))\right).
\end{aligned}$$

We show that $I(s) \le 0$ for any $t > 0$ and $s \in [0,1]$. For the simplicity of notation, we write $\boldsymbol{U} = \boldsymbol{U}(t)$, $\boldsymbol{u}^s = \boldsymbol{u}^s(t)$ and so on. We also set $\boldsymbol{U}_j = (U_j, V_j), \boldsymbol{U} = \boldsymbol{U}_1 - \boldsymbol{U}_3, \boldsymbol{f}(\boldsymbol{u}) = (f_1(\boldsymbol{u}), f_2(\boldsymbol{u}))$. The definitions of f_1 and f_2 in (3.1.4) imply

$$\begin{aligned}
&(1-s)f_1(\boldsymbol{U}_3) + sf_1(\boldsymbol{U}_1) - f_1((1-s)\boldsymbol{U}_3 + s\boldsymbol{U}_1) \\
&= -(1-s)s\Big[(1+s)(U^2+V^2)U + 3U_3U^2 + U_3V^2 + 2V_3UV \\
&\quad +a\omega\Big\{-(1+s)(U^2+V^2)V - V_3U^2 - 2U_3UV - 3V_3V^2\Big\}\Big], \\
&(1-s)f_2(\boldsymbol{U}_3) + sf_2(\boldsymbol{U}_1) - f_2((1-s)\boldsymbol{U}_3 + s\boldsymbol{U}_1) \\
&= -(1-s)s\Big[(1+s)(U^2+V^2)V + 2U_3UV + V_3U^2 + 3V_3V^2 \\
&\quad +a\omega\Big\{-(1+s)(U^2+V^2)U + 3U_3U^2 + 2V_3UV + U_3V^2\Big\}\Big].
\end{aligned}$$

Using $\boldsymbol{n} = (-V, U)^T$, we obtain

$$\begin{aligned}
I(s) &= -s(1-s)(U^2+V^2) \\
&\quad \times\Big[V_3U - U_3V + a\omega\{(1+s)(U^2+V^2) + 3U_3U + 3V_3V\}\Big] \\
&= -s(1-s)(U^2+V^2)(I_1 + a\omega I_2),
\end{aligned}$$

where

$$\begin{aligned} I_1 &:= V_3U - U_3V, \\ I_2 &:= (1+s)(U^2+V^2) + 3U_3U + 3V_3V. \end{aligned}$$

Noting $\boldsymbol{U}_j = R_j(\cos\Theta_j, \sin\Theta_j)$, we have

$$\begin{aligned} I_1 &= (R_1\cos\Theta_1 - R_3\cos\Theta_3)R_3\sin\Theta_3 \\ &\quad -R_3\cos\Theta_3(R_1\sin\Theta_1 - R_3\cos\Theta_3) \\ &= R_1R_3\sin(\Theta_3-\Theta_1), \\ I_2 &\geq (U^2+V^2) + 3U_3U + 3V_3V \\ &= U(U_1+2U_3) + V(V_1+2V_3) \\ &= (R_1\cos\Theta_1 - R_3\cos\Theta_3)(R_1\cos\Theta_1 + 2R_3\cos\Theta_3) \\ &\quad +(R_1\sin\Theta_1 - R_3\sin\Theta_3)(R_1\sin\Theta_1 + 2R_3\sin\Theta_3) \\ &= R_1^2 - R_3^2 + R_3(R_1-R_3) - R_1R_3(1-\cos(\Theta_3-\Theta_1)). \end{aligned}$$

To prove $I(s) \leq 0$, we need to show $I_1 + a\omega I_2 \geq 0$. By the first equation of (3.2.6), $R_1(t) > R_3(t)$. Therefore,

$$I_2 \geq -R_1R_3(1-\cos(\Theta_3-\Theta_1)).$$

Combing the above inequalities, we have

$$I_1 + a\omega I_2 \geq R_1R_3\Big\{\sin(\Theta_3-\Theta_1) - a\omega(1-\cos(\Theta_3-\Theta_1))\Big\}.$$

From (3.2.10) and (3.2.13),

$$\Theta_3(t) - \Theta_1(t) \leq \Theta_3(0) - \Theta_1(0) + 2a\omega\log\frac{R_1(0)}{R_3(0)}, \tag{3.2.14}$$

which implies

$$\Theta_3(t) - \Theta_1(t) \leq \varepsilon + 2a\omega\log(1+\delta).$$

For any positive constant κ, there exists $\xi_*(\kappa) \in (0,\pi)$ such that

$$\sin\xi \geq \kappa(1-\cos\xi) \qquad \text{for } \xi \in [0, \xi_*(\kappa)].$$

Therefore, if positive constants ε and δ are small that

$$\varepsilon + 2a\omega\log(1+\delta) < \xi_*(a\omega), \tag{3.2.15}$$

then

$$\Theta_3(t) - \Theta_1(t) < \xi_*(a\omega) \qquad \text{for any } t > 0.$$

By the choice of $\xi_*(a\omega)$,

$$\sin(\Theta_3 - \Theta_1) - a\omega(1 - \cos(\Theta_3 - \Theta_1)) \geq 0.$$

Therefore, we obtain

$$I_1 + a\omega I_2 \geq 0,$$

which implies

$$\left(\frac{\partial H_2}{\partial t} + \nabla_u H_2 \cdot \boldsymbol{f}(\boldsymbol{u})\right)\Big|_{\boldsymbol{u}=\boldsymbol{u}^s(t)} \leq 0.$$

For any $t > 0$ and any $\boldsymbol{w} \in \partial\Sigma_2(t) \cap \partial\Sigma(t)$, there is $s \in [0,1]$ such that $\boldsymbol{w} = \boldsymbol{u}^s(t)$. The condition (2.5.10) holds when $j = 2$. Hence, the family of $\Sigma(t) = \bigcap_{j=1}^{3} \Sigma_j(t)$ is the invariant set of (3.1.5). □

Remark 3.2.4. *The invariant set $\Sigma(t)$ can intersect with the ball with radius 1, because $\Theta_3 - \Theta_1$ is positive. Especially, for any $R(\cos\Theta, \sin\Theta) \in \Sigma(t)$, we have*

$$\cos\left(\frac{\varepsilon}{2} + a\omega\log(1+\delta)\right) \leq R \leq R_1(t). \tag{3.2.16}$$

Lemma 3.2.5. *Assume $a\omega > 0$. Then the set $\Sigma(t)$ is convex.*

Proof. To show the convexity of $\Sigma(t)$, we only need to show that of $\partial\Sigma_1$, because $\partial\Sigma_2(t) \cap \partial\Sigma(t)$ is a line segment and $\partial\Sigma_3(t) \cap \partial\Sigma(t)$ is a portion of a circle. The boundary $\partial\Sigma_1(t) \cap \partial\Sigma(t)$ is given by $\{\boldsymbol{w} \in \partial\Sigma(t) \mid H_1(\boldsymbol{w}, t) = 0\}$. Thus,

$$\begin{aligned} &\partial\Sigma_1(t) \cap \partial\Sigma(t) \\ &\quad = \{\boldsymbol{w} \in \mathbb{R}^2 \mid H_1(\boldsymbol{w}, t) = 0,\ R_3(t) \leq |\boldsymbol{w}| \leq R_1(t)\} \\ &\quad = \left\{ R(t; r_0)(\cos\theta, \sin\theta) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} \theta = \Theta(t; r_0, \Theta_2(0)), \\ R_3(0) \leq r_0 \leq R_1(0) \end{array} \right\}. \end{aligned}$$

Namely, $\partial\Sigma_1(t) \cap \partial\Sigma(t)$ is a curve parameterized by r_0 for all $t > 0$. Hence the tangent vector of $\partial\Sigma_1(t)$ at

$$\boldsymbol{w} = R(t; r_0)(\cos\Theta(t; r_0, \Theta_2(0)), \sin\Theta(t; r_0, \Theta_2(0)))$$

can be describe as follows:

$$\begin{aligned}\frac{d\boldsymbol{w}}{dr_0} &= \frac{\partial R(t;r_0)}{\partial r_0}(\cos\Theta, \sin\Theta) + R(t;r_0)(-\sin\Theta, \cos\Theta)\frac{\partial\Theta(t;r_0,\Theta_2(0))}{\partial r_0} \\ &= \frac{e^t}{(r_0^2(e^{2t}-1)+1)^{3/2}}\left\{(\cos\Theta, \sin\Theta) + a\omega r_0^2(e^{2t}-1)(\sin\Theta, -\cos\Theta)\right\} \\ &= \frac{e^t}{(r_0^2(e^{2t}-1)+1)^{3/2}}(\cos(\Theta+\alpha), \sin(\Theta+\alpha)),\end{aligned}$$

where α is given by $\tan\alpha := -a\omega r_0^2(e^{2t}-1)$. Here, we used

$$\Theta(t;r_0,\Theta_2(0)) = \Theta_2(0) - a\omega\log(r_0^2(e^{2t}-1)+1).$$

Therefore, the curvature of $\partial\Sigma_1(t)\cap\partial\Sigma(t)$ satisfies

$$\begin{aligned}&\frac{d(\Theta(t;r_0,\Theta_2(0))+\alpha)}{dr_0} \\ &= \frac{\partial\Theta(t;r_0,\Theta_2(0))}{\partial r_0} + \frac{1}{1+\tan^2\alpha}\frac{d\alpha}{dr_0} \\ &= -a\omega r_0(e^{2t}-1)\left(\frac{1}{r_0^2(e^{2t}-1)+1} + \frac{1}{1+\tan^2\alpha}\right).\end{aligned}$$

Since

$$\frac{1}{r_0^2(e^{2t}-1)+1} + \frac{1}{1+\tan^2\alpha} \geq 0,$$

and $a\omega > 0$, the curvature is negative for any $t > 0$. Because the orientation by r_0 is opposite, it is shown that $\partial\Sigma_1(t)\cap\partial\Sigma(t)$ is convex. □

Theorem 3.2.6. *Let ε, δ be chosen as in Proposition 3.2.3. Under the assumptions* (3.2.13) *and $a\omega > 0$, the family of $\Sigma(t)$ is the invariant set of* (3.1.1).

Proof. This theorem immediately follows from Theorem 2.5.3, Proposition 3.2.3 and Lemma 3.2.5. □

As seen in Remark 3.2.1, the invariant set $\Sigma(t)$ converges to the orbit of the limit cycle, namely, $\{\boldsymbol{w} \in \mathbb{R}^2 \mid |\boldsymbol{w}|^2 = 1\}$. Hence, Theorem 3.2.6 implies that the solution of (3.1.1) approaches to the orbit of the limit cycle after a sufficiently long time, even though the image of $(u_0(\boldsymbol{x}), v_0(\boldsymbol{x}))$ is distributed far from the limit cycle. To show the convergence to the limit cycle, we need to study the dynamics of (3.1.1) near the limit cycle.

3.3 Convergence to the spatial homogeneous periodic solution on the limit cycle

To apply Theorem 1.3.1, we first study characteristic multiplier near the periodic solution $\boldsymbol{p}(t)$. Recall that $\boldsymbol{p}(t) = (\cos((1-a)\omega t + \theta_0), \sin((1-a)\omega t + \theta_0))$ is a periodic solution of (3.1.5) as well as (3.1.3). The period of $\boldsymbol{p}$ is given by

$$T := \frac{2\pi}{(1-a)\omega}.$$

To study the stability of $\boldsymbol{p}(t)$, we consider the linearized equation:

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \Delta \boldsymbol{\xi} + \boldsymbol{J}(t)\boldsymbol{\xi}, \qquad \boldsymbol{x} \in \Omega,\ t > 0, \tag{3.3.17}$$

with the Neumann boundary condition where

$$\boldsymbol{J}(t) := \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{p}(t)).$$

Denote the Poincaré map of the above linearized equation by $\Phi(T)$, namely, $\boldsymbol{\xi}(T) = \Phi(T)\boldsymbol{\xi}(0)$. The non-zero eigenvalues of $\Phi(T)$ are called *characteristic multipliers*.

Let ϕ_n be an eigenfunction of $-\Delta$ which satisfies Neumann boundary condition and λ_n be the corresponding eigenvalue related to ϕ_n, namely,

$$-\Delta \phi_n = \lambda_n \phi_n, \quad \lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \cdots.$$

We can expand the solution of the linearized equation (3.3.17) by ϕ_n, namely,

$$\boldsymbol{\xi}(t) = \sum_{n=0}^{\infty} \boldsymbol{c}_n(t)\phi_n$$

with some $\boldsymbol{c}_n(t) \in \mathbb{R}^2$. Substituting the above expansion to (3.3.17) yields

$$\frac{d\boldsymbol{c}_n}{dt} = (\boldsymbol{J}(t) - \lambda_n)\boldsymbol{c}_n \quad (n = 0, 1, 2, \cdots). \tag{3.3.18}$$

We denote the eigenvalues of the above equation by $\mu_1^{(n)}, \mu_2^{(n)}$. Consider the case where $n = 0$. Since $\boldsymbol{p}(t)$ satisfies (3.1.5), the differentiation of (3.1.5) with respect to t yields

$$\frac{d^2\boldsymbol{p}}{dt^2} = \boldsymbol{J}(t)\frac{d\boldsymbol{p}}{dt},$$

which implies that $\boldsymbol{dp}/dt$ is a solution of (3.3.18) with $\lambda_n = 0$. Thus we have

$$\frac{d\boldsymbol{p}}{dt}(T) = \frac{d\boldsymbol{p}}{dt}(0) = \Phi(T)\frac{d\boldsymbol{p}}{dt}(0),$$

which implies that 1 is a characteristic multiplier and $\boldsymbol{dp}/dt(0)$ is the corresponding eigenvector. Then,

$$\mu_1^{(0)}\mu_2^{(0)} = \det\Phi(T) = \exp\int_0^T \mathrm{Tr}(J(t))dt.$$

Setting $\mu_1^{(0)} = 1$, we have

$$\mu_2^{(0)} \quad = \quad \exp\int_0^T \mathrm{Tr}(J(t))dt.$$

Since

$$\begin{aligned}
\mathrm{Tr}(J) &= \left(\frac{\partial f_1}{\partial u} + \frac{\partial f_2}{\partial v}\right)\Big|_{(u,v)=\boldsymbol{p}(t)} \\
&= \left(1 - 3u^2 - v^2 + 2a\omega uv + 1 - 3v^2 - u^2 - 2a\omega uv\right)\Big|_{(u,v)=\boldsymbol{p}(t)} \\
&= 2 - 4(u^2 + v^2)|_{(u,v)=\boldsymbol{p}(t)} = -2 < 0,
\end{aligned}$$

we see that $0 < \mu_2^0 < 1$. Using

$$\frac{d}{dt}(e^{\lambda_n t}\boldsymbol{c_n}) = J(t)e^{\lambda_n t}\boldsymbol{c_n},$$

we get

$$\mu_1^{(n)} = e^{-\lambda_n T}, \quad \mu_2^{(n)} = e^{-\lambda_n T}\mu_2^{(0)}.$$

We can conclude that the characteristic multiplier 1 is an isolated simple eigenvalue and others are contained in the region $|\mu| < e^{-\beta T}$ for some constant $\beta > 0$.

Applying [9, Theorem 8.2.3] to (3.1.3), we see that the limit cycle $\boldsymbol{p}(t)$ is asymptotically stable in the following sense:

Theorem 3.3.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$. *Consider the sectorial operator* $A = -\Delta : D(A) \to X = L^q(\Omega)$ *where*

$$D(A) = \left\{\boldsymbol{u} \in W^{2,q}(\omega); \frac{\partial u_i}{\partial \nu} = 0, \quad i \in [1, n]\right\}.$$

Additionally we set the nonlinear function of the Complex Ginzburg-Landau equation $f : X^{\frac{1}{2}} \to X$. *there are positive constants* ρ *and* C *such that for any*

$\boldsymbol{u}(\boldsymbol{x},0)$ satisfying $\|\boldsymbol{u}(\boldsymbol{x},0)-\boldsymbol{p}(0)\|_{W^{1,q}(\Omega)}<\rho$, the solution $\boldsymbol{u}(\boldsymbol{x},t)$ exists for all $t>0$ and

$$\|\boldsymbol{u}(\boldsymbol{x},t)-\boldsymbol{p}(t-\eta_0)\|_{W^{1,q}(\Omega)}<C\rho e^{-\beta t} \qquad \text{for } t>0 \tag{3.3.19}$$

with some $\eta_0=\eta_0(\boldsymbol{u}(\boldsymbol{x},0))$ where $q>N+1$.

Then,The trajectory $|\boldsymbol{u}|=1$ of spatial homogeneous periodic solution of the Complex Ginzburg-Landau equation is asymptotic stable.

Finally, we give the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. Let $\boldsymbol{w}(\cdot)=(w_1(\cdot),w_2(\cdot))$ be arbitrary initial functions in $C^0(\overline{\Omega})^2$. Let ρ, $\boldsymbol{U}_j$ and $\Sigma(t)$ be as in the previous section. The positive constants ε, δ and the perturbation (K_1,K_2) will be specified later. If $\boldsymbol{w}(\boldsymbol{x})+(K_1,K_2)\in\Sigma(0)$, then $\boldsymbol{u}(\boldsymbol{x},t)\in\Sigma(t)$ for any $t>0$. Set $\boldsymbol{U}_j(t)=(U_j(t),V_j(t))=R_j(t)(\cos\Theta_j(t),\sin\Theta_j(t))$ $(j=1,2,3)$. For any $R(\cos\Theta,\sin\Theta)\in\Sigma(t)$, by (3.2.14), we have

$$0\le\Theta-\Theta_1(t)\le\varepsilon+2a\omega\log(1+\delta).$$

From (3.2.16) and $\lim_{t\to\infty}R_1(t)=1$, there is a positive time T_1 and a positive constant M_0 such that

$$\sup_{\boldsymbol{u}\in\Sigma(t)}|\boldsymbol{u}-\boldsymbol{U}_1(t)|\le M_0\{\varepsilon+2a\omega\log(1+\delta)\}.$$

for any $t\ge T_1$. The solution $\boldsymbol{u}=(u,v)$ of (3.1.2) satisfies

$$\left\{\begin{array}{rcl}(u-U_1)_t&=&\Delta(u-U_1)+f_1(u,v)-f_1(U_1,V_1),\\(v-V_1)_t&=&\Delta(v-V_1)+f_2(u,v)-f_2(U_1,V_1).\end{array}\right. \tag{3.3.20}$$

Note that

$$\sup_{\boldsymbol{x}\in\Omega}|\boldsymbol{u}(\boldsymbol{x},t)-\boldsymbol{U}(t)|\le M_0\{\varepsilon+2a\omega\log(1+\delta)\},$$
$$\sup_{\boldsymbol{x}\in\Omega}|f_j(u(\boldsymbol{x},t),v(\boldsymbol{x},t))-f_j(U_1(t),V_1(t))|\le M_1\{\varepsilon+2a\omega\log(1+\delta)\}$$

for $j=1,2$ and $t\ge T_1$ with some positive constant M_1. Applying the global estimate (for example, see [14, 16]) to (3.3.20), we obtain

$$\sup_{\boldsymbol{x}\in\Omega}|\nabla(\boldsymbol{u}(\boldsymbol{x},t)-\boldsymbol{U}(t))|=\sup_{\boldsymbol{x}\in\Omega}|\nabla\boldsymbol{u}(\boldsymbol{x},t)|\le M_2\{\varepsilon+2a\omega\log(1+\delta)\}$$

for $t\ge T_1+1$ with some positive constant M_2. Choose positive constants ε and δ small that $(M_0+M_2)\{\varepsilon+2a\omega\log(1+\delta)\}|\Omega|^{1/q}\le\rho/2$. From the above inequalities, $\|\boldsymbol{u}(\cdot,t) - U_1(t)\|_{W^{1,q}(\Omega)} \leq \rho/2$ for $t \geq T_1 + 1$. Since $\boldsymbol{U}_1(t)$ converges to the limit cycle as $t \to \infty$, there is a time T_2 and $\boldsymbol{p}(0)$ such that $\|\boldsymbol{u}(\cdot,T_2) - \boldsymbol{p}(0)\|_{W^{1,q}(\Omega)} \leq \rho$. By (3.3.19), there is a constant η_0 and a positive constant C satisfying

$$\|\boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{p}(t-\eta_0)\|_{W^{1,q}(\Omega)} < C\rho e^{-\beta t} \qquad \text{for } t > T_2.$$

We only need to show the existence of a positive constant K_1 such that $\boldsymbol{w}(\boldsymbol{x}) + (K_1, 0) \in \Sigma(0)$. Set

$$R_w := \sup_{\boldsymbol{x}\in\Omega} |\boldsymbol{w}(\boldsymbol{x})| \geq 0.$$

For $K_1 > R_w$, take

$$R_1(0) := R_w + K_1, \quad R_3(0) := \frac{R_1(0)}{1+\delta} = \frac{R_w + K_1}{1+\delta},$$
$$\Theta_2(0) := \arcsin \frac{R_w}{K_1} \in \left(0, \frac{\pi}{2}\right).$$

Draw the tangent line from the point $R_3(0)(\cos\Theta_2(0), \sin\Theta_2(0))$ to the circle centered at $(K_1, 0)$ with radius R_w. Define $\Theta_1(0)$ such that $R_1(0)(\cos\Theta_1(0), \sin\Theta_1(0))$ is the intersection point between the above line and the circle centered at the origin with radius $R_1(0)$. By rescaling by K_1, we see that

$$\frac{R_w}{K_1} \to 0, \quad \frac{R_1(0)}{K_1} \to 1, \quad \frac{R_3(0)}{K_1} \to \frac{1}{1+\delta}$$

as $K_1 \to \infty$, which implies that $\Theta_j(0) \to 0$ $(j = 1, 2)$ as $K_1 \to \infty$. Thus, we can choose K_1 such that $\Theta_2(0) - \Theta_1(0) < \varepsilon$. By the choice of the parameters, $\Sigma(0)$ includes the image of $\boldsymbol{w}(\boldsymbol{x}) + (K_1, 0)$, because the image is included in the disk centered at $(K_1, 0)$ with radius R_w. The proof of Theorem 3.1.1 is complete. □

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