Knowledge Acquisition Based on Fuzzy Switching Functions

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Hiroaki Kikuchi

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Chapter 1

Introduction

1.1 Motivation

The uncertain, inconsistent and incomplete nature of human knowledge makes it difficult to deal with it on computer. To handle the uncertainty, fuzzy set theory has been proposed and developed in several fields[1, 2]. With fuzzy sets, if-then style rules can be a powerful tool to represent knowledge in readable way. The if-then rule is, however, generally derived from a human expert in a heuristic way, and thus some rules may be inconsistent, that is, some conflicts with others. In the practical large scale problem, the complete description of knowledge is impossible, which involves the lack of information. Therefore, fuzzy sets and if-then rules are subject to be tuned by trial and error. The manual derivation of if-then rule is current issue in fuzzy inference methods, and several automated approaches have been attempted, including fuzzy neural network systems, self-learning systems based on a genetic algorithm, and adaptive networks[3, 4].

Neural Networks are also widely used to model uncertain reasoning. In neural networks, human knowledge is used to learn how to behave like the expert. After learning a sufficient number of iterations, the network can approximate any given learning data. The results of this learning are, however, numeric values of weights and thresholds that are meaningless unless interpreted. Moreover, a practical large problem with many inputs requires computational power to learn a knowledge. Acquisition of a underlying principle of knowledge is therefore an important issue.

To overcome these drawbacks, we have proposed new approaches to knowledge acquisition based on fuzzy switching functions[12, 13, 14]. A fuzzy switching function is a mapping $f : [0,1]^n \to [0,1]$ that can be represented by a single logic formula. A logic formula consists of logical connectives *and, or, not* defined by minimum, maximum, and 1 minus. For example, we illustrate a fuzzy switching model in Figure 1.1, where a single logic formula $F = (x \wedge y) \vee (\sim y)$ represents a mapping $F : [0,1]^2 \to [0,1]$. With input of $\boldsymbol{a} = (0.2, 0.7)$, this simple logic formula F replaces x by 0.2, y by 0.7, and gives

$$F(0.2, 0.7) = \max(\min(0.2, 0.7), 1 - 0.7) = 0.3,$$

which can be simulated uncertain reasoning in our mind.

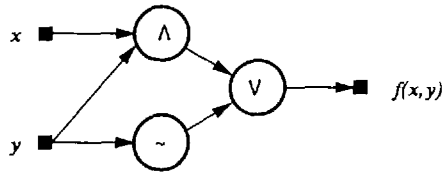

Figure 1.1: Fuzzy switching function model $F = (x \wedge y) \vee (\not{y})$

The study of fuzzy switching function began early in the history of fuzzy theory, and have been investigated: fundamental properties [6, 8]; minimization[7]; necessary and sufficient condition[9]; representation[10]; and quantization[11]. These known properties are useful in deriving a unique logic formula systematically.

In our approach we use a logic formula instead of if-then rules or neural networks model. The logical description of the knowledge is readable as much as if-then rules. The significant feature of description by a single logic formula is *consistency* and *uniqueness*. The inconsistency of a learning data can be canceled in extracting the logic formula, which allows consistent description only. Even if a given knowledge is incomplete, by verifying the condition of uniqueness we can detect the lack of information of the knowledge. We clarify the necessary and sufficient conditions for knowledge to be consistent and to be complete, that is, the logic formula can be determined uniquely.

1.2 Coffee Problem

Before we go into the detail, let us consider a simple problem, called "coffee problem," which would help to understand the problem that we are going to deal with in this thesis.

Coffee Problem

Some people like black coffee, some with milk, sugar, or both. Preference in coffee depends on individuals. Suppose there are three cups of coffee. Cup A is black, cup B has milk and sugar, and cup C just has milk.

A coffee sommelier (?) samples each cup, and replies the three question:

- Do you think there was sugar in the coffee?
- Do you think there was milk in the coffee?
- How good was it?

He replies with the fuzzy truth value shown in Table 1.1, where 1 means *yes*, 0 means *no*, and 0.5 means *unknown*.

Determine the complete range of his liking for coffee. Does he like coffee with sugar but no milk?

In this case, he seems to prefer sugar as in cup B, and hates milk or the lack of sugar as in cup C. Thus, we can infer meaningful conclusions from uncertain information such as that in Table 1.1.

Table 1.1: Preference in coffee

	cup A	cup B	cup C
sugar	0.4	0.8	0.2
milk	0.3	0.6	0.9
taste	0.7	0.8	0.2

1.3 Thesis goals

The coffee problem, a simple uncertain knowledge acquisition, can be represented as a mathematical expression.

Let A and B be a finite set of objects. Knowledge is regarded as a mapping f with domain A and range B, that is,

$$f : A \to B,$$

where f maps each element of A to an element of B. In fact, however, it is impossible to derive the complete range of the mapping f from a human expert as mentioned in the coffee problem. Thus f is restricted to the domain, a subset A' of A, that is,

$$f' : A' \to B,$$

which is called *restriction* of f, and so the three cups of coffee can be expressed by three equations defined by,

$$\begin{aligned} f'(0.4, 0.8) &= 0.2, \\ f'(0.3, 0.6) &= 0.9, \\ f'(0.7, 0.8) &= 0.2, \end{aligned}$$

where $A = \{(0.4, 0.8), (0.3, 0.6), (0.7, 0.8)\} \in V^2$.

The attempt of uncertain knowledge acquisition is to approximate knowledge f only by having f'. To solve the problem, we have several models of f, which formalizes several subgoals: an identification problem of fuzzy switching function, of P-fuzzy switching function, and of Kleenean function; a fitting problem of fuzzy switching function as follows.

1. Identification of fuzzy switching function (Chapter 3)

   Consider a mapping $f' : A \to [0, 1]$ where A is a subset of $[0, 1]^n$.

   (i). Show if there is at least one fuzzy switching function F such that $f'(\boldsymbol{a}) = F(\boldsymbol{a})$ for all $\boldsymbol{a} \in A$ (representable).

   (ii). If so, show the uniqueness of identification.

   (iii). Obtain the logic formula F representing f'.

2. Identification of P-fuzzy switching function (Chapter 4)

   Suppose that mapping f' is a restriction of P-fuzzy switching function that is representable by sum of prime implicants.

(i). Show if there is at least one P-fuzzy switching function F_p such that $f'(\boldsymbol{a}) = F_p(\boldsymbol{a})$ for all $\boldsymbol{a} \in A$ (P-representable).

(ii). Show the uniqueness of F_p, which is called P-uniqueness.

(iii). Obtain the logic formula F_p that consists of prime implicants only.

3. Identification of Kleenean function (Chapter 5)

   Suppose that mapping f' is a restriction of fuzzy switching function with some constant value of $[0, 1]$, which is called Multiple-valued Kleenean function or just Kleenean function.

   (i). Show if there is at least one Kleenean function F_k such that $f'(\boldsymbol{a}) = F_k(\boldsymbol{a})$ for all $\boldsymbol{a} \in A$ (K-representable).

   (ii). Obtain the logic formula F_k that consists of variables, logical connectives, and constants of $[0, 1]$

4. Fitting fuzzy switching function (Chapter 4)

   Suppose that mapping f' is a restriction of fuzzy switching function which includes some noises.

   (i). Find the fuzzy switching function f^* with the shortest distance to the mapping f'.

   (ii). Obtain the best logic formula F^* that approximates mapping f'.

The first problem, identification of fuzzy switching function, is fundamental and thus the result will be used for the other problems. Especially, the properties of simple and complementary phrases should be clarified in order for representation by a logic formula.

In the second problem, We use a P-fuzzy switching function instead of the standard fuzzy switching function. A *P-fuzzy switching function* is a special class of fuzzy switching function that can be represented by a disjunction of prime implicants only. Since P-fuzzy switching function never contains meaningless complementary phrase such as $(\sim x \wedge x)$, it can simplify result of knowledge expression.

In the third problem, we study yet another multiple-valued function. A *multiple-valued Kleenean function* is an extension of fuzzy switching function so that a logic formula consists of any constant value of $[0, 1]$, while traditional fuzzy switching function has no constant except 0 and 1. Clearly, Kleenean function could make an inconsistent restriction of fuzzy switching function be representable with arbitrary constant values.

The last problem is not an identification. We shall notice that some error and incompleteness involved by human response could spoil the consistency of any classes of functions. Hence, we suppose a human expert's response based on a logic formula but with some noise, and then attempt to fit fuzzy switching functions to the underlying knowledge. Since this approach is an approximation, there must exist a unique solution for any given restriction. Hence, we can omit to study a condition for uniqueness. This is a significant feature for practical applications.

1.4 Thesis Structure

The thesis consists of eight chapters.

Chapter 1 *Introduction* addresses the difficulty in modeling uncertain reasoning, and define three simple problems, that are the goals of this thesis.

Chapter 2 *Fuzzy Switching Functions* reviews the fundamental definition and properties examined so far including the monotonicity, the normality, and the quantization theorem.

Chapter 3 *Identification* studies partially specified fuzzy switching functions called restrictions and clarified some important properties which will be used in the following sections. Some of new concepts are taken in this section. Quantized sets, which are sets of ternary elements of $\{0, 0.5, 1\}^n$, is defined by the fuzzy elements of the given learning data. Expansion, which is conjunction with regularity or monotonicity of the restriction, are defined by the corresponding quantized sets. After investigates some properties of the quantized sets and expansions, the necessary and sufficient conditions for a restriction to be represented by a fuzzy switching function are clarified. In addition, the uniqueness, which shows the learning data can be represented unique fuzzy switching function, is also clarified.

Chapter 4 *P-Fuzzy Switching Functions* discusses the simplification problem of the derived logic formula. Since the complementary laws do not hold in fuzzy logic, the extracted logic formula may include meaningless phrase. In this chapter, P-fuzzy switching functions is introduced as a way to eliminate the redundant description and obtain the simplified logic formulae. A P-fuzzy switching function is a meaningful class of fuzzy switching functions that can be represented by prime implicants. The necessary and sufficient conditions for any given learning data to be representable with P-fuzzy switching functions, and to be expressed by a unique logic formula.

Chapter 5 *Kleenean Functions* studies the identification problem of fuzzy switching function with constant values of $[0, 1]$, which is called a Multiple-valued Kleenean function or just a Kleenean function. After some of the fundamental properties of Kleenean functions are clarified, we define some extended quantizations, strong, weak, and quasi quantizations. Main result is Theorem 5.7 which clarifies a necessary and sufficient condition for an identification problem of Kleenean function to be solved.

Chapter 6 *Fitting* studies the identification problem with some errors in learning data. This section provides an algorithm that takes a piece of knowledge, which is to be used as learning data, and calculates the logic formula with the shortest distance to the learning data. The problem is solved in three steps; first, the given data is divided into some small problems, called Q-equivalent classes; second, the local distances between the given data and each local fuzzy switching functions; and the last, the shortest distance is obtained by a modified graph-theoretic algorithm.

Chapter 7 *Examples* demonstrates the proposed algorithm based on the results described in the thesis by simple evaluation problem with four input variables. It shows how knowledge can be extracted from the learning data, and how many inputs can be solved in the proposed algorithm.

Chapter 8 *Conclusion* summarizes, concludes and indicates the direction of future work based this thesis. The main results of this thesis is some necessary and sufficient conditions of restrictions that are used in uncertain knowledge acquisition.

We summarize the types of problem and models discussed in this thesis on Table 1.2.

Table 1.2: Problem types and models

problem	model	chapter
Identification	fuzzy switching function	3
Identification	P-fuzzy switching function	4
Identification	multiple-valued Kleenean function	5
Fitting	fuzzy switching function	6

The major contribution of this thesis is to demonstrate the viability of knowledge acquisition, uncertain reasoning, a new fuzzy inference, expert systems, fuzzy analysis techniques, evaluation problems, and modeling human reasoning.

1.5 Glossary of Symbols

V	Set of truth values, $[0, 1]$
V_2	Set of truth values, $\{0, 1\}$
V_3	Set of truth values, $\{0, 0.5, 1\}$
V^n	n-dimensional Cartesian product
f_0	Restriction, or learning data given by human
A	Domain of restriction f_0
B	Range of restriction f_0
$\boldsymbol{a}$	Element of A, $\boldsymbol{a} = (a_1, \ldots, a_n) \in V^n$
b	Element of B, $b \in V$
$\boldsymbol{c}$	Element of $C(f)$. $\boldsymbol{c} = (c_1, \ldots, c_n) \in V_3^n$
$\boldsymbol{x}$	Variables, $\boldsymbol{x} = (x_1, \ldots, x_n)$
$f\|_A$	Restriction of f by A
$C(f)$	Quantized set of f
$C^*(f)$	Expansion of quantized set of f
$C^p(f)$	P-expansion of quantized set of f
$P(f)$	P-resolution
f_p	P-fuzzy switching function
f_k	Kleenean function
α_a	Simple phrase corresponding to $\boldsymbol{a} \in V_3^n$
β_b	Complementary phrase corresponding to $\boldsymbol{a} \in V_3^n - V_2^n$
F	Logic formula
$\wedge$	Logical operator called *and*
$\vee$	Logical operator called *or*
$\sim$	Logical operator called *not*
$\succeq$	Partially ordered relation of ambiguity
$\succ$	Strict partially ordered relation of ambiguity
$\overline{a}^\lambda$	(Strong) Quantization of a by λ
$\underline{a}_\lambda$	Weak quantization of a by λ
$Q_\lambda(a)$	Quasi-quantization of a by λ
$\approx$	Q-equivalent relation
$[\boldsymbol{a}]$	Q-equivalent class
f_a^b	Partially defined fuzzy switching function
$I_a(f)$	Possibility of function f for $\boldsymbol{a} \in V_3^n$

Chapter 2

Fuzzy Switching Functions

In this chapter we review the basic definition of fuzzy switching functions and summarize some of their important properties; the monotonicity, the normality, the quantization, and the representation theorems. In Chapter 5, we will recall the definition of logic formula so that it can contain any arbitrary constant of $[0, 1]$. While, we use only two constant value of 0 and 1 in this section.

2.1 Basic Definitions

2.1.1 Fuzzy Switching Function

Definition 2.1 Let $V = [0,1]$, $V_2 = \{0,1\}$, and $V_3 = \{0, 0.5, 1\}$ be the sets of truth values. A *logic formula* consists of constants 0 and 1, variables $x_1, \ldots, x_n$, and logical operators *and* ($\wedge$), *or* ($\vee$) and *not* ($\sim$), defined as follows:

$$\begin{aligned} x_i \wedge x_j &= x_i x_j = \min(x_i, x_j), \\ x_i \vee x_j &= \max(x_i, x_j), \\ \sim x_i &= 1 - x_i. \end{aligned}$$

A function from the domain n-dimensional Cartesian product V^n to the range V is called an n-variable *fuzzy function*. A fuzzy function representable by a logic formula is called an n-variable *fuzzy switching function* or *fuzzy logic function*.

To avoid having formulas cluttered with brackets, we adopt the following precedence:

$$\sim \quad \wedge \quad \vee.$$

Thus the logic formula $F = (x \vee ((\sim x) \wedge y))$ can be represented simply as $F = x \vee \sim x \wedge y$ or $F = x \vee \overline{x}y$.

Hereafter, we simply call a fuzzy switching function to mean n-variable fuzzy switching function.

2.1.2 Partially Ordered Relation of Ambiguity

On the set of truth value V, a partially ordered relation with respect to an ambiguity is defined as follows.

Definition 2.2 Let a and b be elements of V. Then, $a \succeq b$ if and only if either $0.5 \geq a \geq b$ or $b \geq a \geq 0.5$.

The relation $\succeq$ can be extended to V^n by letting $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V^n, $\boldsymbol{a} \succeq \boldsymbol{b}$ if and only if $a_i \succeq b_i$ for each i $(i = 1, \ldots, n)$. Any two elements a in $[0, 0.5)$ and b in $(0.5, 1]$ are not comparable with respect to $\succeq$. We denote this by $a \not\succeq b$.

Occasionally we write $a \succ b$ to mean that $a \succeq b$ and $a \neq b$.

Example 2.1 Given a partial order $\succeq$, we can draw a Hasse diagram on a finite set of truth values. The following diagram shows that $0.8 \succeq 1$, $0.4 \succeq 0.2$, and $0.4 \not\succeq 0.8$.

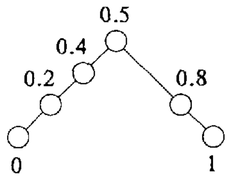

Figure 2.1: The partial ordered relation $\succeq$

Clearly, $a \succeq a$ for any element a of V. The greatest element with respect to $\succeq$ is 0.5 and the least elements are 0 and 1.

2.1.3 Quantization

For a treatment of infinite number of fuzzy truth values with some finite number of elements, we introduce a *quantization* of fuzzy truth value in this section.

A *quantization* is a unary operation[1] that maps a value in V to one of the three values in V_3.

Definition 2.3 Let x and λ be elements of V. A *quantization* $\overline{x}^\lambda$ of x by λ is an element of V_3 defined by:

$$\overline{x}^\lambda = \begin{cases} 0 & \text{if } 0 \leq x \leq \min(\lambda, 1-\lambda) \leq 0.5,\ x \neq 0.5, \\ 1 & \text{if } 0.5 \leq \max(\lambda, 1-\lambda) \leq x \leq 1,\ x \neq 0.5, \\ 0.5 & \text{otherwise.} \end{cases}$$

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an element of V^n. A quantization of $\boldsymbol{x}$ by λ is an element of V_3^n defined by $\overline{\boldsymbol{x}}^\lambda = (\overline{x_1}^\lambda, \ldots, \overline{x_n}^\lambda)$.

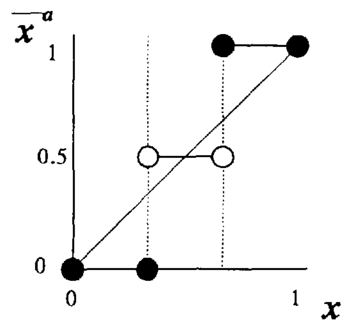

Figure 2.2: The quantization

Figure 2.2 illustrates the mapping made by the quantization. The quantization partitions the set of truth values V into three sets V_3.

Example 2.2

$$\overline{(0.2,1,0.6)}^{0.8} = (0,1,0.5), \qquad \overline{(0.5,1,0)}^{0.5} = (0.5,1,0).$$

Notice that quantizations of 0, 1, and 0.5 are always identical to themselves, that is,

$$\overline{0}^{x} = 0, \quad ,\overline{1}^{x} = 1, \quad \overline{0.5}^{x} = 0.5$$

for any x of V, also

$$\overline{x}^{\lambda} = \overline{x}^{1-\lambda}$$

for any x, especially,

$$\overline{x}^{0} = \overline{x}^{1}.$$

2.1.4 Representation of Fuzzy Switching Functions

A fuzzy switching function can be represented by a disjunctive form which is a disjunction (*or*) of some conjunctions (*and*). However, since the complementary law $(x_i \wedge (\overline{x_i}) = 0)$ does not hold in fuzzy logic, we have two phrase types. One is a *complementary phrase*, which contains a literal and its negation such as $x_i \wedge (\overline{x_i})$, the other is a *simple phrase*. A complementary phrase that contains all variables is called a *complementary minterm*.

These two kinds of phrase correspond to elements of V_3^n in the following way.

Definition 2.4 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be elements of V_3^n and $V_3^n - V_2^n$, respectively. A *simple phrase* α^{a} corresponding to $\boldsymbol{a}$ is defined by

$$\alpha^{a} = x_1^{a_1} \wedge \cdots \wedge x_n^{a_n},$$

[1] Mukaidono has defined another quantization for $\lambda \in [0, 0.5)$[10], which is essentially equal to Definition 2.3. This allows λ to take any value in $[0, 1]$ in order to simplify later discussions.

where for every i ($i = 1, \ldots, n$),

$$x_i^{a_i} = \begin{cases} x_i & \text{if } a_i = 1 \\ \overline{x_i} & \text{if } a_i = 0 \\ 1 & \text{if } a_i = 0.5. \end{cases}$$

A complementary phrase β^b corresponding to $\boldsymbol{b}$ is defined by

$$\beta^b = x_1^{b_1} \wedge \cdots \wedge x_n^{b_n},$$

where for every i ($i = 1, \ldots, n$),

$$x_i^{b_i} = \begin{cases} x_i & \text{if } b_i = 1 \\ \overline{x_i} & \text{if } b_i = 0 \\ x_i\,\overline{x_i} & \text{if } b_i = 0.5. \end{cases}$$

Example 2.3 An element $\boldsymbol{a} = (0, 0.5, 1) \in V_3^3$ corresponds to the simple phrase $\alpha^a = \sim x_1 \wedge x_3$ and to the complementary phrase $\beta^a = \sim x_1 \wedge x_2 \wedge \sim x_2 \wedge x_3$.

2.2 Properties of Fuzzy Switching Functions

From Definition 2.1, the following property is trivially true.

Theorem 2.1 (Normality) Let f be a fuzzy switching function.

$$\boldsymbol{a} \in V_2^n \;\Rightarrow\; f(\boldsymbol{a}) \in V_2.$$

Theorem 2.2 (Monotonicity) [8] Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be elements of V^n, and f be a fuzzy switching function.

$$\boldsymbol{a} \succeq \boldsymbol{b} \;\Rightarrow\; f(\boldsymbol{a}) \succeq f(\boldsymbol{b})$$

Corollary 2.1 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be elements of V^n, and f be a fuzzy switching function.

$$\boldsymbol{a} \succ \boldsymbol{b},\, f(\boldsymbol{a}) \in \{0, 1\} \Rightarrow f(\boldsymbol{b}) = f(\boldsymbol{a}),$$
$$\boldsymbol{a} \succ \boldsymbol{b},\, \boldsymbol{a} \succ \boldsymbol{c},\, f(\boldsymbol{b}) = 1, f(\boldsymbol{c}) = 0 \Rightarrow f(\boldsymbol{a}) = 0.5.$$

Theorem 2.3 (Quantization) [8] Let f be a fuzzy switching function, and $\boldsymbol{a}$ be an element of V^n. Then

$$\overline{f(\boldsymbol{a})}^{\lambda} = f(\overline{\boldsymbol{a}}^{\lambda})$$

for all λ of V.

Theorem 2.4 [10] Let f and g be fuzzy switching functions. Then, $f(\boldsymbol{a}) = g(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V_3^n, if and only if $f(\boldsymbol{a}) = g(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V^n.

This theorem shows that a fuzzy switching function is uniquely determined by its values on the elements of V_3^n. We, therefore, can identify several fuzzy switching functions by examining whether they are equal or not on the ternary truth table in a finite number of steps.

Lemma 2.1 For any simple phrase α^a corresponding to $\boldsymbol{a} \in V_3^n$,

$$\alpha^a(\boldsymbol{a}) = 1.$$

For any complementary phrase β^b corresponding to $\boldsymbol{b} \in V_3^n - V_2^n$,

$$\beta^b(\boldsymbol{b}) = 0.5.$$

Proof. Proof is omitted. □

Example 2.4 An element $\boldsymbol{a} = (0.5, 1, 0) \in V_3^n$ corresponds to a simple phrase $\alpha^a = x_2 \wedge \sim x_3$ and a complementary phrase $\beta^a = x_1 \wedge \sim x_1 \wedge x_2 \wedge x_3$.

Then,

$$\alpha^a(\boldsymbol{a}) = 1, \quad \beta^a(\boldsymbol{a}) = 0.5.$$

Lemma 2.2 Let α^a and β^b be a simple phrase and a complementary phrase corresponding to $\boldsymbol{a} \in V_3^n$ and $\boldsymbol{b} \in V_3^n - V_2^n$, respectively.

$$\boldsymbol{a} \succ \boldsymbol{b} \quad \Leftrightarrow \quad \alpha^a(\boldsymbol{b}) = 1$$

Proof. When $\boldsymbol{a} \succ \boldsymbol{b}$, $\alpha^a(\boldsymbol{a}) \succ \alpha^a(\boldsymbol{b})$ [Theorem 2.2]. Hence $\alpha^a(\boldsymbol{b}) = 1$, since $\alpha^a(\boldsymbol{a}) = 1$ [Lemma 2.1].

Conversely, suppose that $\alpha^a(\boldsymbol{b}) = 1$. If $a_i = 1$ then $b_i = 1$, hence $a_i \succ b_i$. If $a_i = 0$ then $b_i = 0$, hence $a_i \succ b_i$. If $a_i = 0.5$ then $a_i \succ b_i$ for any b_i. Consequently $a_i \succ b_i$ for each i, which means $\boldsymbol{a} \succ \boldsymbol{b}$. □

Lemma 2.3 [12] Let α^a be a simple phrase corresponding to $\boldsymbol{a} \in V_3^n$ and $\boldsymbol{b}$ be an element of $V_3^n - V_2^n$, respectively. If $\alpha^a(\boldsymbol{b}) = 0.5$, there exists $\boldsymbol{c} \in V_3^n$ such that

$$\boldsymbol{a} \succ \boldsymbol{c}, \ \boldsymbol{b} \succ \boldsymbol{c}.$$

Proof. Suppose that $\alpha^a(\boldsymbol{b}) = 0.5$. If $a_i = 0$ then $b_i \in \{0.5, 0\}$. If $a_i = 1$ then $b_i \in \{0.5, 1\}$. If $a_i = 0.5$ then $b_i \in V_3$. Thus, for each i, at least either $a_i \succ b_i$ or $b_i \succ a_i$. Hence, a_i and b_i are always comparable with respect to $\succ$ for every i. Therefore there exists an element $\boldsymbol{c} = (c_1, \ldots, c_n)$ of V_3^n such that $c_i = \text{glb}_{\succ}\{a_i, b_i\}$ where glb is the greatest lower bound. The $\boldsymbol{c}$ holds $\boldsymbol{a} \succ \boldsymbol{c}$ and $\boldsymbol{b} \succ \boldsymbol{c}$. □

Example 2.5 Let $\boldsymbol{a} = (0.5, 0)$ and $\boldsymbol{b} = (1, 0.5)$ in V_2. Then, $\boldsymbol{a}$ corresponds to a simple phrase $\alpha^a(x, y) = \sim y$, and so

$$\alpha^a(\boldsymbol{b}) = 1 - 0.5 = 0.5,$$

hence, there exists $\boldsymbol{c} \in V_3^n$ which holds

$$\boldsymbol{a} \succeq \boldsymbol{b}, \quad \boldsymbol{b} \succeq \boldsymbol{c}.$$

Lemma 2.4 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be elements of V^n, β^b be a complementary phrase corresponding to $\boldsymbol{b}$.

$$\boldsymbol{a} \succ \boldsymbol{b} \quad \Leftrightarrow \quad \beta^b(\boldsymbol{a}) = 0.5.$$

Proof. When $\boldsymbol{a} \succ \boldsymbol{b}$, $\beta^b(\boldsymbol{a}) \succ \beta^b(\boldsymbol{b})$ [Theorem 2.2]. Therefore $\alpha^a(\boldsymbol{b}) = 0.5$, since $\beta^b(\boldsymbol{b}) = 0.5$ [Lemma 2.1].

Conversely, let $\beta^b(\boldsymbol{a}) = 0.5$. If $b_i = 1$ then $a_i \in \{0.5, 1\}$. If $b_i = 0$ then $a_i \in \{0.5, 0\}$. If $b_i = 0.5$ then $a_i = 0.5$. Thus, $a_i \succ b_i$ for every i, that is, $\boldsymbol{a} \succ \boldsymbol{b}$. □

Lemma 2.2 and Lemma 2.4 lead to the following corollary.

Corollary 2.2

$$\boldsymbol{a} \succ \boldsymbol{b} \quad \Leftrightarrow \quad \alpha^a(\boldsymbol{b}) = 1 \quad \Leftrightarrow \quad \beta^b(\boldsymbol{a}) = 0.5$$

Theorem 2.5 (Representation Theorem) Any fuzzy switching function f is represented by a logic formula F:

$$F = \bigvee_{\boldsymbol{a} \in f^{-1}(1)} \alpha^a \bigvee_{\boldsymbol{b} \in f^{-1}(0.5)} \beta^b$$

where $f^{-1}(1)$ and $f^{-1}(0.5)$ are subsets of V_3^n defined by

$$f^{-1}(i) = \{\boldsymbol{a} \in V_3^n | f(\boldsymbol{a}) = i\} \quad i = 1, 0.5,$$

α^a is a simple phrase corresponding to $\boldsymbol{a}$, and β^b is a complementary phrase corresponding to $\boldsymbol{b}$.

Proof. We prove that $f(\boldsymbol{a}) = F(\boldsymbol{a})$ for all $\boldsymbol{a} \in V_3^n$ using three cases: $f(\boldsymbol{a}) = 1$; $f(\boldsymbol{a}) = 0.5$; $f(\boldsymbol{a}) = 1$.

When $f(\boldsymbol{a}) = 1$, $\boldsymbol{a}$ belongs to f^{-1}, thereby, there exists the corresponding simple phrase α^a in F such that $\alpha^a(\boldsymbol{a}) = 1 \leq F(\boldsymbol{a})$.

When $f(\boldsymbol{a}) = 0.5$, there exists a complementary phrase β^a in F that corresponds to $\boldsymbol{a}$. By Lemma 2.1, $\beta^a(\boldsymbol{a}) = 0.5 \leq F(\boldsymbol{a})$. There is, however, no simple phrase α' in F such that $\alpha'(\boldsymbol{a}) = 1$. Otherwise, from Lemma 2.2, we have the corresponding element $\boldsymbol{a}'$ such that $\boldsymbol{a}' \succeq \boldsymbol{a}$. Thereby, however, $f(\boldsymbol{a}') = 1$ and $f(\boldsymbol{a}) = 0.5$ violate monotonicity of f.

When $f(\boldsymbol{a}) = 0$, there is no simple phrase α' such that $\alpha'(\boldsymbol{a}) \geq 0.5$. Otherwise, from Lemma 2.3, for the corresponding element $\boldsymbol{a}'$ of α', there exists $\boldsymbol{c} \in V_3^n$ such that $\boldsymbol{a}' \succeq \boldsymbol{c}$ and $\boldsymbol{a} \succeq \boldsymbol{c}$. However, $f(\boldsymbol{a}') = 1$, $f(\boldsymbol{a}') = 0$ cannot satisfy monotonicity with $\boldsymbol{c}$.

Moreover, there is no complementary phrase β in F such that $\beta(\boldsymbol{a}) = 0.5$ Otherwise, by Lemma 2.4 we have $\boldsymbol{a} \succeq \boldsymbol{b}$ for the the corresponding $\boldsymbol{b}$ of β. However, since $f(\boldsymbol{a}) = 0$ and $f(\boldsymbol{b}) \leq 0.5$, this violates the monotonicity of f.

As we have shown, $f(\boldsymbol{a}) = F(\boldsymbol{a})$ for all $\boldsymbol{a} \in V_3^n$. Therefore, by Theorem 2.4, we have $f(\boldsymbol{a}) = F(\boldsymbol{a})$ for all elements of V^n. This implies that that logic formula F represents fuzzy switching function f. □

Note that the logic formula F defined here is not in *fuzzy canonical disjunctive form* in the sense of Mukaidono[10], but F is also uniquely determined for f. By applying the absorption law, F can be simplified. We show that by an example.

Example 2.6 For a fuzzy switching function f given by the ternary truth table in Table 2.1, we have

$$\begin{aligned} f^{-1}(1) &= \{(0,0),(0.5,0),(1,0),(1,1)\} \\ f^{-1}(0.5) &= \{(0,0.5),(0.5,0.5),(1,0.5),(0.5,1)\} \end{aligned}$$

$$\begin{aligned} \bigvee_{\boldsymbol{a}\in f^{-1}(1)} \alpha^{a} &= \sim x \sim y \vee \sim y \vee x \sim y \vee xy \\ \bigvee_{\boldsymbol{b}\in f^{-1}(0.5)} \beta^{b} &= \sim xy \sim y \vee x \sim xy \sim y \vee xy \sim y \vee x \sim y \vee x \sim xy \end{aligned}$$

In accordance with Theorem 2.5 and the absorption law, we have a simple logic formula

$$\begin{aligned} F &= \bigvee_{\boldsymbol{a}\in f^{-1}} \alpha^{a} \vee \bigvee_{\boldsymbol{b}\in f^{-1}(0.5)} \beta^{b} \\ &= \sim y \vee xy. \end{aligned}$$

Table 2.1: ternary truth value f

$y\backslash x$	0	.5	1
0	1	1	1
.5	.5	.5	.5
1	0	.5	1

Chapter 3

Identification

The purpose of this section is to establish a general method to find a fuzzy switching function representing a given restriction. The method can be considered as an approximate reasoning. This is the first step to uncertain reasoning based on fuzzy switching functions.

The main results are the necessary and sufficient conditions for a restriction to be represented by a fuzzy switching function, and for the fuzzy switching function to be uniquely determined. These conditions make the proposal different than the conventional inference methods such as neural networks or fuzzy inference. This difference will be given as a comparison with conventional methods.

3.1 Introduction

Approximate reasoning is currently being studied as a way of dealing with uncertain knowledge. It is extremely difficult to make an exact model of human knowledge, because of its essential indefiniteness and uncertainty. Thus several methods to approximate uncertain knowledge have been proposed such as neural networks and a fuzzy inference. Here we regard our exact knowledge as a mapping f from a finite set A to B, that is,

$$f : A \rightarrow B,$$

then what we can extract from f is just a partial mapping f' which is restricted to the domain A', a subset of A, that is,

$$f' : A' \rightarrow B.$$

The methods of approximate reasoning attempt to infer or reconstruct f from f'. For example[1], when a mapping f outputs as the following,

$$\begin{aligned} f(0.4, 0.3) &= 0.7, \\ f(0.8, 0.6) &= 0.8, \\ f(0.2, 0.9) &= 0.2, \end{aligned}$$

[1]An answer using fuzzy switching functions will be shown in section 3.4.

what is a value of output $f(x, y)$ for any (x, y) in V^2 ? In this example, three pairs of inputs and outputs represent a partical mapping f', which is called a *restriction.* A restriction can be regarded as a IF-THEN rule base in fuzzy inference, or as a learning data in neural networks.

Some of the methods have been applied to a practical use, however, they do not include two important conditions , *representability* and *uniqueness.*

Representability

A system in which any function of a class is representable is called functionally unique for the class. In this sense, neither fuzzy inference nor neural networks is unique for n-variable functions. (For example, consider any discontinuous function). This means these methods cannot always approximate a given restriction f' that might be entirely inconsistent. However what we want to point out is not the uniquness itself, but that they have no means to determine whether a function f' is representable. There may be no answer for a given f'.

We say that a restriction f' is *representable* if f' is representable by fuzzy switching functions.

Uniqueness

Even if f' is representable, there may not be enough information to determine f uniquely. Nevertheless, a neural network system always outputs definite values. There is no difference between the certainty of output values learned by a large number of inputs and the certainty by only one input.

We say a restriction f' is *unique* if a fuzzy switching function f is uniquely determined by f'.

In this section, we will attempt to establish a new approximate reasoning based on some properties of a fuzzy switching function that finds the fuzzy switching function representing a given restriction. The main results are the necessary and sufficient conditions for a restriction to be consistent and unique. These conditions make this different from the conventional approximate reasoning methods such as neural networks or fuzzy inference.

First, we will introduce the concept of *quantized sets* in order to characterize some conditions. Next, we will show necessary and sufficient condition for restrictions to be representable, and show a certain set in order for quantized sets. Finally, we will show necessary and sufficient condition to be unique which gives a solution of the identification problem of a fuzzy switching function.

3.2 Representability

Let f be a mapping $f : V^n \to V$, and A be a non-empty subset of V^n. A *restriction* of a fuzzy switching function f to A is a mapping $f|_A$ defined by $f|_A(\boldsymbol{a}) = f(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of A.

Here, let us consider the conditions necessary for a mapping f' from A to V to be a restriction $f|_A$, which means that f' is representable by a logic formula.

3.2.1 Necessary and Sufficient Conditions for Fuzzy Switching Functions

The following theorem shows a necessary and sufficient condition for a mapping $f' : V^n \to V$ to be a fuzzy switching function. (Note that this f' is defined on V^n, not on A).

Theorem 3.1 [21] A mapping $f : V^n \to V$ is a fuzzy switching function if and only if :

$$\begin{array}{lll} (1)\,\text{normality} & \forall \boldsymbol{a} \in V_2^n & f(\boldsymbol{a}) \in V_2, \\ (2)\,\text{Quantization Theorem} & \forall \lambda \in V & \overline{f(\boldsymbol{a})}^{\lambda} = f(\overline{\boldsymbol{a}}^{\lambda}). \end{array}$$

From these two conditions (1) and (2), one can derive the following property[21],

$$(3)\,\text{monotonicity} \quad \boldsymbol{a} \succeq \boldsymbol{b} \Rightarrow f(\boldsymbol{a}) \succeq f(\boldsymbol{b}).$$

However, when a mapping is given partially by $f' : A \to V$, conditions (1) and (2) are insufficient to be a fuzzy switching function. For instance, let us consider a mapping f' from $A = \{0, 0.5, 1\}$ to V such that:

$$f'(0) = 0,\ f'(0.5) = 1,\ f'(1) = 1.$$

Since 0, 1, and 0.5 are unchanged for any quantization by λ in V, it is clear that f' satisfies condition (2). In addition, condition (1) also holds in f'. Although f' satisfies both conditions, there is no one-variable fuzzy switching function that satisfies the above equation.

Notice that condition (3), monotonicity for ambiguity, does not hold in f'. Furthermore, we cannot believe condition (2) for a restriction, because there might be an element $\boldsymbol{a}$ of A such as $\overline{\boldsymbol{a}}^{\lambda} \notin A$ for which f' cannot map to any element of V. As we have seen, it necessary for a restriction to define the fourth condition instead of the Quantization Theorem.

3.2.2 Quantized Sets

In this section, we will introduce the concept of *quantized sets*, which characterizes a restriction with some subsets of V_3^n.

Definition 3.1 Let $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ be a subset of V^n, and f be a mapping $f : A \to V$. *Quantized sets* of f are subsets of V_3^n $C_1(f)$, $C_0(f)$, $C_U(f)$ and $C(f)$ defined as follows:

$$\begin{aligned} C_1(f) &= \{\overline{\boldsymbol{a}_i}^{\lambda} \in V_3^n \mid \exists \boldsymbol{a}_i \in A, \exists \lambda \in V, \overline{f(\boldsymbol{a}_i)}^{\lambda} = 1\} \\ C_0(f) &= \{\overline{\boldsymbol{a}_i}^{\lambda} \in V_3^n \mid \exists \boldsymbol{a}_i \in A, \exists \lambda \in V, \overline{f(\boldsymbol{a}_i)}^{\lambda} = 0\} \\ C_U(f) &= \{\overline{\boldsymbol{a}_i}^{\lambda} \in V_3^n\ , \mid \exists \boldsymbol{a}_i \in A, \exists \lambda \in V, \overline{f(\boldsymbol{a}_i)}^{\lambda} = 0.5\} \\ C(f) &= C_1(f) \cup C_0(f) \cup C_U(f) \\ &= \{\overline{\boldsymbol{a}_i}^{\lambda} \in V_3^n \mid \exists \boldsymbol{a}_i \in A, \exists \lambda \in V\}. \end{aligned}$$

Example 3.1 The quantized sets of a mapping $f(0.3) = 0.7$, $f(0.6) = 0.4$ are as follows.

$$\begin{aligned} C_1(f) &= \{\overline{0.3}^{0.3}\} = \{0\}, \\ C_0(f) &= \{\overline{0.6}^{0.6}\} = \{1\}, \\ C_U(f) &= \{\overline{0.3}^{1}\} = \{\overline{0.6}^{0}\} = \{0.5\}, \\ C(f) &= \{0, 0.5, 1\}. \end{aligned}$$

A fuzzy switching function f satisfies the following property concerning quantized sets.

Theorem 3.2 (disjoint) Any quantized set of a fuzzy switching function $C_1(f)$, $C_0(f)$ and $C_U(f)$ is disjoint to any other quantized set:

$$C_i(f) \cap C_j(f) = \emptyset \qquad (i \neq j \in \{0, 1, U\}).$$

Proof. Suppose that $\boldsymbol{b} \in C_1(f) \cap C_U(f)$. For the $\boldsymbol{b}$, there exist elements $\boldsymbol{a}$ and $\boldsymbol{c}$ in V^n and elements λ and τ of V such that : $\boldsymbol{b} = \overline{\boldsymbol{a}}^{\lambda} = \overline{\boldsymbol{c}}^{\tau}$, $\overline{f(\boldsymbol{a})}^{\lambda} = 1$, $\overline{f(\boldsymbol{b})}^{\tau} = 0.5$. By applying the Quantization Theorem, we have the following contradiction:

$$f(\overline{\boldsymbol{a}}^{\lambda}) = f(\boldsymbol{b}) = 1 \neq f(\overline{\boldsymbol{c}}^{\tau}) = f(\boldsymbol{b}) = 0.5.$$

In other cases contradictions can be similarly derived. Hence there is no element that belongs to multiple quantized sets. □

3.2.3 Expansions of Quantized Sets

Disjointness is a necessary condition for a mapping to be a fuzzy switching funtion. But it is not a sufficient condition. For instance, consider mapping f defined by

$$f(0.7, 0.5) = 0.3 \qquad f(0.5, 0.8) = 0.8.$$

Clearly, the quantized sets of f:

$$\begin{aligned} C_1(f) &= \{\overline{(0.5, 0.8)}^{0.7}\} = \{(0.5, 1)\} \\ C_0(f) &= \{\overline{(0.7, 0.5)}^{0.6}\} = \{(1, 0.5)\} \\ C_U(f) &= \{\overline{(0.5, 0.8)}^{1}\} = \{(0.5, 0.5)\} \end{aligned}$$

are disjoint to each other. However, there is no two variable fuzzy switching function that satisfies f. In this section, to show the above case we introduce an *expansion* of quantized sets.

Definition 3.2 Let f be a mapping $f : A \to V$, $C_1(f)$, $C_0(f)$, $C_U(f)$ and $C(f)$ be quantized sets of f. Then, the *expansions* of quantized sets are subsets of V_3^n defined as follows:

$$\begin{aligned} C_1^*(f) &= C_1(f) \cup \{\boldsymbol{a} \in V_3^n \mid \boldsymbol{b} \in C_1(f), \boldsymbol{b} \succeq \boldsymbol{a}\}, \\ C_0^*(f) &= C_0(f) \cup \{\boldsymbol{a} \in V_3^n \mid \boldsymbol{b} \in C_0(f), \boldsymbol{b} \succeq \boldsymbol{a}\}, \\ C_U^*(f) &= C_U(f) \cup \left\{\boldsymbol{a} \in V_3^n \;\middle|\; \begin{array}{l} \boldsymbol{b} \in C_1^*(f),\ \boldsymbol{a} \succeq \boldsymbol{b}, \\ \boldsymbol{c} \in C_0^*(f),\ \boldsymbol{a} \succeq \boldsymbol{c} \end{array}\right\}, \\ C^*(f) &= C_1^*(f) \cup C_0^*(f) \cup C_U^*(f). \end{aligned}$$

Example 3.2 The quantized sets:

$$C_1(f) = \{(0.5, 0)\},\ C_0(f) = \{(1,1)\},\ C_U(f) = \{(0.5, 0.5)\}$$

has the expansions such that:

$$\begin{aligned} C_1^*(f) &= \{(0.5, 0), (\mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{0})\}, \\ C_0^*(f) &= \{(1,1)\}, \\ C_U^*(f) &= \{(0.5, 0.5), (\mathbf{1}, \mathbf{0.5})\}. \end{aligned}$$

Expansions of quantized sets involve a property called *regularity*. A ternary function f is regular if and only if

$$f(\boldsymbol{a}) \in \{0, 1\} \Rightarrow f(\boldsymbol{a}) = f(\boldsymbol{b}) \text{ for every } \boldsymbol{b} \text{ such as } \boldsymbol{a} \succeq \boldsymbol{b}.$$

It has been already proved by M.Mukaidono[10] that monotonicity is equivalent to regularity for the ternary function. Even in fuzzy switching functions, the following theorem shows the relation between monotonicity and regularity.

Theorem 3.3 Let f be a restriction $f : A \to V$, $C_1(f)$, $C_0(f)$, $C_U(f)$ and $C(f)$ be quantized sets of f, $C_1^*(f)$, $C_0^*(f)$, $C_U^*(f)$ and $C^*(f)$ be expansions. If $C_i^*(f) \cap C_j^*(f) = \emptyset$ for every $i \neq j$, then

$$\overline{\boldsymbol{a}}^\lambda \succeq \overline{\boldsymbol{b}}^\tau \Rightarrow \overline{f(\boldsymbol{a})}^\lambda \succeq \overline{f(\boldsymbol{b})}^\tau$$

for every two elements $\overline{\boldsymbol{a}}^\lambda$ and $\overline{\boldsymbol{b}}^\tau$ of $C(f)$.

Proof. Suppose that $C_i^*(f) \cap C_j^*(f) = \emptyset$. We show monotonicity for all elements of $C(f)$ in four cases.

(i). For an element $\boldsymbol{a}$ in $C_1(f)$, there is no element $\boldsymbol{b}$ in $C_U(f)$ such as $\boldsymbol{a} \succeq \boldsymbol{b}$. Because if $\boldsymbol{a} \succeq \boldsymbol{b}$, then $\boldsymbol{b} \in C_1^*(f)$, which is contradictory to regularity. Therefore $\boldsymbol{a}$ and $\boldsymbol{b}$ holds either $\boldsymbol{b} \succeq \boldsymbol{a}$ or $\boldsymbol{a} \not\succeq \boldsymbol{b}$, that is, they cannot violate monotonicity.

(ii). For an element $\boldsymbol{a}$ in $C_0(f)$, it is shown in the similar manner that f cannot violate monotonicity.

(iii). For an element $\boldsymbol{a}$ in $C_1(f)$, if an element $\boldsymbol{b}$ of $C_0(f)$ holds $\boldsymbol{a} \succeq \boldsymbol{b}$, then $\boldsymbol{b} \in C_1^*(f)$. If $\boldsymbol{b} \succeq \boldsymbol{a}$, then $\boldsymbol{a} \in C_0^*(f)$. Thus for any $\boldsymbol{b}$ of $C_0(f)$, $\boldsymbol{a} \not\succeq \boldsymbol{b}$. That means they cannot violate monotonicity.

(iv). Two elements $\boldsymbol{a}$ and $\boldsymbol{b}$ belonging to the same quantized set $C_i(f)$ apparently always hold monotonicity.

Consequently, any two of $C(f)$ hold monotonicity. □

3.2.4 Choice of Quantizing values

Next, we show an efficient way to obtain quantized sets. In a restriction defined over a finite set A, the quantization should be taken not for every element in infinite set V, but only for a few elements characterized in this section.

Lemma 3.1 Let a, b, and c be elements of V.

$$\overline{a}^b \in \{0,1\}, \overline{b}^c \in \{0,1\} \Rightarrow \overline{a}^c \in \{0,1\}.$$

Proof. Since $\overline{a}^b \neq 0.5$, then $a \neq 0.5$, and similarly $b \neq 0.5$. Thus, the following always holds according to Definition 2.3.

$$\min(a, 1-a) \leq \min(b, 1-b) \leq \min(c, 1-c)$$

Therefore, $\min(a, 1-a) \leq \min(c, 1-c)$, which means $\overline{a}^c \in \{0,1\}$. □

Theorem 3.4 Let $A = \{a_1, \ldots, a_n, 1\}$ be a subset of V. For any λ in V, there exists an element $a^\star$ of A such that $\overline{a_i}^\lambda = \overline{a_i}^{a^\star}$.

Proof. Let us define a subset A_λ of A by

$$A_\lambda = \{a \in A \mid \overline{a}^\lambda \in \{0,1\}\},$$

where $A_\lambda \neq \emptyset$ since at least $1 \in A$. Notice that 0.5 does not belong to A_λ, because $\overline{0.5}^\lambda = 0.5$ for any $\lambda \in V$.

Let $a^\star$ be an element of A_λ such that $\overline{a}^{a^\star} \in \{0,1\}$ for every $a \in A_\lambda$. Suppose that there exist an element a of A such that $\overline{a}^\lambda \neq \overline{a}^{a^\star}$. There exists at least one $a^\star$, which is the most similar element for A_λ.

(i). If $\overline{a}^\lambda = 0$ then $a \in A_\lambda$. Thus, according to the definition of $a^\star$, $\overline{a}^{a^\star} \in \{0,1\}$. Hence, $\overline{a}^{a^\star} = 1$, since $\overline{a}^\lambda \neq \overline{a}^{a^\star}$, which follows

$$0 \leq a \leq \min(\lambda, 1-\lambda) \leq 0.5 \leq \max(a^\star, 1-a^\star) \leq a \leq 1$$

therefore $a = 0.5$. However there is no λ which satisfies $\overline{0.5}^\lambda = 0$.

(ii). If $\overline{a}^\lambda = 1$, in the same manner as i., there is no λ in V.

(iii). If $\overline{a}^\lambda = 0.5$ then $\overline{a}^{a^\star} \in \{0,1\}$. While $\overline{a^\star}^\lambda \in \{0,1\}$ since $a^\star \in A_\lambda$. By applying Lemma 3.1, $\overline{a}^\lambda \in \{0,1\}$. This contradicts $\overline{a}^\lambda = 0.5$.

Finally, over (i), (ii) and (iii), $\boldsymbol{a}$ always contradicts the assumption. □

This theorem follows the following corollary, which presents the condition for sufficient elements in order to obtain quantized sets.

Corollary 3.1 Let f be a mapping $f : A \to V$, $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of A. $B(f(\boldsymbol{a}))$ is a subset of V defined by

$$B(f(\boldsymbol{a})) = \{a_1, \ldots, a_n, f(\boldsymbol{a}), 1\}.$$

For any element λ in V, there exists an element τ of $B(f(\boldsymbol{a}))$ such that

$$\overline{f(\boldsymbol{a})}^\lambda = \overline{f(\boldsymbol{a})}^\tau, \;\; \overline{\boldsymbol{a}}^\lambda = \overline{\boldsymbol{a}}^\tau.$$

3.2.5 Representability

Here we will clarify the necessary and sufficient condition for a given restriction $f : A \to V$ to be representable by a fuzzy switching function.

Lemma 3.2 Let A be a subset of V^n, f be a restriction $f : A \to V$. If $C_i(f) \cap C_j(f) = \emptyset$ for every $i \neq j$ in $\{0, 1, U\}$, then

$$\boldsymbol{a} \in A \cap V_3^n \quad \Rightarrow \quad f(\boldsymbol{a}) \in V_3.$$

Proof. Assume that there exist an element $\boldsymbol{a}$ of $A \cap V_3^n$ such that $f(\boldsymbol{a}) \notin V_3$. Since $\boldsymbol{a} \in V_3^n$, for any λ in V, $\overline{\boldsymbol{a}}^\lambda = \boldsymbol{a}$. While for 0.5 and 1 in V, $\overline{f(\boldsymbol{a})}^{0.5} \in \{0, 1\}$ and $\overline{f(\boldsymbol{a})}^1 = 0.5$ since $f(\boldsymbol{a}) \notin V_3$. Thereby

$$\begin{aligned} \overline{\boldsymbol{a}}^{0.5} &= \boldsymbol{a} \in C_1(f) \cup C_0(f), \\ \overline{\boldsymbol{a}}^1 &= \boldsymbol{a} \in C_U(f). \end{aligned}$$

Thus, $\boldsymbol{a} \in C_1(f) \cap C_U(f)$ or $\boldsymbol{a} \in C_0(f) \cap C_U(f)$. This conflicts with the hypothesis. Therefore we have $f(\boldsymbol{a}) \in V_3$. □

Lemma 3.3 Let A be a subset V^n, f be a restriction $f : A \to V$, $C(f)$ be a quantized set of f. A fuzzy switching function F holds

$$F(\overline{\boldsymbol{a}}^\lambda) = \overline{f(\boldsymbol{a})}^\lambda$$

for all elements $\overline{\boldsymbol{a}}^\lambda$ of $C(f)$, if and only if

$$F(\boldsymbol{a}) = f(\boldsymbol{a})$$

for all elements $\boldsymbol{a}$ of A.

Proof. When $F(\overline{\boldsymbol{a}}^\lambda) = \overline{f(\boldsymbol{a})}^\lambda$ for every $\overline{\boldsymbol{a}}^\lambda \in C(f)$, there is no element $\boldsymbol{a}$ of A such as $F(\boldsymbol{a}) \neq f(\boldsymbol{a})$. Otherwise, for the $\boldsymbol{a}$, either $\lambda = F(\boldsymbol{a})$ or $\lambda = f(\boldsymbol{a})$ holds that $\overline{f(\boldsymbol{a})}^\lambda \neq \overline{F(\boldsymbol{a})}^\lambda = F(\overline{\boldsymbol{a}}^\lambda)$, where $\overline{\boldsymbol{a}}^\lambda$ is in $C(f)$. Thereby it is contradictory to the hypothesis.

Conversely, suppose that $F(\boldsymbol{a}) = f(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of A. For any element $\overline{\boldsymbol{a}}^\lambda$ of $C(f)$, there exist $\boldsymbol{a} \in A$ and $\lambda \in V$ such that $f(\boldsymbol{a}) = F(\boldsymbol{a})$. By quantizing both sides by λ, we have $\overline{f(\boldsymbol{a})}^\lambda = \overline{F(\boldsymbol{a})}^\lambda = F(\overline{\boldsymbol{a}}^\lambda)$. □

Theorem 3.5 (representability) Let A be a subset of V^n, $C_1(f)$, $C_0(f)$, $C_U(f)$, and $C(f)$ be the quantized sets of a restriction $f : A \to V$, and $C_1^*(f)$, $C_0^*(f)$, $C_U^*(f)$, and $C^*(f)$ be their expansions. There exists a fuzzy switching function F such that $F(\boldsymbol{a}) = f(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of A, if and only if f satisfies both of the following conditions:

R (*Regularity*)	$C_i^*(f) \cap C_j^*(f) = \emptyset$	$(i \neq j), i, j \in V_3$
N (*Normality*)	$C_U(f) \cap V_2^n = \emptyset.$	

Proof. It is evident that f satisfies conditions R and N by Theorem 2.1 and Theorem 3.2. We will show that F represents f when both conditions hold.

Consider a logic formula F defined by

$$F = \bigvee_{a \in C_1(f)} \alpha^a \vee \bigvee_{b \in C_U(f)} \beta^b$$

where α^a is a simple phrase corresponding to an element $\boldsymbol{a}$ of $C_1(f)$ and β^b is a simple phrase corresponding to an element $\boldsymbol{b}$ of $C_U(f)$. Then, F represents a fuzzy switching function. Here we show F satisfies that $F(\overline{\boldsymbol{a}}^\lambda) = \overline{f(\boldsymbol{a})}^\lambda$ for all elements $\overline{\boldsymbol{a}}^\lambda$ of $C(f)$ in the following cases.

(i). For an element $\boldsymbol{a}$ of $C_1(f)$, there exists a simple phrase α^a corresponding to $\boldsymbol{a}$ in F such that $\alpha^a(\boldsymbol{a}) = 1$ [Lemma 2.1]. Therefore, $F(\boldsymbol{a}) = 1$ since F contains at least one phrase that is 1.

(ii). For an element $\boldsymbol{b}$ of $C_U(f)$, there exists a complementary phrase β^b corresponding to $\boldsymbol{b}$ in F. Since $\boldsymbol{b}$ is an element of $V_3^n - V_2^n$ by condition N, we have $\beta^b(\boldsymbol{b}) = 0.5$ from Lemma 2.1.

While there is no simple phrase α^a in F such as $\alpha^a(\boldsymbol{b}) = 1$. This is because the $\boldsymbol{a}$ and $\boldsymbol{b}$ hold $\boldsymbol{a} \succeq \boldsymbol{b}$ [Lemma 2.2], and thereby $\boldsymbol{b} \in C_1^*(f)$. Since $\boldsymbol{b} \in C_U(f)$, too, which contradicts condition R.

Next, there is no element $\boldsymbol{c}$ of $C_U(f)$ such that $\beta^c(\boldsymbol{b}) = 1$, because by condition N, $\boldsymbol{c}$ contains at least one c_i such that $c_i = 0.5$, that is, β^c is a complementary phrase which contains $x_i \wedge (\sim x_i)$. Any complementary phrase can never take value of 1.

Thus for any element $\boldsymbol{b}$ of $C_U(f)$, $F(\boldsymbol{b}) = 0.5$.

(iii). For an element $\boldsymbol{c}$ of $C_0(f)$, there is no simple phrase α^a in F such that $\alpha^a(\boldsymbol{c}) = 1$. Because by Lemma 2.2, $\boldsymbol{a} \succeq \boldsymbol{c}$, thereby $\boldsymbol{c} \in C_1^*(f)$. Thus $\mathbf{c}$ belongs both $C_0(f)$ and $C_1^*(f)$. This contradicts condition R.

Next, there is no simple phrase in F such that $\alpha^a(\boldsymbol{c}) = 0.5$. Otherwise, by Lemma 2.3, there exists an element $\boldsymbol{d}$ of V_3^n such that $\boldsymbol{a} \succeq \boldsymbol{d}$ and $\boldsymbol{c} \succeq \boldsymbol{d}$. According to Definition 3.2, we have $\boldsymbol{d} \in C_1^*(f)$ and $\boldsymbol{d} \in C_0^*(f)$, which contradicts condition R.

Finally, there is no complementary phrase such that $\beta^b(\boldsymbol{c}) \in \{0.5, 1\}$. $\beta^b(\boldsymbol{c})$ can never take the value of 1, since it is a complementary phrase by condition N. Even if $\beta^b(\boldsymbol{c}) = 0.5$, by Lemma 2.4, $\mathbf{c} \succeq \boldsymbol{b}$ and thereby $\boldsymbol{b} \in C_0^*(f)$. Since $\boldsymbol{b}$ belongs also $C_U(f)$, this is contradictory to the condition R.

Consequently, for the $\mathbf{c}$ any phrase in F talks neither 0.5 nor 1, that means, $F(\boldsymbol{c}) = 0$.

As we have seen over (i), (ii), and (iii), for any element of $C(f)$,

$$F(\boldsymbol{a}) = \begin{cases} 1 & \text{for every } \boldsymbol{a} \in C_1(f) \\ 0 & \text{for every } \boldsymbol{a} \in C_0(f) \\ 0.5 & \text{for every } \boldsymbol{a} \in C_U(f). \end{cases}$$

Hence by Lemma 3.3, we have, $F(\boldsymbol{a}) = f(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ in A. □

Theorem 3.5 means that regularity and normality for quantized sets are essential properties for a restriction of a fuzzy switching function.

Example 3.3 Let $A = \{(0.3), (0.8)\}$ be a subset of V. Even though a mapping $f : A \to V$ such as $f(0.3) = 0.7$, $f(0.8) = 0.1$ satisfies normality and monotonicity for ambiguity, f does not satisfy regularity, as follows:

$$C_1(f) = \{0\},\ C_0(f) = \{1, 0.5\},\ C_U(f) = \{0.5\}$$

where 0.5 belongs to both $C_0(f)$ and $C_U(f)$. Therefore, there is no one-variable fuzzy switching function representing f by Theorem 3.5.

3.3 Uniqueness

Even if a restriction satisfies every condition of Theorem 3.5, the fuzzy switching function representing f might be indefinite, that is, there can be several fuzzy switching functions for one restriction. For instance, the f such as $f(0.5) = 0.5$ which apparently satisfies conditions in Theorem 3.5 can be represented by the following four fuzzy switching functions:

$$F_1 = x,\ F_2 = {\sim}x,\ F_3 = x \wedge {\sim}x,\ F_4 = x \vee {\sim}x.$$

In this section, we clarify the condition for a fuzzy switching function to be uniquely determined for a restriction.

3.3.1 Necessary and Sufficient Condition for Restrictions to be Unique

Lemma 3.4 Let F and G be fuzzy switching functions, A be a subset of V^n, $C(F|_A)$ be a quantized set of a restriction $F|_A$. Then,

$$F(\boldsymbol{a}) = G(\boldsymbol{a})$$

for all elements $\boldsymbol{a}$ of A if and only if

$$F(\boldsymbol{a}) = G(\boldsymbol{a})$$

for all elements $\boldsymbol{a}$ in $C(F|_A)$.

Proof. Suppose that $F(\boldsymbol{a}) = G(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ in A. Then, for any element $\boldsymbol{b}$ of $C(F|_A)$, there exists an element $\boldsymbol{a} \in A$ and $\lambda \in V$ such that $\boldsymbol{b} = \overline{\boldsymbol{a}}^{\lambda}$. For the $\boldsymbol{a}$, $F(\boldsymbol{a}) = G(\boldsymbol{a})$ by the hypothesis. By quantizing both sides by λ, we obtain the former half:

$$\overline{F(\boldsymbol{a})}^{\lambda} = F(\overline{\boldsymbol{a}}^{\lambda}) = F(\boldsymbol{b}) = G(\boldsymbol{b}) = G(\overline{\boldsymbol{a}}^{\lambda}) = \overline{G(\boldsymbol{a})}^{\lambda}.$$

Conversely, when $F(\boldsymbol{a}) = G(\boldsymbol{a})$ for every $\boldsymbol{a} \in C(F|_A)$, let us suppose that there exist an element $\boldsymbol{b}$ of A such as $F(\boldsymbol{b}) \neq G(\boldsymbol{b})$. Then, there exists a certain λ in V such that:

$$\overline{F(\boldsymbol{b})}^{\lambda} = F(\overline{\boldsymbol{b}}^{\lambda}) \neq \overline{G(\boldsymbol{b})}^{\lambda} = G(\overline{\boldsymbol{b}}^{\lambda}).$$

This contradicts the hypothesis since $\overline{\boldsymbol{b}}^{\lambda}$ belongs to $C(F|_A)$. □

Lemma 3.5 Let F and G be fuzzy switching functions, A be a subset of V^n, $C(F|_A)$ be a quantized set of a restriction $F|_A$, $C^*(F|_A)$ be an expansion of $C(F|_A)$. Then

$$F(\boldsymbol{a}) = G(\boldsymbol{a})$$

for all elements $\boldsymbol{a}$ in $C(F|_A)$ if and only if

$$F(\boldsymbol{a}) = G(\boldsymbol{a})$$

for all elements $\boldsymbol{a}$ in $C^*(F|_A)$.

Proof. Suppose that $F(\boldsymbol{a}) = G(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ in $C(F|_A)$. Obviously, it is sufficient to show that any element $\boldsymbol{a}$ of the difference set $C^*(F|_A) - C(F|_A)$ implies $F(\boldsymbol{a}) = G(\boldsymbol{a})$.

(i). For $\boldsymbol{a} \in C_1^*(F|_A) - C_1(F|_A)$, there exists a $\boldsymbol{b}$ in $C_1(F|_A)$ such that $\boldsymbol{b} \succeq \boldsymbol{a}$ and $F(\boldsymbol{b}) = 1$. Hence, by Corollary 2.1,

$$F(\boldsymbol{a}) = 1 = G(\boldsymbol{a}).$$

(ii). For $\boldsymbol{a} \in C_0^*(F|_A) - C_0(F|_A)$, in the same manner as (i), we have $F(\boldsymbol{b}) = 0 = G(\boldsymbol{b})$.

(iii). For $\boldsymbol{a} \in C_U^*(F|_A) - C_U(F|_A)$, there exists a $\boldsymbol{b} \in C_1(F|_A)$ and $\boldsymbol{c} \in C_0(F|_A)$ such that $\boldsymbol{a} \succeq \boldsymbol{b}$ and $\boldsymbol{a} \succeq \boldsymbol{c}$. Hence by (i), (ii), and Corollary 2.1, we have

$$F(\boldsymbol{a}) = 0.5 = G(\boldsymbol{a}).$$

As we have seen over (i), (ii), and (iii), for any element $\boldsymbol{a}$ of $C^*(F|_A)$, we have $F(\boldsymbol{a}) = G(\boldsymbol{a})$. The latter half is evident since $C^*(F|_A)$ subsumes $C(F|_A)$. □

Here we show the necessary and sufficient condition for a fuzzy switching function F representing a given restriction $F|_A$ to be uniquely determined.

Theorem 3.6 (uniquness) Let F and G be fuzzy switching functions, $F|_A$ be a restriction of F to a subset A of V^n, $C^*(F|_A)$ be an expansion of quantized set of $F|_A$. When $C^*(F|_A)$ satisfies a condition U:

$$\text{U } (\textit{Uniqueness}) \qquad C^*(F|_A) = C_1^*(F|_A) \cup C_0^*(F|_A) \cup C_U^*(F|_A) = V_3^n$$

then restriction $F|_A$ is unique, that is,

$$\begin{array}{ll} \forall \boldsymbol{a} \in A & F(\boldsymbol{a}) = G(\boldsymbol{a}) \\ & \Updownarrow \\ \forall \boldsymbol{a} \in V^n & F(\boldsymbol{a}) = G(\boldsymbol{a}). \end{array}$$

Proof. The followings are equivalent.

$$\begin{array}{lll}
\forall \boldsymbol{a} \in A & F(\boldsymbol{a}) = G(\boldsymbol{a}) & \\
 & \Updownarrow & \text{[Lemma 3.4]} \\
\forall \boldsymbol{a} \in C(F|_A) & F(\boldsymbol{a}) = G(\boldsymbol{a}) & \\
 & \Updownarrow & \text{[Lemma 3.5]} \\
\forall \boldsymbol{a} \in C^*(F|_A) & F(\boldsymbol{a}) = G(\boldsymbol{a}) & \\
 & \Updownarrow & \text{[Condition U]} \\
\forall \boldsymbol{a} \in V_3^n & F(\boldsymbol{a}) = G(\boldsymbol{a}) & \\
 & \Updownarrow & \text{[Theorem 2.4]} \\
\forall \boldsymbol{a} \in V^n & F(\boldsymbol{a}) = G(\boldsymbol{a}). &
\end{array}$$

□

The condition U implies unique fuzzy switching function can be decided for the given restriction $F|_A$.

Here, we have clarified two important conditions, representablity and uniquness, which are characterized by some classes of fuzzy switching functions, disjoint, regular, an' ɪor-mal. These relationship is illustrated on a Venn diagram in Figure 3.1, where N, D, and R indicates sets of normal, disjoint, and regular restrictions, respectively. Note that the set of regular restrictions is a subset of that of disjoint restrictions. Thus there is no restriction that is not disjoint but regular.

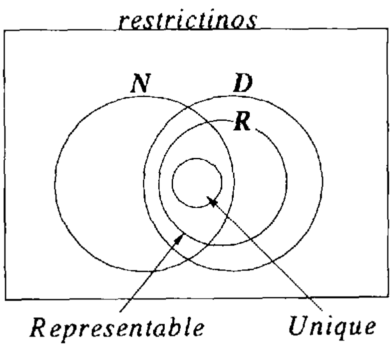

Figure 3.1: Some classes of restrictions

Example 3.4 Here are seven restrictions with domain of subset of V^2. Consider the following restrictions of two variables.

$$f_1(0.3, 0.1) = 0.5 \tag{3.1}$$

$$f_2(0.4, 0.2) = 0.8 \tag{3.2}$$
$$f_2(0.8, 0.7) = 0.2 \tag{3.3}$$

$$f_3(0.4, 0.8) = 0.3 \tag{3.4}$$

$$f_4(0.8, 0.3) = 0.8 \quad (3.5)$$
$$f_4(0.4, 0.7) = 0.4 \quad (3.6)$$
$$f_5(0.8, 0.5) = 0.8 \quad (3.7)$$
$$f_5(0.4, 0.9) = 0.1 \quad (3.8)$$
$$f_6(0.9, 0.9) = 0.8 \quad (3.9)$$
$$f_7(0.6, 0.3) = 0.7 \quad (3.10)$$
$$f_7(0.2, 0.9) = 0.1 \quad (3.11)$$

The quantized sets and expansions are illustrated on ternary truth tables in Figure 3.2, where the notation a/b denotes duplicate element that belongs both C_a and C_b.

Table 3.1 shows properties satisfied for each restrictions. We can verify that the normality is independent of the property of regularity, though the regularity depends on whether the quantized sets are disjoint.

class	D	R	N	Representable	Unique
f_1	√	√			
f_2	√		√		
f_3			√		
f_4	√	√	√	√	
f_5	√				
f_6					
f_7	√	√	√	√	√

Table 3.1: characteristics of restrictions

$y\backslash x$	0	0.5	1
0	0.5	0.5	
0.5		0.5	
1			

$C(f_1)$

$y\backslash x$	0	0.5	1
0	0.5	0.5	
0.5		0.5	
1			

$C^*(f_1)$

$y\backslash x$	0	0.5	1
0	1	1	
0.5		0.5	0
1			0

$C(f_2)$

$y\backslash x$	0	0.5	1
0	1	1	1/0
0.5		0.5	0
1			0

$C^*(f_2)$

$y\backslash x$	0	0.5	1
0			
0.5		0.5	
1	0	0/0.5	

$C(f_3)$

$y\backslash x$	0	0.5	1
0			
0.5		0.5	
1	0	0/0.5	0

$C^*(f_3)$

$y\backslash x$	0	0.5	1
0			1
0.5		0.5	1
1	0	0.5	

$C(f_4)$

$y\backslash x$	0	0.5	1
0			1
0.5		0.5	1
1	0	0.5	1

$C^*(f_4)$

$y\backslash x$	0	0.5	1
0			
0.5		0.5	1
1	0	0	

$C(f_5)$

$y\backslash x$	0	0.5	1
0			1
0.5		0.5	1
1	0	0	1/0

$C^*(f_5)$

$y\backslash x$	0	0.5	1
0			
0.5		0.5	
1			1/0.5

$C(f_6)$

$y\backslash x$	0	0.5	1
0			
0.5		0.5	
1			1/0.5

$C^*(f_6)$

$y\backslash x$	0	0.5	1
0		1	1
0.5		0.5	
1	0	0	

$C(f_7)$

$y\backslash x$	0	0.5	1
0	1	1	1
0.5	0.5	0.5	0.5
1	0	0	0

$C^*(f_7)$

Figure 3.2: quantized sets and expansions

3.4 Comparison with conventional methods

This section compares our proposed fuzzy switching function with two conventional inference methods, neural networks and fuzzy inference, from an engineering viewpoint. They are compared on this problem, which was defined in the introduction:

> Let f be a mapping $f : V^2 \to V$, $A = \{(0.4, 0.3), (0.8, 0.6), (0.2, 0.9)\}$ be a subset of V^2. For each elements of A, f maps as follows:
>
> $$\begin{aligned} f(0.4, 0.3) &= 0.7, \\ f(0.8, 0.6) &= 0.8, \\ f(0.2, 0.9) &= 0.2. \end{aligned}$$
>
> Find values $f(\boldsymbol{a})$ for any element of V^2, and approximate the whole f by each method.

3.4.1 Fuzzy Switching Function

In fuzzy switching function, there are the following three questions:

- Is there any fuzzy switching function representing f ?
- If possible, is it determined uniquely ?
- If it is unique, what is the logic formula representing f ?

First of all, we get quantized sets for $f|_A$. According to Theorem 3.1, it is sufficient to take a quantized value for A by each element of set $B(f|_A)$:

$$B(f|_A) = \{0.2, 0.3, 0.4, 0.6, 0.7, 0.8, 0.9, 1\},$$

and we can thereby effectively obtain the quantized sets as follows,

$$\begin{aligned} C_1(f|_A) &= \{(0.5, 0), (0, 0), (1, 0.5), (1, 1)\}, \\ C_0(f|_A) &= \{(0, 1)\}, \\ C_U(f|_A) &= \{(0.5, 0.5), (0.5, 1)\}. \end{aligned}$$

Next, for the quantized sets, we have the expansions as follows:

$$\begin{aligned} C_1^*(f|_A) &= C_1(f|_A) \cup \{(1, 0)\}, \\ C_0^*(f|_A) &= C_0(f|_A), \\ C_U^*(f|_A) &= C_U(f|_A) \cup \{(0, 0.5)\}. \end{aligned}$$

Table 3.2, 3.3, and 3.4 illustrate the given restrictin $f|_A$, the quantized sets $C(f|_A)$, and the expansions $C^*(f|_A)$ on truth tables, respectively.

Table 3.2: Restiction f

$y\backslash x$	0	0.2	0.4	0.5	0.8	1
0						
0.3			0.7			
0.5						
0.6					0.8	
0.9		0.2				
1						

Table 3.3: Quantized set $C(f)$

$y\backslash x$	0	0.5	1
0	1	1	
0.5		0.5	1
1	0	0.5	1

As shown in the figure, clearly, they satisfy condition D of Theorem 3.5:

$$\begin{aligned} C_1^*(f|_A) \cap C_0^*(f|_A) &= \emptyset \\ C_1^*(f|_A) \cap C_U^*(f|_A) &= \emptyset \\ C_0^*(f|_A) \cap C_U^*(f|_A) &= \emptyset \end{aligned}$$

and also satisfy condition R as we have shown before, there must be a certain fuzzy switching function F such as $F(\boldsymbol{a}) = f(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of A. In addition, they satisfy the conditions in Theorem 3.6,

$$C_1^*(f|_A) \cup C_0^*(f|_A) \cup C_U^*(f|_A) = V_3^2.$$

So we know that F is the only fuzzy switching function representing f.

Finally, we obtain the logic formula representing F. The result that $C^*(f) = V_3^n$ follows the correspondence:

$$f^{-1}(1) = C_1^*(f), \quad f^{-1}(0.5) = C_U^*(f).$$

Hence, by applying Theorem 2.5 here, we have

$$F(x,y) \;=\; \bigvee_{a \in C_1^*(f)} \alpha^a \vee \bigvee_{b \in C_U^*(f)} \beta^b$$

Table 3.4: Expansion $C^*(f)$

$y\backslash x$	0	0.5	1
0	1	1	1
0.5	0.5	0.5	1
1	0	0.5	1

$$\begin{aligned} &= \bigvee\{(\sim y),(\sim x\sim y),(x),(xy),(x\sim y)\} \\ &\quad \bigvee\{(x\sim xy\sim y),(x\sim x\sim y),(\sim xy\sim y)\} \\ &= x \vee \sim y \quad \text{(by the absorption law)} \end{aligned}$$

In this way, we can obtain the fuzzy switching function F representing f for all elements of A. Now we can know any value $f(\boldsymbol{a})$ for all $\boldsymbol{a}$ in V^2 by using this logic formula F (Figure 3.3).

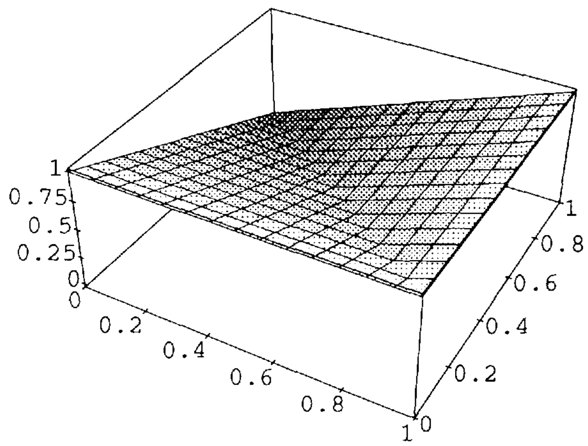

Figure 3.3: Mapping F by fuzzy switching function

3.4.2 Neural Networks

In neural networks, the restriction f' can be considered as a learning data by which weights and thresholds are settled down. The mapping f is represented by a neural network N that have two units x, y in the input layer, one unit u in a hidden layer and one unit $f(x, y)$ in the output layer. Value $f(x, y)$ is computed as follows:

$$\begin{aligned} f(x,y) &= s(\omega_0^2 u(x,y) + \theta_2), \\ u(x,y) &= s(\omega_0^1 x + \omega_1^1 y + \theta_1), \\ s(z) &= \frac{1}{1+e^{-z}}. \end{aligned}$$

Figure 3.4 and Figure 3.5 illustrate the neural network model and the behavior of function f.

By applying back propagation with f' to N, weights ω_i^j of unit i at layer j and thresholds θ_i at layer i are trained to the following values:

$$\omega_0^1 = -3.751357,\ \omega_1^1 = 2.603957,\ \omega_0^2 = -4.338762,$$

$$\theta_1 = -0.173989,\ \theta_2 = 2.107396$$

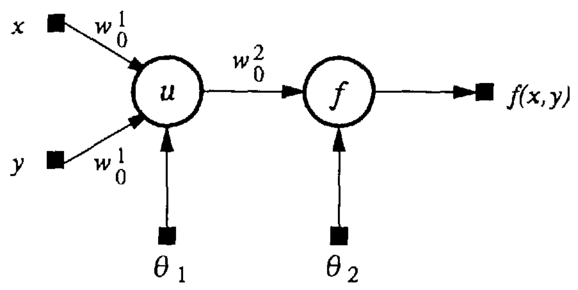

Figure 3.4: Neural network model

The results of calculation for V^2 are illustrated on Figure 3.6. Although this indeed coincides with the restriction f' within 0.00001 error, we should mention a few disadvantages.

- This method needs computational power that increases with the number of inputs.
- It cannot be guaranteed that learning will converge in a finite number of iterations. It implies an adjustment of some parameters or a rearrangement of the network's topology. The condition of the existence of a significant solution as Theorem 3.5 has not been clarified.

3.4.3 Fuzzy Inference

In fuzzy inference, we regard the restriction f' as the following IF-THEN rule base.

$$\begin{array}{llll} \text{rule 1}: & \text{IF } x = \tilde{0.4} & \text{and } y = \tilde{0.3} & \text{Then } f(x,y) = \tilde{0.7} \\ \text{rule 2}: & \text{IF } x = \tilde{0.8} & \text{and } y = \tilde{0.6} & \text{Then } f(x,y) = \tilde{0.8} \\ \text{rule 3}: & \text{IF } x = \tilde{0.2} & \text{and } y = \tilde{0.9} & \text{Then } f(x,y) = \tilde{0.2} \end{array}$$

where $\tilde{a}$ is a fuzzy set characterized by the following membership function μ_a:

$$\mu_a(x) = 1 - |a - x| \quad a, x \in V,$$

which is illustrated in Figure 3.8.

Then, the result $f(x, y)$ is calculated by

$$f(x,y) = \frac{\Sigma z \mu_{B(x,y)}(z)}{\Sigma \mu_{B(x,y)}(z)}$$

where membership function $\mu_{B(x,y)}$ is defined by

$$\mu_{B(x,y)}(z) = \max \left\{ \begin{array}{l} \min(\mu_{0.4}(x), \mu_{0.3}(y), \mu_{0.7}(z)), \\ \min(\mu_{0.8}(x), \mu_{0.6}(y), \mu_{0.8}(z)), \\ \min(\mu_{0.2}(x), \mu_{0.9}(y), \mu_{0.2}(z)) \end{array} \right\}$$

Figure 3.7 shows the consequence of fuzzy inference for V. Even recently when some products using fuzzy inference were developed, the similarities to neural network still exist and a number of variations have been proposed. Some fundamental properties of fuzzy inference such as the optimal definition of membership functions or composition of consequence are still unknown.

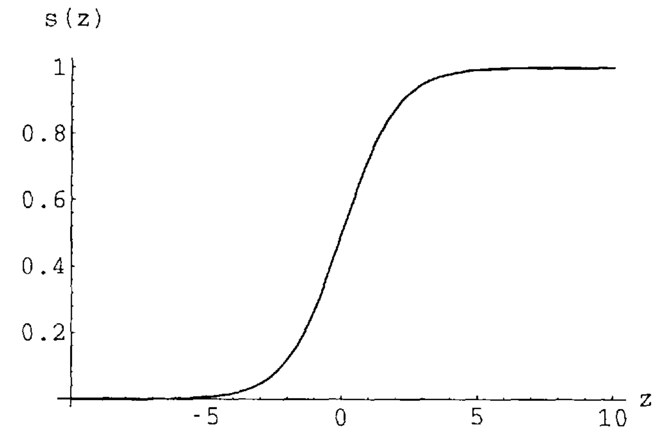

Figure 3.5: Function $s(z)$

3.5 Conclusion

We have clarified some fundamental and important properties of a restriction of fuzzy switching functions, that make it possible to extract essential information from incomplete and uncertain knowledge, and to identify a whole mapping with a fuzzy switching function. This is the first attempt to consider a fuzzy switching function as a method for approximate reasoning.

The necessary and sufficient condition in order for a restriction to be a fuzzy switching functions (Theorem 3.5) and the necessary and sufficient condition for fuzzy switching functions to be uniquely determined by a restriction (Theorem 3.6) have been clarified. We can see in a finite number of steps whether a given restriction has a solution as a fuzzy switching function, and whether the solution is determined uniquely or not. From the point of view of inference systems, this works much more effectively than conventional approximate methods that involves trial and error.

However, the condition for a fuzzy switching function seems too strong to be a model for our natural inferences, which is inexact and changeable. Therefore, we should make them weaker and investigate some more general logical systems that deal with uncertainty

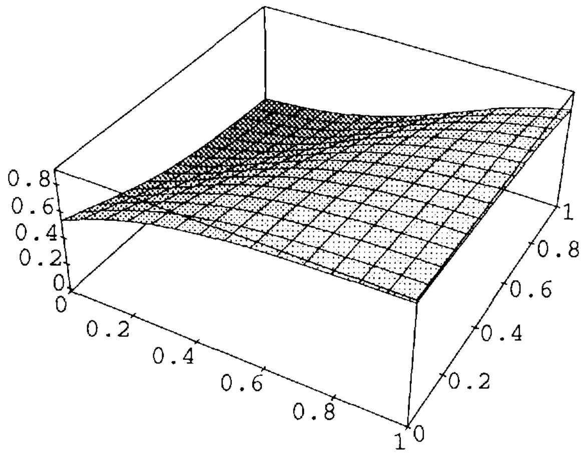

Figure 3.6: Calculation in a neural network N

including *unknown* or *contradiction*.

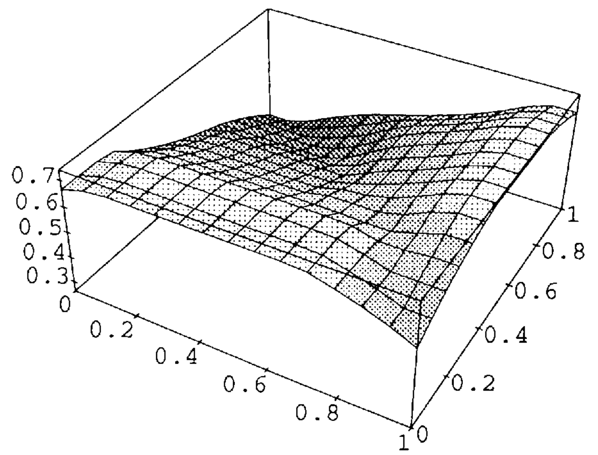

Figure 3.7: Consequences of fuzzy inference

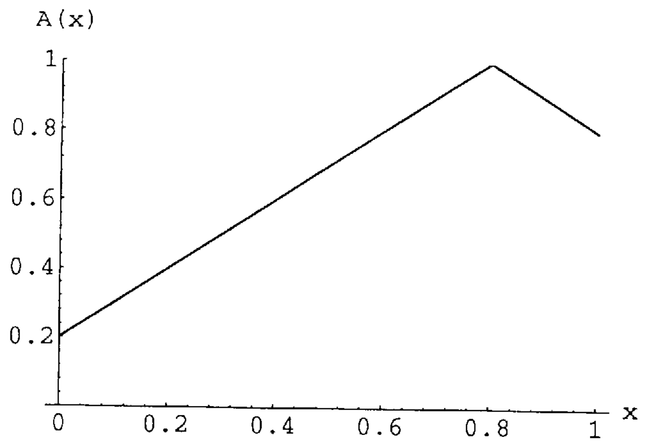

Figure 3.8: Membership function $\mu_A(x)$

Chapter 4

P-Fuzzy Switching Functions

A *P-Fuzzy Switching Function* is a meaningful class of fuzzy switching functions that can be represented by a logic formula of prime implicants.

We show how knowledge can be extracted and represented as prime implicants from given learning data. We derive necessary and sufficient conditions for the learning data to be representable with a P-fuzzy switching function, and to be expressed by a unique logic formula.

4.1 Introduction

In the previous chapter, we have studies the identification of fuzzy swiching function. However, since the complementary laws, $x \vee \sim x = 1$ and $x \wedge \sim x = 0$ do not hold in fuzzy logic, the algorithm could produce non classical logic formulae. For example, with the following learning data f,

$$\begin{cases} f(0.7, 0.1) = 0.9, \\ f(0.4, 0.7) = 0.4, \end{cases} \tag{4.1}$$

illustrated in Table 4.1, we find the nonunique result of the following three logic formulae f_1, f_2 and f_3:

$$\begin{aligned} f_1(x,y) &= x \vee \overline{y}, \qquad (4.2)\\ f_2(x,y) &= xy \vee \overline{y}, \\ f_3(x,y) &= x\overline{x}y \vee \overline{y} \end{aligned}$$

As the ternary truth tables in Table 4.2 show, these logic formulae are almost equivalent, differing only for $(1, 0.5)$ and $(1, 1)$. Formulae f_2 and f_3, however, are much more complicated than f_1. The complementary phrase $x\overline{x}y$ in f_3 is a contradiction and becomes 0 in binary logic. Formulae such as f_2 and f_3 are not, therefore, appropriate to modeling human knowledge.

In this chapter, we introduce *P-fuzzy switching functions* as a way to eliminate the redundant formulae from the possible solutions and obtain the simplest logic formula. A P-fuzzy switching function[17] can be represented by a disjunction of prime implicants only. Any given learning data can be represented by P-fuzzy switching functions without any complementary phrase.

Table 4.1: Learning data $f(x, y)$

$y \backslash x$	$0 \cdots$	.4	$\cdots$	.7	$\cdots 1$
0					
$\vdots$					
.1				.9	
$\vdots$					
.7		.4			
$\vdots$					
.1					

In this chapter, we will be trying to extract knowledge represented as P-fuzzy switching functions from given learning data. The main results are the necessary and sufficient conditions for the learning data to be representable with P-fuzzy switching functions, and to be determined by a unique logic formula.

Firstly, we define P-fuzzy switching functions, and clarify their fundamental properties. Second, we discuss restrictions of fuzzy switching functions, which are used as learning data. Finally, we will clarify the necessary and sufficient conditions for the given learning data to be representable with P-fuzzy switching functions using ternary subsets characterized by the learning data called P-resolutions.

Table 4.2: Truth tables of f_1, f_2 and f_3

$f_1(x, y) = xy \vee \overline{y}$

$y \backslash x$	0	.5	1
0	1	1	1
.5	.5	.5	.5
1	0	.5	1

$f_2(x, y) = x \vee \overline{y}$

$y \backslash x$	0	.5	1
0	1	1	1
.5	.5	.5	1
1	0	.5	1

$f_3(x, y) = \overline{y} \vee x\overline{x}y$

$y \backslash x$	0	.5	1
0	1	1	1
.5	.5	.5	.5
1	0	.5	0

4.2 P-Fuzzy Switching Functions

4.2.1 Prime Implicants

A *literal* is a variable x_i, or its negation $\overline{x_i}$. A *phrase* is a conjunction of one or more literals. There are two kinds of phrase: one is a *complementary phrase* which contains both a variable and its negation, for at least one variable, while the other is a *simple phrase* which does not.

Definition 4.1 An *implicant* of a fuzzy switching function f is a phrase α such that $\alpha(\boldsymbol{a}) \leq f(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V_2^n. $I(f)$ is a set of all implicants of f.

A *simple implicant* of f is a simple phrase in $I(f)$. A *fuzzy implicant* of f is an implicant α such that $\alpha(\boldsymbol{a}) \leq f(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V^n. $SI(f)$ and $FI(f)$ are phrase sets of all simple implicants of f and all fuzzy implicants of f, respectively.

A *prime implicant* of f is a simple implicant that is not an implicant of any element of $SI(f)$. A *fuzzy prime implicant* of f is a fuzzy implicant that is not a fuzzy implicant of any element of $I(f)$. $PI(f)$ and $FPI(f)$ are phrase sets of all prime implicants of f and fuzzy prime implicants of f, respectively.

Example 4.1 Sets of implicants of the fuzzy switching function $f(x, y) = x\overline{y} \vee xy \vee \overline{x}y\overline{y}$ are illustrated in Figure 4.1. Note that phrase x is an implicant of f, but is not a fuzzy

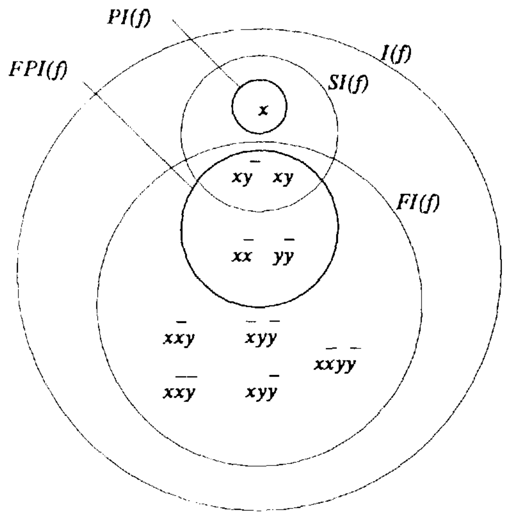

Figure 4.1: Implicants of $f(x, y) = x\overline{y} \vee xy \vee \overline{x}y\overline{y}$

implicant of f as follows:

$$f(1, 0.5) = 0.5 < x(1, 0.5) = 1 \tag{4.3}$$

Definition 4.2 A *P-fuzzy switching function* f_p is a fuzzy switching function which can be represented by the disjunction of all prime implicants of f_p.

Note that if f_p is a P-fuzzy switching function, then every element of $SI(f_p)$ is a fuzzy

4.2.2 Properties of P-Fuzzy Switching Functions

Definition 4.3 (Uniform) Let $\boldsymbol{a}$ be an element of V^n and B_a be a subset of V_2^n defined by

$$B_a = \{\boldsymbol{b} \in V_2^n | \boldsymbol{a} \succ \boldsymbol{b}\}.$$

A fuzzy switching function f is *uniform* if and only if

$$\begin{array}{lcl} f(\boldsymbol{a}) = 0 & \Leftrightarrow & f(B_a) = \{0\}, \\ f(\boldsymbol{a}) = 1 & \Leftrightarrow & f(B_a) = \{1\}, \end{array} \tag{4.4}$$

where $f(B_a)$ is an image of f by B_a.

Example 4.3 The fuzzy switching functions f_1 and f_3 in Eq.(4.2) are not uniform, because for $\boldsymbol{a} = (1, 0.5)$, $f(B_a) = \{1\}$, but $f(\boldsymbol{a}) = 0.5 \neq 1$ and $\boldsymbol{b} = (0.5, 1)$, $f_3(B_b) = \{0\}$, but $f_3(\boldsymbol{b}) = 0.5 \neq 0$. f_2 is, however, uniform.

We can see this by looking at the ternary truth tables in Table 4.3. Columns and rows in which both ends have the same value are circled, which shows the condition in Eq.(4.4), $f(B_a) = \{1\}$ or $f(B_a) = \{0\}$. The hatched cells violate uniformity.

Table 4.3: uniformity

$f_1(x, y) = xy \vee \overline{y}$

y \ x	0	.5	1
0	1	1	1
.5	.5	.5	1
1	0	.5	1

$f_2(x, y) = x \vee \overline{y}$

y \ x	0	.5	1
0	1	1	1
.5	.5	.5	[hatched]
1	0	.5	1

$f_3(x, y) = \overline{y} \vee x\overline{x}y$

y \ x	0	.5	1
0	1	1	1
.5	.5	.5	.5
1	0	[hatched]	0

Lemma 4.1 Let f be a fuzzy switching function. For any $\boldsymbol{a} \in V_3^n$ and B_a, the following hold.

$$\begin{array}{lcl} f(\boldsymbol{a}) = 0.5 & \Leftarrow & f(B_a) = \{0, 1\} \\ f(\boldsymbol{a}) = 0 & \Rightarrow & f(B_a) = \{0\} \\ f(\boldsymbol{a}) = 1 & \Rightarrow & f(B_a) = \{1\} \end{array}$$

Proof. The proof is straightforward from Corollary 2.1. □

Theorem 4.1 A fuzzy switching function f is uniform if and only if for all $\boldsymbol{a} \in V_3^n$,

$$f(\boldsymbol{a}) = 0.5 \Rightarrow f(B_a) = \{0, 1\}. \tag{4.5}$$

Proof. Let us assume $f(B_a) = \{0\}$ when Eq. (4.5) holds for every element of V_3^n. Hence, if $f(\boldsymbol{a}) = 0.5$ then $f(B_a) = \{0, 1\}$. Moreover, if $f(\boldsymbol{a}) = 1$ then $f(B_a) = 1$ by Lemma 4.1. Therefore, $f(\boldsymbol{a}) = 0$. Similarly, if $f(B_a) = \{1\}$ then $f(\boldsymbol{a}) = 1$. That is, f is uniform.

Conversely, let us suppose that f is uniform and $f(\boldsymbol{a}) = 0.5$. If $f(B_a) = \{0\}$ then $f(\boldsymbol{a}) = 0$. Hence, $f(B_a) \neq \{0\}$. Similarly $f(B_a) \neq \{1\}$. We therefore have $f(B_a) = \{0, 1\}$. □

Lemma 4.2 Let f_p be a P-fuzzy switching function, and $\boldsymbol{a}$ be an element of V_3^n and B_a.

$$f_p(B_a) = \{1\} \Rightarrow f_p(\boldsymbol{a}) = 1$$

Proof. We will show that the simple phrase α^a corresponding to $\boldsymbol{a}$ is an implicant of f_p when $f_p(B_a) = \{1\}$. For every $\boldsymbol{b} \in V_2^n$ such that $\alpha^a(\boldsymbol{b}) = 1$, we have $\boldsymbol{a} \succ \boldsymbol{b}$ by Lemma 2.2. It follows that $f_p(\boldsymbol{b}) = 1$ because $f_p(B_a) = \{1\}$. Thus, we have $f_p(\boldsymbol{b}) \geq \alpha^a(\boldsymbol{b})$, which means α^a is an implicant of f_p. Thereby, there exists a prime implicant α' in f_p such that $\alpha' \geq \alpha^a$, because f_p is a P-fuzzy switching function that contains all prime implicants. For the α', $f_p(\boldsymbol{a}) \geq \alpha'(\boldsymbol{a}) \geq \alpha^a(\boldsymbol{a}) = 1$. Consequently, $f_p(\boldsymbol{a}) = 1$, and this proves the Lemma. □

Lemma 4.3 Let f_p be a P-fuzzy switching function, and $\boldsymbol{a}$ be an element of V_3^n.

$$f_p(B_a) = \{0\} \Rightarrow f_p(\boldsymbol{a}) = 0.$$

Proof. We will show $f_p(\boldsymbol{a}) \neq 1$ when $f_p(B_a) = \{0\}$. If $f_p(\boldsymbol{a}) = 1$, then $f_p(B_a) = \{1\}$ by Lemma 4.1, which contradicts the hypothesis. Thus, $f_p(\boldsymbol{a}) \neq 1$.

Next, we will show $f_p(\boldsymbol{a}) \neq 0.5$ when $f_p(B_a) = \{0\}$. Suppose that there is a simple implicant $\alpha^{a'}$ such that $\alpha^{a'}(\boldsymbol{a}) = 0.5$. Then, by Lemma 2.3, there exists $\boldsymbol{c} \in V_3^n$ such that $\boldsymbol{a} \succ \boldsymbol{c}$, $\boldsymbol{a}' \succ \boldsymbol{c}$. Thereby, from Lemma 2.2 and Theorem 2.2, we have $f_p(\boldsymbol{a}') = 1 \succ f_p(\boldsymbol{c}) = 1$, which contradicts $f_p(\boldsymbol{c}) = 0$ for every $\boldsymbol{c}$ such that $\boldsymbol{a} \succ \boldsymbol{c}$. Hence, there is no simple implicant such that $\alpha^{a'}(\boldsymbol{a}) = 0.5$.

Since f_p is a P-fuzzy switching function that is representable with prime implicants, there is no complementary phrase with the value of 0.5. Therefore, $f_p(\boldsymbol{a}) = 0$. □

Theorem 4.2 Any P-fuzzy switching function f_p is representable by logic formula F:

$$F = \bigvee_{\boldsymbol{a} \in f^{-1}(1)} \alpha^a,$$

where $f^{-1}(1)$ is a subset of all $\boldsymbol{a} \in V_3^n$ such that $f(\boldsymbol{a}) = 1$.

Proof. We prove that $f(\boldsymbol{a}) = F(\boldsymbol{a})$ for all $\boldsymbol{a} \in V_3^n$ using three cases: $f(\boldsymbol{a}) = 1$; $f(\boldsymbol{a}) = 0.5$; $f(\boldsymbol{a}) = 0$.

When $f(\boldsymbol{a}) = 1$, there exists a simple phrase α^a in F, thereby, $\alpha^a(\boldsymbol{a}) = 1 = F(\boldsymbol{a})$.

When $f(\boldsymbol{a}) = 0.5$, there exists an implicant α^b of f such that $\alpha^b(\boldsymbol{a}) = 0.5$. Since $\alpha^b(\boldsymbol{b}) = 1$, the α^b also exists in F. There is, however, no simple phrase α^c in F such that $\alpha^c(\boldsymbol{a}) = 1$. Otherwise, for an element $\boldsymbol{c}$ corresponding to α^c, we have $f(\boldsymbol{c}) = 1$.

Then, from Lemma 2.2, we have $\boldsymbol{c} \succ \boldsymbol{a}$. Thereby, however, $f(\boldsymbol{c}) = 1 \succ f(\boldsymbol{a}) = 1$. This contradicts the hypothesis that $f(\boldsymbol{a}) = 0.5$.

When $f(\boldsymbol{a}) = 0$, there is no simple phrase α^b such that $\alpha^b(\boldsymbol{a}) \geq 0.5$. For an element $\boldsymbol{b} \in V_3^n$ corresponding to the α^b, from Lemma 2.3, there exists $\boldsymbol{c} \in V_2^n$ such that $\boldsymbol{b} \succ \boldsymbol{c}$ and $\boldsymbol{a} \succ \boldsymbol{c}$. Thereby, $\alpha^b(\boldsymbol{b}) = 1 = f(\boldsymbol{b}) \succ f(\boldsymbol{c}) = 1$ and $f(\boldsymbol{a}) \succ f(\boldsymbol{c}) = 1$, and hence $f(\boldsymbol{a}) \geq 0.5$. This contradicts the hypothesis that $f(\boldsymbol{a}) = 0$. There is, therefore, no simple phrase such as α^b in F, either.

As we have shown, $f(\boldsymbol{a}) = F(\boldsymbol{a})$ for all elements of V_3^n. Therefore, from Theorem 2.4, we have $f(\boldsymbol{a}) = F(\boldsymbol{a})$ for all elements of V^n. This implies that F represents f. □

Theorem 4.3 Uniform fuzzy switching function is a P-fuzzy switching function.

Proof. We prove that the uniform fuzzy switching function f can be represented by the logic formula $F = \bigvee_{\boldsymbol{a} \in f^{-1}(1)} \alpha^a$, by verifying $f(\boldsymbol{a}) = F(\boldsymbol{a})$ for every $\boldsymbol{a} \in V_3^n$.

We consider three cases: (i) f($\boldsymbol{a}$)=1; (ii) f($\boldsymbol{a}$)=0.5; (iii) f($\boldsymbol{a}$)=0.

(i) If $f(\boldsymbol{a}) = 1$, then $\boldsymbol{a} \in f^{-1}(1)$ and so $\alpha^a(\boldsymbol{a}) = 1 \leq F(\boldsymbol{a}) = 1$.

(ii) If $f(\boldsymbol{a}) = 0.5$, then from Theorem 4.1, we have $f(B_a) = \{0, 1\}$. Thus, there exists $\boldsymbol{b} \in B_a$ for which $f(\boldsymbol{b}) = 1$. Since $\boldsymbol{b} \in B_a$, we have $\boldsymbol{a} \succ \boldsymbol{b}$, which follows $\alpha^b(\boldsymbol{a}) = 0.5$ from Lemma 2.2. Hence, $F(\boldsymbol{a}) \leq 0.5$.

Consider $\boldsymbol{b} \in f^{-1}(1)$ such that $\alpha^b(\boldsymbol{a}) = 1$. Then, from Lemma 2.2, we have $\boldsymbol{a} \succ \boldsymbol{b}$. However, $f(\boldsymbol{a}) = 0.5$ and $f(\boldsymbol{b}) = 1$ violate the monotonicity of f. Hence, $F(\boldsymbol{a}) < 1$. Finally, we have $F(\boldsymbol{a}) = 0.5$ whenever $f(\boldsymbol{a}) = 0.5$.

(iii) If $f(\boldsymbol{a}) = 0$, then there is no simple phrase α^b in F such that $\alpha^b(\boldsymbol{a}) \leq 0.5$. Otherwise, from Lemma 2.3, there must exist $\boldsymbol{c} \in V_3^n$ such that $\boldsymbol{a} \succ \boldsymbol{c}$ and $\boldsymbol{b} \succ \boldsymbol{c}$. However, no value for $f(\boldsymbol{c})$ satisfies monotonicity together with $f(\boldsymbol{a}) = 0$ and $f(\boldsymbol{b}) = 1$. Therefore, $F(\boldsymbol{a}) = 0$.

In all three cases, therefore, $f(\boldsymbol{a}) = F(\boldsymbol{a})$ for all $\boldsymbol{a} \in V_3^n$, which leads to $f(\boldsymbol{a}) = F(\boldsymbol{a})$ for all $\boldsymbol{a} \in V^n$. From Theorem 4.2, f is a P-fuzzy switching function. □

Theorem 4.4 Let f be a fuzzy switching function. The following three propositions are then equivalent.

(i). f is a P-fuzzy switching function.

(ii). f is a uniform fuzzy switching function. That is, for any $\boldsymbol{a} \in V_3^n$

$$\begin{cases} f(\boldsymbol{a}) = 0 & \Leftrightarrow \quad f(B_a) = \{0\} \\ f(\boldsymbol{a}) = 1 & \Leftrightarrow \quad f(B_a) = \{1\} \end{cases}$$

(iii). For any $\boldsymbol{a} \in V_3^n$

$$f(\boldsymbol{a}) = 0.5 \Leftrightarrow f(B_a) = \{0, 1\}$$

Proof. First, from Theorem 4.3, if fuzzy switching function f is uniform, then f is a P-fuzzy switching function.

Conversely, Lemmas 4.1, 4.2, and 4.3 imply that the P-fuzzy switching function is uniform. Conditions 1 and 2 are, therefore, equivalent. Condition 3 is straightforward from Theorem 4.1. □

4.3 Restrictions of P-Fuzzy Switching Function

Let f be a mapping $f : V^n \to V$, A be a non-empty subset of V^n. A *restriction* $f|_A$ of f to A is a mapping defined by $f|_A(\boldsymbol{a}) = f(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of A.

In the preceding section, we have introduced the concept of *quantized sets*, which characterize a restriction with some subsets of V_3^n. Let us remind of the definition of the quantized sets.

Let $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ be a subset of V^n, f be a mapping $f : A \to V$. *Quantized sets* of f are subsets of V_3^n defined as follows:

$$\begin{aligned}
C_1(f) &= \{\overline{\boldsymbol{a}_i}^\lambda \mid \boldsymbol{a}_i \in A,\, \lambda \in V,\, \overline{f(\boldsymbol{a}_i)}^\lambda = 1\}\\
C_0(f) &= \{\overline{\boldsymbol{a}_i}^\lambda \mid \boldsymbol{a}_i \in A,\, \lambda \in V,\, \overline{f(\boldsymbol{a}_i)}^\lambda = 0\}\\
C_U(f) &= \{\overline{\boldsymbol{a}_i}^\lambda \mid \boldsymbol{a}_i \in A,\, \lambda \in V,\, \overline{f(\boldsymbol{a}_i)}^\lambda = 0.5\}\\
C(f) &= C_1(f) \cup C_0(f) \cup C_U(f).
\end{aligned}$$

Expansions of quantized sets are subsets of V_3^n defined as follows:

$$\begin{aligned}
C_1^*(f) &= C_1(f) \cup \{\boldsymbol{a} \in V_3^n \mid \boldsymbol{b} \in C_1(f), \boldsymbol{b} \succ \boldsymbol{a}\},\\
C_0^*(f) &= C_0(f) \cup \{\boldsymbol{a} \in V_3^n \mid \boldsymbol{b} \in C_0(f), \boldsymbol{b} \succ \boldsymbol{a}\},\\
C_U^*(f) &= C_U(f) \cup \left\{\boldsymbol{a} \in V_3^n \middle| \begin{array}{c}\boldsymbol{b} \in C_1^*(f), \boldsymbol{c} \in C_0^*(f)\\ \boldsymbol{a} \succeq \boldsymbol{b},\, \boldsymbol{a} \succeq \boldsymbol{c}\end{array}\right\}\\
C^*(f) &= C_1^*(f) \cup C_0^*(f) \cup C_U^*(f).
\end{aligned}$$

4.3.1 P-Resolutions

Definition 4.4 Let $C_0^*(f)$ $C_1^*(f)$ be expansions of mapping f. *P-resolutions* $P_1(f)$ and $P_0(f)$ of f are subsets of V_2^n defined as follows:

$$\begin{aligned}
P_1^0(f) &= C_1^*(f) \cap V_2^n\\
P_0^0(f) &= C_0^*(f) \cap V_2^n\\
P_1^{i+1}(f) &= P_1^i(f) \cup \left\{\boldsymbol{a} \in V_2^n \middle| \begin{array}{c}\boldsymbol{b} \in C_U(f), \boldsymbol{b} \succ \boldsymbol{a},\\ B_b - \{\boldsymbol{a}\} \subset P_0^i(f)\end{array}\right\}\\
P_0^{i+1}(f) &= P_0^i(f) \cup \left\{\boldsymbol{a} \in V_2^n \middle| \begin{array}{c}\boldsymbol{b} \in C_U(f), \boldsymbol{b} \succ \boldsymbol{a},\\ B_b - \{\boldsymbol{a}\} \subset P_1^i(f)\end{array}\right\}
\end{aligned}$$

$$\begin{aligned}
P_1(f) &= P_1(f)^i = P_1(f)^{i+1} = \cdots\\
P_0(f) &= P_0(f)^i = P_0(f)^{i+1} = \cdots
\end{aligned}$$

We often call $P_0^i(f)$ and $P_1^i(f)$ i-th P-resolutions of f.

Example 4.4 Given the followings expansions of a P-fuzzy switching function,

$$\begin{aligned} C_1^*(f) &= \emptyset, \\ C_0^*(f) &= \{(0,0)\}, \\ C_U^*(f) &= \{(0.5,0),(0.5,0.5),(0.5,1),(1,0.5)\} \end{aligned}$$

the P-resolutions are defined one after another as follows.

$$\begin{aligned} P_1^0(f) &= \emptyset \\ P_0^0(f) &= \{(0,0)\} \\ P_1^1(f) &= C_1^*(f) \cup \{(1,0)\} \\ P_0^1(f) &= C_0^*(f) \\ P_1^2(f) &= P_1^1(f) \\ P_0^2(f) &= P_0^1(f) \cup \{(1,1)\} \\ P_1^3(f) &= P_1^2(f) \cup \{(0,1)\} \\ P_0^3(f) &= P_0^2(f) \\ P_1^4(f) &= P_1^5(f) = \cdots = P_1(f) \\ P_0^4(f) &= P_0^5(f) = \cdots = P_0(f) \end{aligned}$$

We demonstrate P-resolutions in the ternary truth tables in Table 4.4, in which elements of $C_U^*(f)$, $P_1^i(f)$, and $P_0^i(f)$ are .5, 1, and 0, respectively.

Table 4.4: Truth tables of P-resolutions

$C^*(f)$

$y\backslash x$	0	.5	1
0	0	.5	
.5		.5	.5
1		.5	

$P_1^1(f)$ and $P_0^1(f)$

$y\backslash x$	0	.5	1
0	0	.5	**1**
.5		.5	.5
1		.5	

$P_1^2(f)$ and $P_0^2(f)$

$y\backslash x$	0	.5	1
0	0	.5	1
.5		.5	.5
1		.5	**0**

$P_1^3(f)$ and $P_0^3(f)$

$y\backslash x$	0	.5	1
0	0	.5	1
.5		.5	.5
1	**1**	.5	0

4.3.2 P-Representability

We say a restriction $f|_A$ of a fuzzy switching function f is *P-representable* if there is a P-fuzzy switching function f_p such that $f(\boldsymbol{a}) = f_p(\boldsymbol{a})$ for all $\boldsymbol{a} \in A$.

Theorem 4.5 (P-representability) Let f be a restriction of fuzzy switching functions, and $P_1(f)$ and $P_0(f)$ be P-resolutions of f. Then, f is a P-fuzzy switching function if and only if

$$P_0(f) \cap P_1(f) = \emptyset. \tag{4.6}$$

Proof. We prove by contradiction that there is no element which belongs to both P-resolutions of P-fuzzy switching function f. Suppose that $\boldsymbol{a} \in P_1(f) \cap P_0(f)$. Since f is a restriction of a fuzzy switching function, from Theorem 3.2 $C_0^*(f) \cap C_1^*(f) = \emptyset$. Hence, $\boldsymbol{a} \notin P_0^0(f) \cap P_1^0(f) = \emptyset$.

Let $\boldsymbol{a}$ be an element of $P_0^i(f)$ and of $P_1^{i+1}(f) - P_1^i(f)$ without loss of generality. There then exists a $\boldsymbol{b} \in C_U^*(f)$ such that $\boldsymbol{b} \succ \boldsymbol{a}$ and $B_b \subset P_0^i$. This implies $f(B_b) = \{0\}$ and $f(\boldsymbol{b}) = 0.5$, which violates uniformity of f, and hence contradicts the hypothesis that f is a P-fuzzy switching function. We therefore have $P_1(f) \cap P_0(f) = \emptyset$.

Conversely, let us suppose f is not a P-fuzzy switching function when $P_1(f) \cap P_0(f) = \emptyset$. Consider $\boldsymbol{a} \in C_U^*(f)$ such that $f(B_a) \neq \{0,1\}$, which violates uniformity of f. If $f(B_a) = \{1\}$ then there must exist $\boldsymbol{b} \in P_1^i(f)$ and $\boldsymbol{b} \in P_0^{i+1}(f)$ for an element $\boldsymbol{b} \in B_a$. This contradicts the hypothesis that $P_1(f) \cap P_0(f) = \emptyset$. Furthermore, if $f(B_a) = \{0\}$ then, in a similar way, we have $\boldsymbol{b} \in P_0(f) \cap P_1(f)$ for any $\boldsymbol{b} \in B_a$. Consequently, $f(B_a) = \{0,1\}$ for any $\boldsymbol{a} \in C_U^*(f)$, so f is a P-fuzzy switching function. □

4.3.3 P-Uniqueness

We say a restriction $f|_A$ of a fuzzy switching function is *P-unique* if there is a unique P-fuzzy switching function f_p such that $f(\boldsymbol{a}) = f_p(\boldsymbol{a})$ for every $\boldsymbol{a} \in A$.

Lemma 4.4 Let f_p and g_p be P-fuzzy switching functions. Then, $f_p(\boldsymbol{a}) = g_p(\boldsymbol{a})$ for all $\boldsymbol{a} \in V_2^n$ if and only if $f_p(\boldsymbol{a}) = g_p(\boldsymbol{a})$ for all $\boldsymbol{a} \in V^n$.

Proof. This is evident since all implicants are determined uniquely by elements of V_2^n only. □

Lemma 4.5 Let f_p be P-fuzzy switching functions, and $C^*(f_p|_A)$, $P_1(f_p|_A)$ and $P_0(f_p|_A)$ be the expansion and the P-resolutions of a restriction $f_p|_A$, respectively. Then, $f_p(\boldsymbol{a}) = 1$ for all $\boldsymbol{a} \in P_1(f_p)$ and $f_p(\boldsymbol{b}) = 0$ for all $\boldsymbol{b} \in P_0(f_p)$.

Proof. We prove this by induction by verifying it for all i-th P-resolutions.

It is clearly true for $P_0^0(f_p)$ and $P_1^0(f_p)$, since $P_0^0(f_p) \subset C_0^*(f_p)$ and $P_1^0(f_p) \subset C_1^*(f_p)$.

Assume the lemma hold for i-th P-resolutions, that is, $f_p(\boldsymbol{a}) = 1$ for all $\boldsymbol{a} \in P_1^i(f_p)$ and $f_p(\boldsymbol{b}) = 0$ for all $\boldsymbol{b} \in P_0^i(f_p)$.

For $\boldsymbol{a} \in P_1^{i+1}(f_p)$, by Definition 4.4, there exists $\boldsymbol{b} \in C_U(f_p)$ such that $\boldsymbol{b} \succ \boldsymbol{a}$ and $B_b - \{\boldsymbol{a}\} \subset P_0^i(f_p)$. Our assumption gives us $f_p(\boldsymbol{b}) = 0.5$ and $f_p(\boldsymbol{c}) = 0$ for all $\boldsymbol{c} \in B_b - \{\boldsymbol{a}\}$, which is a subset of $P_0^i(f_p)$. Because of uniformity of f_p, we have $f_p(B_b) = \{0,1\}$, and hence, $f_p(\boldsymbol{a}) = 1$. In the same manner, $f_p(\boldsymbol{a}) = 0$ for all $\boldsymbol{a} \in P_0^{i+1}(f_p)$. The lemma, therefore holds for $i+1$-th P-resolutions whenever the lemma holds for i-th P-resolutions. By the Principle of Mathematical Induction, the lemma holds for all i-th P-resolutions. □

Theorem 4.6 (P-uniqueness) Let f_p and g_p be P-fuzzy switching functions. Then, $f_p(\boldsymbol{a}) = g_p(\boldsymbol{a})$ for every $\boldsymbol{a} \in V^n$ if and only if

$$1. \quad f_p(\boldsymbol{a}) = g_p(\boldsymbol{a}) \quad \forall \boldsymbol{a} \in A, \tag{4.7}$$

$$2. \quad P_0(f_p|_A) \cup P_1(f_p|_A) = V_2^n. \tag{4.8}$$

Proof. Condition 1 is now equivalent to the following.

$$\begin{array}{ll} \forall \boldsymbol{a} \in A & f_p(\boldsymbol{a}) = g_p(\boldsymbol{a}) \\ & \Updownarrow \quad \text{[Lemma 3.5]} \\ \forall \boldsymbol{a} \in C^*(f_p|_A) & f_p(\boldsymbol{a}) = g_p(\boldsymbol{a}) \\ & \Updownarrow \quad \text{[Lemma 4.5]} \\ \forall \boldsymbol{a} \in P_1(f_p|_A) \cup P_0(f_p|_A) & f_p(\boldsymbol{a}) = g_p(\boldsymbol{a}) \\ & \Updownarrow \quad \text{[Condition 2]} \\ \forall \boldsymbol{a} \in V_2^n & f_p(\boldsymbol{a}) = g_p(\boldsymbol{a}) \\ & \Updownarrow \quad \text{[Lemma 4.4]} \\ \forall \boldsymbol{a} \in V^n & f_p(\boldsymbol{a}) = g_p(\boldsymbol{a}) \end{array}$$

□

By verifying the conditions in Eq.(4.6) and Eq.(4.7), we can see whether any given restriction $A \rightarrow V$ is P-consistent or P-unique. Figure 4.2 illustrates the relationship between some classes of restrictions, where the set of P-unique restrictions is shaded. Note that any restriction that is unique and P-consistent is also P-unique.

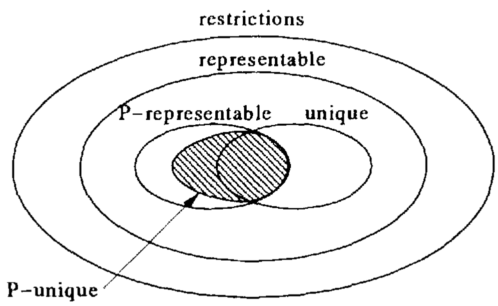

Figure 4.2: Classes of restrictions

4.3.4 Representation of P-unique restriction

Definition 4.5 (P-expansion) Let $C_1^*(f)$, $C_0^*(f)$, and $C_U^*(f)$ be expansions of a fuzzy switching function f, and $P_1(f)$ and $P_0(f)$ be P-resolutions of f. *P-expansions* of f are subsets of V_3^n defined as follows:

$$\begin{aligned} C_1^P(f) &= C_1^*(f) \cup \{\boldsymbol{a} \in V_3^n \mid B_a - \{\boldsymbol{a}\} \subset P_1(f)\} \\ C_0^P(f) &= C_0^*(f) \cup \{\boldsymbol{a} \in V_3^n \mid B_a - \{\boldsymbol{a}\} \subset P_0(f)\} \end{aligned}$$

$$\begin{aligned} C_U^P(f) &= C_U^*(f) \cup \left\{ \boldsymbol{a} \in V_3^n \middle| \begin{array}{l} \boldsymbol{b}_0 \in P_0(f),\ \boldsymbol{a} \succ \boldsymbol{b}_0, \\ \boldsymbol{b}_1 \in P_1(f),\ \boldsymbol{a} \succ \boldsymbol{b}_1 \end{array} \right\} \\ C^P(f) &= C_1^P(f) \cup C_0^P(f) \cup C_U^P(f) \end{aligned}$$

Theorem 4.7 For any P-expansions of P-unique restriction f,

$$C_0^P(f) \cup C_1^P(f) \cup C_U^P(f) = V_3^n$$

Proof. It is straightforward from Theorem 4.4. □

Theorem 4.8 (Representation of P-unique restriction) Let f be a P-unique restriction of a fuzzy switching function, and $C_1^P(f) = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ be a P-expansion of f. The P-fuzzy switching function is represented by the logic formula

$$F = \alpha^{a_1} \vee \alpha^{a_2} \vee \cdots \vee \alpha^{a_m},$$

where α^{a_i} is a simple phrase corresponding to $\boldsymbol{a}_i$.

Proof. From Theorem 4.2, it is sufficient to represent a P-fuzzy switching function to collect only elements of V_3^n that have the value 1. □

4.3.5 Example

We are now ready to obtain a P-fuzzy switching function from any restriction of a fuzzy switching function. Let us look at the problem in the Introduction again.

Problem Let A be a subset of V^2 such that $A = \{(0.7, 0.1), (0.4, 0.7)\}$, and f be a mapping $f : A \to V$ defined as follows:

$$f(0.7, 0.1) = 0.9, \tag{4.9}$$
$$f(0.4, 0.7) = 0.4. \tag{4.10}$$

Question 1: is there a P-fuzzy switching function such that $F(\boldsymbol{a}) = f(\boldsymbol{a})$ for every $\boldsymbol{a} \in A$?

Question 2: What logic formula does represent f ?

We start by getting quantized sets for the restriction f as follows:

$$\begin{aligned} C_1(f) &= \{\overline{(0.7, 0.1)}^{0.2}, \overline{(0.7, 0.1)}^{0.6}\} && (4.11) \\ &= \{(0.5, 0), (1, 0)\}, && (4.12) \\ C_0(f) &= \{\overline{(0.4, 0.7)}^{0.5}\} = \{(0, 1)\}, && (4.13) \\ C_U(f) &= \{\overline{(0.7, 0.1)}^{0}, \overline{(0.4, 0.7)}^{0.7}\} && (4.14) \\ &= \{(0.5, 0.5), (0.5, 1)\}. && (4.15) \end{aligned}$$

Next, for the quantized sets, we have the following expansions:

$$C_1^*(f) = C_1(f) \cup \{(0,0)\}, \quad (4.16)$$
$$C_0^*(f) = C_0(f), \quad (4.17)$$
$$C_U^*(f) = C_U(f) \cup \{(0,0.5)\}. \quad (4.18)$$

Since they satisfy condition 1 of Theorem 3.5:

$$C_1^*(f) \cap C_0^*(f) = \emptyset \quad (4.19)$$
$$C_1^*(f) \cap C_U^*(f) = \emptyset \quad (4.20)$$
$$C_0^*(f) \cap C_U^*(f) = \emptyset \quad (4.21)$$

and also condition 2, there must be at the least one fuzzy switching function F such as $F(\boldsymbol{a}) = f(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of A. For these expansions, we have the following P-resolutions,

$$P_1(f) = C_1^*(f) \cup \{(1,1)\}, \quad (4.22)$$
$$P_0(f) = C_0^*(f), \quad (4.23)$$

that satisfy Eq. (4.6) and Eq. (4.7):

$$P_1(f) \cap P_0(f) = \emptyset, \quad (4.24)$$
$$P_1(f) \cup P_0(f) = V_2^2. \quad (4.25)$$

Thus, by Theorem 4.5 and Theorem 4.6, there is a unique P-fuzzy switching function for f. With the following P-expansions:

$$C_1^P(f) = C_1^*(f) \cup \{(1,0.5),(1,1)\}, \quad (4.26)$$
$$C_0^P(f) = C_0^*(f), \quad (4.27)$$
$$C_U^P(f) = C_U^*(f), \quad (4.28)$$

we can uniquely determine the logic formula F which represents f by applying Theorem 4.8 as follows:

$$F = \alpha^{(0,0)} \vee \alpha^{(0.5,0)} \vee \alpha^{(1,0)} \vee \alpha^{(1,0.5)} \vee \alpha^{(1,1)} \quad (4.29)$$
$$= \overline{x}\,\overline{y} \vee \overline{y} \vee x\,\overline{y} \vee x \vee x\,y \quad (4.30)$$
$$= \overline{y} \vee x \qquad \text{[by absorption laws]} \quad (4.31)$$

This is the only P-fuzzy switching function which satisfies Eq. (4.9). We demonstrate this mapping in Figure 4.3. We illustrate these steps in Table 4.5, which shows particular elements in each step in boldface.

4.4 Conclusion

We have studied the properties of P-fuzzy switching functions, and clarified the necessary and sufficient conditions for restrictions to be P-representable in Theorem 4.5 and to be P-unique in Theorem 4.6. These conditions are useful for automatically deriving knowledge represented as simple logic formula from any given learning data. We also described a way to represent P-fuzzy switching functions from any P-unique restriction in Theorem 4.8.

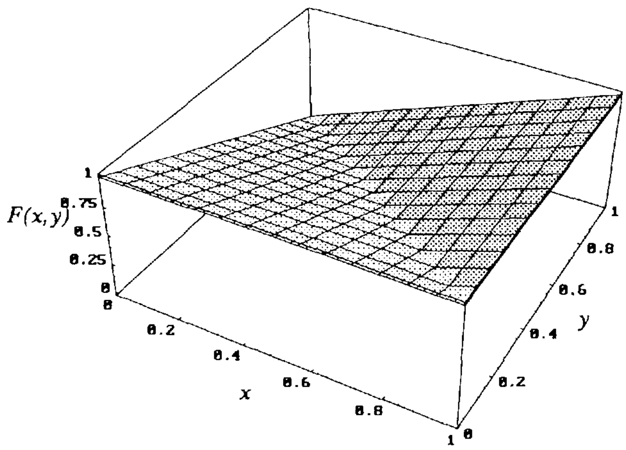

Figure 4.3: P-fuzzy switching function F

Table 4.5: Truth tables of quantized sets

$C(f)$

$y \backslash x$	0	.5	1
0		1	1
.5		.5	
1	0	.5	

$C^*(f)$

$y \backslash x$	0	.5	1
0	**1**	1	1
.5	**.5**	.5	
1	0	.5	

$P_0(f)$ and $P_1(f)$

$y \backslash x$	0	1
0	1	1
1	0	**1**

$C^P(f)$

$y \backslash x$	0	.5	1
0	1	1	1
.5	.5	.5	**1**
1	0	.5	1

Chapter 5

Identification of Kleenean Functions

Some properties of the partially specified fuzzy logic functions with constants are investigated and the identification problem of logic formula is solved. A fuzzy logic (switching) function is a mapping represented by means of logic formula which consists n variables, three logical connectives, and two constants of 0 and 1. In this chapter, any truth values of $[0, 1]$ are allowed to be constants in logic formula. The fuzzy logic function with arbitrary constants is called Multiple-valued Kleenean function. Main result is Theorem 5.7 which clarifies a necessary and sufficient condition for an identification problem of Kleenean function to be solved.

5.1 Introduction

Fuzzy switching function is a simple mapping represented by a logic formula, and thus can be used for logical expression of knowledge. We have studied the identification problem of a partially specified P-fuzzy switching function and clarified the necessary and sufficient condition for a given partial mapping to be represented by a single logic formula in the preceding chapter. The results follow an effective algorithm of uncertain knowledge acquisition that fits the best logic formula into a partial mapping specified by typical learning data of human experts[14].

In practice, fuzzy switching function, however, is not robust, that is, any small noise of learning data could spoil the consistency because no constant except 0 and 1 is allowed in logic formula. For example, consider a single variable fuzzy switching function f. For an element of 0.2, the value of $f(0.2)$ must be either 0, 1, $x = 0.2$, or $\sim x = 0.8$. None of fuzzy switching functions can take a value of 0.3, 0.1 or 0.20001.

In this chapter, we allow any constant of $[0, 1]$ to be in logic formula such as

$$f_1 = 0.3x{\sim}y, \quad f_2 = 0.20001 \vee x,$$

which represents a special class of multiple-valued functions called "n-variable multiple-valued Kleenean function." Hereafter, we often omit "n-variable" and just say a Kleenean function to mean it rather than fuzzy switching function with constants. From a multiple-valued logical view point, Kleenean function has been studied and the fundamental properties including the monotonicity, the canonical form and the representation have been clarified in [20, 24, 25].

We study some properties of Kleenean function which is partially specified by a subset $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ of $[0,1]^n$ and subset $B = \{b_1, \ldots, b_m\}$ of V with $f(\boldsymbol{a}_i) = b_i$ for all $i = 1, \ldots, m$. The identification problem of Kleenean function is to find a logic formula F such that

$$f(\boldsymbol{a}_i) = F(\boldsymbol{a}_i) \text{ for all } \boldsymbol{a}_i \in A$$

from given $f : A \rightarrow B$. An inconsistent mapping f could involve that there is no Kleenean function satisfying f. Hence, the condition for an existence of Kleenean function representing a given f is important issue and should be clarified.

In this paper, after reviewing some fundamental definitions and properties of Kleenean functions, we define some variations of quantization, *strong, weak,* and *quasi quantization.* Main result is the necessary and sufficient condition for existence of Kleenean function that satisfies a given identification problem $f : A \rightarrow B$, which provides a knowledge of given f expressed in a logic formula.

5.2 Kleenean Functions

This section gives fundamental definition of Kleenean function though almost all definitions are the same as those of fuzzy switching function except arbitrary constant values of $[0,1]$. The biggest differnce between them is the quantization theorem of fuzzy switching function does not hold in Kleenean function.

5.2.1 Basic Definitions

Definition 5.1 Let $V = [0,1]$, $V_2 = \{0,1\}$ and $V_3 = \{0, 0.5, 1\}$ be the sets of truth values. A *logic formula* consists of constants of $[0,1]$, n variables x_i $(i = 1, \ldots, n)$, and three kinds of logic connectives *and*($\wedge$), *or*($\vee$) and *not*($\sim$), that are defined by $x_i \wedge x_j = x_i x_j = \min(x_i, x_j)$, $x_i \vee x_j = \max(x_i, x_j)$, and $\sim x_i = \overline{x_i} = 1 - x_i$.

A *fuzzy logic function with constants* or *Kleenean function* is a mapping from an n-dimensional Cartesian product V^n to V which is represented by a logic formula.

Definition 5.2 Let a and b be elements of V. Then, $a \succeq b$ if and only if either $0.5 \geq a \geq b$ or $b \geq a \geq 0.5$. The relation $\succeq$ can be extended to V^n by letting $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V^n, $\boldsymbol{a} \succeq \boldsymbol{b}$ if and only if $a_i \succeq b_i$ for each i $(i = 1, \ldots, n)$. Any two elements a in $[0, 0.5)$ and b in $(0.5, 1]$ are not comparable with respect to $\succeq$. We denote this by $a \not\succeq b$. We write $a \succ b$ to mean that $a \succeq b$ and $a \neq b$.

Kleenean function is representable by a disjunctive form which is a disjunction ($\vee$) of some conjunctions ($\wedge$). There are two types of phrases. One is called a *complementary phrase* which contains a literal and its negation such as $x_i \wedge (\sim x_i)$ for some x_i. The other is called a *simple phrase* which does not.

Definition 5.3 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be elements of V_3^n and $V_3^n - V_2^n$. A *simple phrase* α^a corresponding to $\boldsymbol{a}$ and a complementary phrase β^b corresponding to $\boldsymbol{b}$ are defined by $\alpha^a = x_1^{a_1} \wedge \cdots \wedge x_n^{a_n}$ and $\beta^b = x_1^{b_1} \wedge \cdots \wedge x_n^{b_n}$ where

$$x_i^{a_i} = \begin{cases} x_i & \text{if } a_i = 1 \\ \overline{x_i} & \text{if } a_i = 0 \\ 1 & \text{if } a_i = 0.5, \end{cases} \qquad x_i^{b_i} = \begin{cases} x_i & \text{if } b_i = 1 \\ \overline{x_i} & \text{if } b_i = 0 \\ x_i\,\overline{x_i} & \text{if } b_i = 0.5 \end{cases}$$

for every i ($i = 1, \ldots, n$).

Note that the above definition of simple and complementary phrases do not contain constants of $[0,1]$ except 1.

5.2.2 Fundamental Properties

Theorem 5.1 (Monotonicity) [24] Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be elements of V^n and f be a Kleenean function.

$$\boldsymbol{a} \succeq \boldsymbol{b} \;\Rightarrow\; f(\boldsymbol{a}) \succeq f(\boldsymbol{b})$$

Theorem 5.2 [24] Let f and g be Kleenean functions. Then, $f(\boldsymbol{a}) = g(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V_3^n, if and only if $f(\boldsymbol{a}) = g(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V^n.

Theorem 5.3 (Representation) [25] Any Kleenean function f is represented by logic formula F:

$$F(\boldsymbol{x}) = \bigvee_{a \in V_3^n} \left\{ \bigwedge_{a \succeq b} f(\boldsymbol{b})\alpha_a(\boldsymbol{x}) \vee f(\boldsymbol{a})\beta_a(\boldsymbol{x}) \right\}$$

where α^a is a simple phrase corresponding to $\boldsymbol{a}$ and β^a is a complementary phrase corresponding to $\boldsymbol{a}$ (let $\beta_a(\boldsymbol{x}) = 0$ if $\mathbf{c} \in V_2^n$).

Corollary 5.1 [25] Let f be a Kleenean function, $\boldsymbol{a} = (a_1, \ldots, a_n)$ a subset of V^n. For any i ($i = 1, \ldots, n$), we have

$$f(\boldsymbol{a}) = F_0{\sim}a_i \vee F_1 a_i \vee F_0 F_1 F_{0.5} \vee F_{0.5} a_i {\sim} a_i$$

where F_0, F_1 and $F_{0.5}$ denote

$$\begin{aligned} F_0 &= f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n), \\ F_1 &= f(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n), \\ F_{0.5} &= f(a_1, \ldots, a_{i-1}, 0.5, a_{i+1}, \ldots, a_n). \end{aligned}$$

Proposition 5.1 [10] For any simple phrase α^a corresponding to $\boldsymbol{a} \in V_3^n$,

$$\alpha^a(\boldsymbol{a}) = 1.$$

For any complementary phrase β^b corresponding to $\boldsymbol{b} \in V_3^n - V_2^n$,

$$\beta^b(\boldsymbol{b}) = 0.5.$$

Proposition 5.2 [10] Let α^a be a simple phrase corresponding to $\boldsymbol{a} \in V_3^n$.

$$\begin{aligned} \boldsymbol{a} \succeq \boldsymbol{b} \quad &\Leftrightarrow \quad \alpha^a(\boldsymbol{b}) = 1 \\ \boldsymbol{b} \succeq \boldsymbol{a} \quad &\Rightarrow \quad \alpha^a(\boldsymbol{b}) = 0.5 \end{aligned}$$

Proposition 5.3 [12] Let α^a be a simple phrase corresponding to $\boldsymbol{a} \in V_3^n$ and $\boldsymbol{b}$ be an element of $V_3^n - V_2^n$, respectively. If $\alpha^a(\boldsymbol{b}) = 0.5$, there exists $\boldsymbol{c} \in V_3^n$ such that

$$\boldsymbol{a} \succeq \mathbf{c},\ \boldsymbol{b} \succeq \mathbf{c}.$$

Proposition 5.4 [10] Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be elements of V^n, β^b be a complementary phrase corresponding to $\boldsymbol{b}$.

$$\boldsymbol{a} \succeq \boldsymbol{b} \quad \Leftrightarrow \quad \beta^b(\boldsymbol{a}) = 0.5.$$

5.3 Identification

5.3.1 Quantizations

A quantization, a useful unary operation, is frequently used in conventional fuzzy logic. In this paper, we define new types of quantizations, called strong, weak, and quasi quantization.

Definition 5.4 Let x and λ be elements of V. A *strong quantization* $\overline{x}^\lambda$ of x by λ is an element of V_3 defined by:

$$\overline{x}^\lambda = \begin{cases} 0 & \text{if } 0 \leq x \leq \min(\lambda, 1-\lambda) \leq 0.5,\ x \neq 0.5, \\ 1 & \text{if } 1 \geq x \geq \max(\lambda, 1-\lambda) \geq 0.5,\ x \neq 0.5, \\ 0.5 & \text{otherwise.} \end{cases}$$

A *weak quantization* $\underline{x}_\lambda$ of x by λ is an element of V_3 defined by:

$$\underline{x}_\lambda = \begin{cases} 0 & \text{if } 0 \leq x < \min(\lambda, 1-\lambda), \\ 1 & \text{if } \max(\lambda, 1-\lambda) < x \leq 1, \\ 0.5 & \text{otherwise.} \end{cases}$$

These two quantizations are illustrated in Figure 5.1 (where we suppose $\lambda < 0.5$.)

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an element of V^n. Strong and weak quantizations $\overline{\boldsymbol{x}}^\lambda$ and $\underline{\boldsymbol{x}}_\lambda$ of $\boldsymbol{x}$ by λ are elements of V_3^n defined by:

$$\begin{aligned} \overline{\boldsymbol{x}}^\lambda &= (\overline{x_1}^\lambda, \ldots, \overline{x_n}^\lambda), \\ \underline{\boldsymbol{x}}_\lambda &= (\underline{x_1}_\lambda, \ldots, \underline{x_n}_\lambda). \end{aligned}$$

Example 5.1 The strong and weak quantizations of $\boldsymbol{a} = (0.2, 0.7, 0.3)$ are as follows:

$$\begin{aligned} \overline{\boldsymbol{a}}^{0.3} &= \overline{\boldsymbol{a}}^{0.7} = (0, 1, 0), \\ \underline{\boldsymbol{a}}_{0.3} &= \underline{\boldsymbol{a}}_{0.7} = (0, 0.5, 0.5). \end{aligned}$$

Note that both quantizations by $\lambda = 0.4$ are equal as follows:

$$\overline{\boldsymbol{a}}^{0.4} = \underline{\boldsymbol{a}}_{0.4} = (0, 1, 0).$$

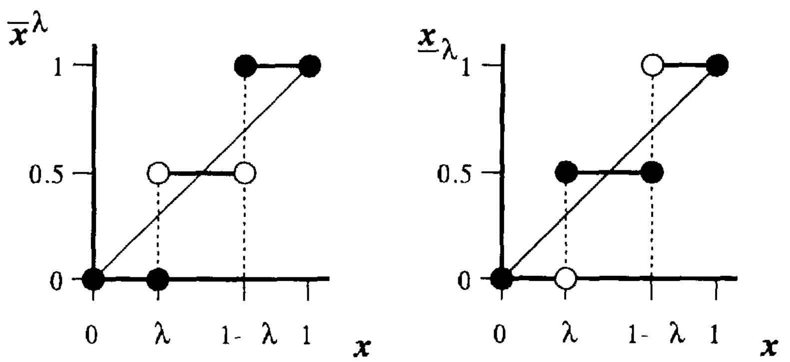

Figure 5.1: Strong and weak quantizations

Proposition 5.5 Let $\boldsymbol{a}$ be in V^n and λ, τ be in V.

$$\lambda \succeq \tau \quad \Rightarrow \quad \overline{\boldsymbol{a}}^\tau \succeq \overline{\boldsymbol{a}}^\lambda,\ \underline{\boldsymbol{a}}_\tau \succeq \underline{\boldsymbol{a}}_\lambda.$$

Proof. The proof is omitted. □

Definition 5.5 Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in V^n$, and $\lambda \in V$. A *quasi quantization* of $\boldsymbol{a}$ by λ is a subset of V^n defined by

$$Q_\lambda(\boldsymbol{a}) = (Q_\lambda(a1), \ldots, Q_\lambda(a_n)),$$

where

$$Q_\lambda(a_i) = \begin{cases} \overline{a_i}^\lambda & \text{if } \lambda \neq a_i, \text{ and } \lambda \neq 1 - a_i, \\ a_i & \text{otherwise.} \end{cases}$$

Note that either strong or weak quantization can be used in the above definition because $\overline{a_i}^\lambda = \underline{a_i}_\lambda$ holds whenever $\lambda \neq a_i$ and $\lambda \neq 1 - a_i$ hold.

Theorem 5.4 Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in V^n$, and f be a Kleenean function such that $f(\boldsymbol{a}) = b$. For any a_i, we have

$$f(a_1, \ldots, Q_b(a_i), \ldots, a_n) = b.$$

Proof. If $a_i = b$ or $a_i = 1 - b$ then $Q_{f(a)}(a_i) = a_i$, and thus the lemma is trivially true. So we assume that $a_i \neq b$ and $a_i \neq 1 - b$. We shall prove by only 8 cases: (i) $0.5 \leq b < a_i$; (ii) $0.5 \leq a_i < b$; (iii) $0.5 \leq 1 - b < a_i$; (iv) $a_i < b \leq 0.5$; (v) $b < a_i \leq 0.5$; (vi) $0.5 \leq 1 - a_i < b$; (vii) $0.5 \leq b < 1 - a_i$; (viii) $0.5 \leq a_i < 1 - b$.

(i) Suppose $0.5 \leq b < a_i$. Then, $Q_b(\boldsymbol{a}) = (a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n) = F_1$. By Corollary 5.1, we have

$$f(\boldsymbol{a}) = F_0{\sim}a_i \vee F_1 a_i \vee F_0 F_1 F_{0.5} \vee F_{0.5} a_i {\sim} a_i = b. \tag{5.1}$$

$F_1 a_i$ and $F_0 F_1 F_{0.5}$ become less than b, that is, a contradiction. So $F_1 \geq b$. Therefore, we have $F_1 = b$, i.e., $f(a_1, \ldots, Q_f(a_i), \ldots, a_n) = b$.

(ii) Suppose $0.5 \leq a_i < b$. Then, $Q_b(\boldsymbol{a}) = (a_1, \ldots, a_{i-1}, 0.5, a_{i+1}, \ldots, a_n) = F_1$. As the same way in (i), we have that both of a_i and $\sim a_i$ are less than b, and thus, phrase $F_0 F_1 F_{0.5}$ must be equal to b. If $F_{0.5} > b$ then by the monotonicity we have $F_1 > b$ and $F_0 > b$, which lead to $f(\boldsymbol{a}) \geq F_1 F_0 F_{0.5} > b$ and so contradiction. Therefore, we have $F_{0.5} = b = f(a_1, \ldots, Q_f(a_i), \ldots, a_n)$.

(iii) When $0.5 \leq 1 - b < a_i$ the theorem can be proved as in the case (i).

(iv) Suppose $a_i < b \leq 0.5$. Then, $F(Q_b(\boldsymbol{a})) = F(a_1, \ldots, 0, \ldots, a_n) = F_0$. By equation (5.1) and $a_i < b$, either phrase $F_0 \sim \boldsymbol{a}_i$ or $F_0 F_1 F_{0.5}$ must be b. If $F_0 > b$ then $f(\boldsymbol{a}) \geq F_0 \sim a_i > b$, so $F_0 \leq b$. While, if $F_0 < b$ then the other phrase $F_0 F_1 F_{0.5}$ also becomes less than b, hence, $F_0 \geq b$. Accordingly, only b can be equal to F_0.

(v) Suppose $b < a_i \leq 0.5$. Then, $F(Q_b(\boldsymbol{a})) = F(a_1, \ldots, 0.5, \ldots, a_n) = F_{0.5}$. Since now both a_i and $\sim a_i$ are greater than b, all of four phrases in equation (5.1) can be b. However, letting $F_{0.5} < b$ leads to contradiction because the monotonicity forces $F_1 < b$ and $F_0 < b$, that is, all of phrases are less than b. Similarly, $F_{0.5} > b$ is also impossible because the fourth phrase $F_{0.5} a_i \sim a_i$ becomes greater than b, which arises the contradiction $f(\boldsymbol{a}) > F_{0.5} a_i > b$. Therefore, we obtain $F_{0.5} = b$.

(vi) (vii) (viii) The case when $0.5 \leq 1 - a_i < b$, $0.5 \leq b < 1 - a_i$, and $0.5 \leq a_i < 1 - b$ can be proved in the same as (ii), (i), and (v), respectively.

Thus we show the theorem holds for all cases.

□

Example 5.2

$$\begin{aligned} f(0.2, 0.6, 1) &= f(Q_{0.3}(0.2), 0.6, 1) = f(0.5, 0.6, 1) \\ &= f(0.2, Q_{0.3}(1), 0.9) = f(0.2, 0.5, 1) \\ &= f(0.2, 0.6, Q_{0.3}(1)) = f(0.2, 0.6, 1) \\ &= \cdots = f(0, 0.5, 1) = 0.3. \end{aligned}$$

Corollary 5.2 Let f be a Kleenean function with $f(\boldsymbol{a}) = b$. For any $\boldsymbol{c} \in V_3^n$,

$$\begin{aligned} \boldsymbol{c} \succeq Q_b(\boldsymbol{a}) &\Rightarrow f(\boldsymbol{c}) \succeq b \\ \boldsymbol{c} = Q_b(\boldsymbol{a}) &\Rightarrow f(\boldsymbol{c}) = b \\ Q_b(\boldsymbol{a}) \succeq \boldsymbol{c} &\Rightarrow b \succeq f(\boldsymbol{c}) \end{aligned}$$

Proof. Theorem 5.4 shows $f(\boldsymbol{c}) = b$. Since $\boldsymbol{c} \succeq Q_b(\boldsymbol{a}) \succeq \boldsymbol{c}'$, we conclude that $f(\boldsymbol{c}) \succeq b \succeq f(\boldsymbol{c}')$ from the monotonicity of f. □

5.3.2 Possibilities of Truth Values

Definition 5.6 Let f be a mapping such that $f(\boldsymbol{a}) = b$ for some $\boldsymbol{a} \in V^n$. For $f(\boldsymbol{a}) = b$, we define

$$\begin{aligned} \boldsymbol{a}^* &= \overline{\boldsymbol{a}}^{f(a)}, \\ \boldsymbol{a}_* &= \underline{\boldsymbol{a}}_{f(a)}. \end{aligned}$$

Note that $\boldsymbol{a}_* \succeq \boldsymbol{a}^*$ always holds, and Corollary 5.2 gives that

$$\begin{aligned} f(\boldsymbol{a}) &\succeq f(\boldsymbol{a}_*), \\ f(\boldsymbol{a}^*) &\succeq f(\boldsymbol{a}). \end{aligned}$$

Definition 5.7 Let $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ be a subset of V^n, and f be a mapping $f : A \rightarrow V$. For $\boldsymbol{a} \in V^n$, $\boldsymbol{a}^*$ and $\boldsymbol{a}_* \in V_3^n$, a subset $C_f(\boldsymbol{a})$ of V_3^n is defined by

$$C_f(\boldsymbol{a}) = \{\boldsymbol{c} \in V_3^n | \boldsymbol{a}^* \succeq \boldsymbol{c}\} \cup \{\boldsymbol{a}_*\}$$

and for A,

$$C_f(A) = C_f(\boldsymbol{a}_1) \cup \cdots \cup C_f(\boldsymbol{a}_m).$$

Example 5.3 Let $\boldsymbol{a} = (0.3, 0.4, 0.9)$ and f be a Kleenean function such that $f(\boldsymbol{a}) = 0.3$. Then,

$$\begin{aligned} \boldsymbol{a}^* &= (0, 0.5, 1), \\ Q_b(\boldsymbol{a}) &= (0.3, 0.5, 1), \\ \boldsymbol{a}_* &= (0.5, 0.5, 1), \\ C_f(\boldsymbol{a}) &= \{\boldsymbol{a}^*, \boldsymbol{a}_*, (0,0,1), (0,1,1)\}. \end{aligned}$$

We illustrate the result on a Hasse diagram in Figure 5.2.

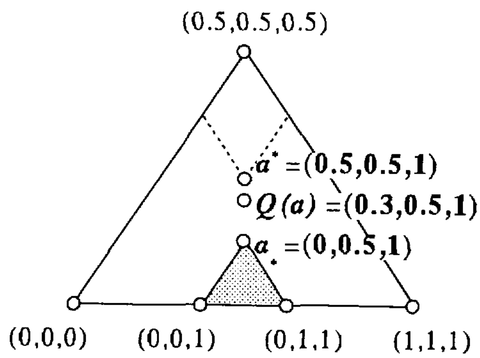

Figure 5.2: Hasse diagram of $C_f(\boldsymbol{a})$

Definition 5.8 Let f be a mapping $f(\boldsymbol{a}) = b$ for some $\boldsymbol{a} \in V^n$. For $\boldsymbol{c} \in C_f(\boldsymbol{a})$, a *possibility* of truth value is a subset of V defined by

$$I_a(\boldsymbol{c}) = \begin{cases} \{x \in V | b \succeq x\} & \text{if } \boldsymbol{a}^* \succeq \boldsymbol{c}, \\ \{x \in V | x \succeq b\} & \text{if } \boldsymbol{c} \succeq \boldsymbol{a}_* \end{cases}$$

if $\boldsymbol{a}^* \neq \boldsymbol{a}_*$; otherwise

$$I_a(\boldsymbol{c}) = \begin{cases} \{x \in V | b \succeq x\} & \text{if } \boldsymbol{a}^* \succ \boldsymbol{c}, \\ \{x \in V | x \succeq b\} & \text{if } \boldsymbol{c} \succ \boldsymbol{a}_*, \\ \{b\} & \text{if } \boldsymbol{c} = \boldsymbol{a}^* = \boldsymbol{a}_*. \end{cases}$$

Example 5.4 Let $f(0.2, 0.3) = 0.3$. Then,

$$\begin{aligned} \boldsymbol{a}^* &= \overline{(0.2, 0.3)}^{0.3} = (0,0), \\ \boldsymbol{a}_* &= \underline{(0.2, 0.3)}_{0.3} = (0, 0.5), \end{aligned}$$

and we have

$$\begin{aligned} C_f(0.2, 0.3) &= \{(0,0), (0, 0.5)\}, \\ I_{(0.2,0.3)}(0,0) &= \{x \in V | 0.3 \succeq x\} = [0, 0.3], \\ I_{(0.2,0.3)}(0, 0.5) &= \{x \in V | x \succeq 0.3\} = [0.3, 0.5], \end{aligned}$$

where a notation $[n, p]$ shows a subset $\{x \in V | n \leq x \leq p\}$. Figure 5.3 illustrates these possibilities.

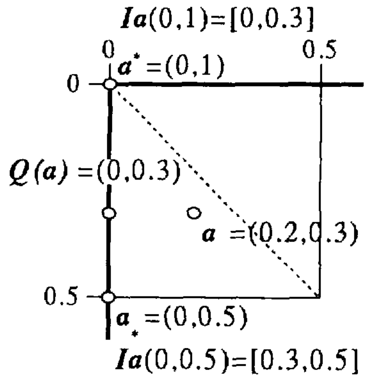

Figure 5.3: Possibilities for $f(0.2, 0.3) = 0.3$

Definition 5.9 Let $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ be a subset of V^n. We write $I_A(\boldsymbol{c})$ to mean that:

$$I_A(\boldsymbol{c}) = \bigcap_{a_i \in C_f^{-1}(c)} I_{a_i}(\boldsymbol{c}),$$

where $\bigcap_{a_i \in C_f^{-1}(c)}$ is a subset of V_3^n such that $\{\boldsymbol{a} \in V | \boldsymbol{c} \in C_f(\boldsymbol{a})\}$.

Note that a subset $C_f(\boldsymbol{a})$ depends on the value of $f(\boldsymbol{a})$. For example, $f(\boldsymbol{a}) = f(0.2, 0.3) = b$ have the following $C_f(\boldsymbol{a})$:

$$\begin{aligned} C_f(\boldsymbol{a}) &= \{(0,0)\} \quad \text{if } b = 0.4, \\ C_f(\boldsymbol{a}) &= \{(0,0), (0, 0.5)\} \quad \text{if } b = 0.3, \\ C_f(\boldsymbol{a}) &= \{(0,0), (0,1), (0, 0.5)\} \quad \text{if } b = 0.25, \\ C_f(\boldsymbol{a}) &= V_3^n \quad \text{if } b = 0.4. \end{aligned}$$

5.3.3 K-Representable

We say a given function $f : A \to V$ is "representable" to mean there exists at least one Kleenean function F such that

$$\text{C1:} \quad f(\boldsymbol{a}) = F(\boldsymbol{a}) \text{ for all } \boldsymbol{a} \in A.$$

In this section, we shall show the necessary and sufficient condition for K-representable (C1) is that every possibility I_A has at least one element, that is,

$$\text{C3:} \quad I_A(\boldsymbol{c}) \neq \emptyset \text{ for all } \boldsymbol{c} \in C_f(A).$$

To prove (C1) and (C3) are equivalent, we consider an assertion:

$$\text{C2:} \quad F(\boldsymbol{c}) \in I_A(\boldsymbol{c}) \text{ for all } \boldsymbol{c} \in C_f(A).$$

First, we show (C1) and (C2) are equivalent by the following lemma and theorem.

Lemma 5.1 Let $\boldsymbol{a}, b$ and $\boldsymbol{c}$ be elements of V^n, V and V_3^n, f a mapping such that $f(\boldsymbol{a}) = b$, $I_a(\boldsymbol{c})$ a possibility for $\boldsymbol{c}$. Then, Kleenean function F satisfies $F(\boldsymbol{a}) = b$ if and only if $F(\boldsymbol{c}) \in I_a(\boldsymbol{c})$ for all $\boldsymbol{c} \in C_f(\boldsymbol{a})$.

Proof. Suppose that $F(\boldsymbol{a}) = b$ and there is $\boldsymbol{c} \in C_f(\boldsymbol{a})$ such that $F(\boldsymbol{c}) \notin I_a(\boldsymbol{c})$. For the $\boldsymbol{c}$ we consider three cases: (i) $\boldsymbol{a}^* \succeq \boldsymbol{c}$; (ii) $\boldsymbol{c} \succeq \boldsymbol{a}_*$; (iii) $\boldsymbol{a}^* = \boldsymbol{a}_* = \boldsymbol{c}$. (Note that we do not have to consider a case $\boldsymbol{a}^* \not\succeq \boldsymbol{c}$ because of the definition of $C_f(\boldsymbol{a})$.)

(i) If $\boldsymbol{a}^* \succeq \boldsymbol{c}$ then the monotonicity leads to $b \succeq F(\boldsymbol{a}^*(\boldsymbol{c})) \succeq F(\boldsymbol{c})$, and thereby $F(\boldsymbol{c}) \in I_a(\boldsymbol{c})$.

(ii) Similarly, if $\boldsymbol{c} \succeq \boldsymbol{a}_*$ then $F(\boldsymbol{c}) \succeq F(\boldsymbol{a}_*) \succeq b$, and thus $F(\boldsymbol{c}) \in I_a(\boldsymbol{c})$.

(iii) If $\boldsymbol{c} = \boldsymbol{a}^* = \boldsymbol{a}_*$ then Corollary 5.2 follows $F(\boldsymbol{c}) = F(\boldsymbol{a}^*) = b \in I_a(\boldsymbol{c}) = \{b\}$. Therefore $F(\boldsymbol{c})$ is always in $I_a(\boldsymbol{c})$.

Conversely, we suppose that $F(\boldsymbol{c}) \neq b$ when $F(\boldsymbol{c}) \in I_a(\boldsymbol{c})$ holds for all $\boldsymbol{c} \in C_f(\boldsymbol{a})$. Let $F(\boldsymbol{a}) = b'$, $\boldsymbol{a}'^* = \overline{\boldsymbol{a}}^{b'}$ and $\boldsymbol{a}'_* = \overline{\boldsymbol{a}}^{b'}$.

(i) If $b \succ b'$ then Proposition 5.5 shows $\boldsymbol{a}'_* \succeq \boldsymbol{a}'^* \succeq \boldsymbol{a}^*$, and hence $F(\boldsymbol{a}'_*) \succeq F(\boldsymbol{a}'^*) \succeq F(\boldsymbol{a}^*)$. For $\boldsymbol{a}'^*$, F must satisfy both $b' \succeq F(\boldsymbol{a}'^*)$ and

$$F(\boldsymbol{a}'^*) \in I_a(\boldsymbol{a}'^*) = \{x \in V | x \succeq b\}.$$

However, no value can be $F(\boldsymbol{a}'^*)$ because now $b \succ b'$. Similarly, (ii) $b \succ b'$ follows $F(\boldsymbol{a}'_*) \notin I_a(\boldsymbol{a}'_*)$.

(iii) If $b \not\succeq b'$ then there exists $\boldsymbol{c} \in C_f(\boldsymbol{a})$ such that $\boldsymbol{a}^* \succeq \boldsymbol{c}$ and $\boldsymbol{a}'^* \succeq \boldsymbol{c}$, which involve

$$b' \succeq F(\boldsymbol{c}),\ b \succeq F(\boldsymbol{c}).$$

Thereby, $F(\boldsymbol{c})$ cannot take any value since $b \not\succeq b'$.

Therefore, in all cases, $F(\boldsymbol{c}) \in I_a(\boldsymbol{c})$ holds, and thus we have the lemma. □

Theorem 5.5 Let $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ be subset of V^n, $B = \{b_1, \ldots, b_m\}$, and $f : A \to B$ a mapping such that $f(\boldsymbol{a}_i) = b_i$ for all $i = 1, \ldots, m$, respectively. There is a Kleenean function F such that $f(\boldsymbol{a}) = F(\boldsymbol{a})$ for all $\boldsymbol{a} \in A$ if and only if $F(\boldsymbol{c}) \in I_A(\boldsymbol{c})$ for all $\boldsymbol{c} \in C_f(A)$.

Proof.

$$\begin{aligned} & \forall \boldsymbol{c} \in C_f(A)\ F(\boldsymbol{c}) \in I_A(\boldsymbol{c}) \\ \Leftrightarrow\ & \forall \boldsymbol{a} \in A\ \forall \boldsymbol{c} \in C_f(\boldsymbol{a})\ F(\boldsymbol{c}) \in I_a(\boldsymbol{c}) \\ \Leftrightarrow\ & \forall \boldsymbol{a} \in A\ F(\boldsymbol{a}) = f(\boldsymbol{a}) \end{aligned}$$

□

Example 5.5 Let $\boldsymbol{a} = (0.2, 0.3)$ and $f(\boldsymbol{a}) = 0.3$. All of the following Kleenean functions satisfy the condition (C2), that is, $f_i(0,0) \in I_a(0,0) = [0, 0.3]$ and $f_i(0, 0.5) \in I_a(0, 0.5) = [0.3, 0.5]$ whenever $f_i(0.2, 0.3) = 0.3$.

$$\begin{aligned} f_1 &= 0.3{\sim}xy{\sim}y \\ f_2 &= y{\sim}y \\ f_3 &= 0.3{\sim}x{\sim}y \\ f_4 &= y \\ f_5 &= {\sim}xy{\sim}y \\ f_6 &= 0.3{\sim}x \end{aligned}$$

Here we show that condition (C1) implies (C3) by the following lemma.

Lemma 5.2 Let A and B be subsets of V^n and V, f_k be a Kleenean function $f_k : A \to B$, and I_A be the possibility of f, respectively. For all $\boldsymbol{c} \in C_f(A)$, we have

$$I_A(\boldsymbol{c}) = \bigcap_{a \in C_f^{-1}(a)} I_a(\boldsymbol{c}) \neq \emptyset.$$

Proof. Suppose that $I_A(\boldsymbol{c}) = \emptyset$ for $\boldsymbol{c} \in C_f(A)$. There must be $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$ in A such that $\boldsymbol{c} \in C_f(a_i) \cap C_f(a_j) = \emptyset$.

(i) If $f_k(a_i) \not\succeq f_k(a_j)$ then $\boldsymbol{a}_i^* \succeq \boldsymbol{c}$ and $\boldsymbol{a}_j^* \succeq \boldsymbol{c}$ and hence $f_k(\boldsymbol{a}_i^*) \succeq f_k(\boldsymbol{c})$ and $f_k(\boldsymbol{a}_j^*) \succeq f_k(\boldsymbol{c})$. But no value $f_k(\boldsymbol{c})$ can satisfy both.

(ii) Without loss of generality, we let $f_k(\boldsymbol{a}_i) \succ f_k(\boldsymbol{a}_j)$, $I_{a_i}(\boldsymbol{c}) = \{x \in V | x \succeq f(a_i)\}$ and $I_{a_j}(\boldsymbol{c}) = \{x \in V | f(a_j) \succeq x\}$. Hence, $\boldsymbol{c} \succeq \boldsymbol{a}_{i*}$ and $\boldsymbol{a}_j^* \succeq \boldsymbol{c}$, and thus $\boldsymbol{a}_j^* \succeq \boldsymbol{a}_{i*}$. However, from the monotonicity and Corollary 5.2, we have $f(\boldsymbol{a}_j) \succeq f(\boldsymbol{a}_j^*) \succeq f(\boldsymbol{a}_{i*}) \succeq f(\boldsymbol{a}_i)$, which conflicts the hypothesis. □

Here shows condition (C3) implies (C2). Since $F(\boldsymbol{c}) \in I_A(\boldsymbol{c})$ implicitly shows $I_A(\boldsymbol{c}) \neq \emptyset$, the converse, i.e., (C2) implies (C3), is trivially true.

Theorem 5.6 Let A and B be subsets of V^n and V, and f be a mapping $f : A \to B$. If

$$I_A(\boldsymbol{c}) \neq \emptyset \quad \text{for all } \boldsymbol{c} \in C_f(A)$$

then there is at least one Kleenean function F such that

$$F(\boldsymbol{c}) \in I_A(\boldsymbol{c}) \quad \text{for all } \boldsymbol{c} \in C_f(A).$$

Proof. Let F be a logic formula defined by

$$F = \bigvee_{a \in A} f(\boldsymbol{a})\gamma_a,$$

where

$$\gamma_a = \begin{cases} \alpha_{a^*} & \text{if } f(\boldsymbol{a}) > 0.5, \\ \beta_{a_*} & \text{if } f(\boldsymbol{a}) \leq 0.5, \end{cases}$$

where α_{a^*} is a simple phrase corresponding to $\boldsymbol{a}^*$, and β_{a_*} is a complemental phrase corresponding to $\boldsymbol{a}_*$. Notice that β_{a_*} could be a simple phrase when $\boldsymbol{a}_* \in V_2^n$. Since F is a logic formula containing constants of V, it defines a Kleenean functions, and so the monotonicity must hold here.

We shall prove $F(\boldsymbol{c}) \in I_A(\boldsymbol{c})$ under an assumption that $I_A(\boldsymbol{c}) \neq \emptyset$ for all $\boldsymbol{c} \in C_f(A)$. According to the definition of $C_f(A)$ (Definition 5.7), we prove by three cases: (1) $\boldsymbol{c} = \boldsymbol{a}_* = \boldsymbol{a}^*$; (2) $\boldsymbol{c} = \boldsymbol{a}_{i*}$; (3) $\boldsymbol{a}_i^* \succeq \boldsymbol{c}$, where $\boldsymbol{a}_i^*$ and $\boldsymbol{a}_{i*}$ are elments of $C_f(A)$ such that

$$C_f(\boldsymbol{a}) = \{\boldsymbol{c} \in V_3^n | \boldsymbol{a}^* \succeq \boldsymbol{c}\} \cup \{\boldsymbol{a}_*\}.$$

(i). If $\boldsymbol{c} = \boldsymbol{a}_i^* = \boldsymbol{a}_{i*}$ then $I_A(\boldsymbol{c}) = \{f(a_i)\}$. There are two cases: (a) $f(\boldsymbol{a}_i) > 0.5$; (b) $f(\boldsymbol{a}_i) \leq 0.5$.

(a) If $f(\boldsymbol{a}_i) > 0.5$ then there exists a simple phrase γ_{a_i} in F such that $\gamma_{a_i} = \alpha_{a_i^*}(\boldsymbol{c}) = \alpha_{a_i^*}(\boldsymbol{a}_i^*) = 1$ from Proposition 5.1. Hence

$$F(\boldsymbol{c}) \geq f(\boldsymbol{c})\alpha_{a_i^*}(\boldsymbol{c}) = f(\boldsymbol{c}) = f(\boldsymbol{a}_i).$$

Then, we show there is no phrase in F such that $f(\boldsymbol{c}')\alpha_{c'}(\boldsymbol{c}) > f(\boldsymbol{c}) > 0.5$. Otherwise $\alpha_{c'}(\boldsymbol{c})$ must be greater than $\alpha_c(\boldsymbol{c}) = 1$, thereby we have $\boldsymbol{c}' \succeq \boldsymbol{c}$ and $I_{a'}(\boldsymbol{c}) = \{x | f(\boldsymbol{c}') \succeq x\}$ from Proposition 5.2. While, since $f(\boldsymbol{c}') > f(\boldsymbol{c})$, we have

$$\begin{aligned} I_A(\boldsymbol{c}) &= I_{a_i}(\boldsymbol{c}) \cap I_{a_j}(\boldsymbol{c}) \\ &= \{f(\boldsymbol{a}_i)\} \cap [f(a_j), 1] = \emptyset, \end{aligned}$$

which contradicts the hypothesis. Therefore, $F(\boldsymbol{c}) \in I_A(\boldsymbol{c})$.

(b) If $f(\boldsymbol{a}_i) \leq 0.5$ then there is γ_{a_i} in F such that $\gamma_{a_i} = \beta_{a_i^*}(\boldsymbol{c}) = \beta_{a_i^*}(\boldsymbol{a}_i^*) = 0.5$. We thus have

$$F(\boldsymbol{c}) \leq f(\boldsymbol{a}_i)\beta_c(\boldsymbol{c}) = f(\boldsymbol{a}_i).$$

We shall show that there is no phrase in F such that $f(\boldsymbol{a}_j)\gamma_{a_j^*}(\boldsymbol{c}) > f(\boldsymbol{c})$ by possible three cases: (i) $f(\boldsymbol{a}_j)\alpha_{a_j^*}(\boldsymbol{c}) \geq 0.5 > f(\boldsymbol{c})$; (ii) $f(\boldsymbol{a}_j)\beta_{a_{j*}}(\boldsymbol{c}) > f(\boldsymbol{c})$ and $\boldsymbol{a}_{j*} \in V_3^n - V_2^n$; (iii) $f(\boldsymbol{a}_j)\beta_{a_{j*}}(\boldsymbol{c}) > f(\boldsymbol{c})$ and $\boldsymbol{a}_{j*} \in V_2^n$.

i. Suppose there is a simple phrase in F such that $f(\boldsymbol{a}_j)\alpha_{a_j^*}(\boldsymbol{c}) \geq 0.5 > f(\boldsymbol{c})$. Since $\alpha_{a_j^*}(\boldsymbol{c}) \geq 0.5$, there must exist $\boldsymbol{c}' \in V_3^n$ such that

$$\boldsymbol{a}_i^* \succeq \boldsymbol{c}', \quad \boldsymbol{a}_j^* \succeq \boldsymbol{c}'.$$

by Proposition 5.3. By Definition 5.7, $\boldsymbol{c}'$ belongs to $C_f(A)$. For $\boldsymbol{c}'$,

$$I_{a_i}(\boldsymbol{c}') = [0, f(\boldsymbol{a}_i)], \quad I_{a_j}(\boldsymbol{c}') = [f(\boldsymbol{a}_j), 1],$$

and thus $I_A(\boldsymbol{c}') = \emptyset$, which conflicts the hypothesis.

ii. Suppose there is a complementary phrase in F such that $f(\boldsymbol{a}_j)\beta_{a_{j*}}(\boldsymbol{c}) > f(\boldsymbol{a}_i)$. Since $\beta_{a_j}(\boldsymbol{c}) = \beta_{a_j}(\boldsymbol{a}_{j*}) = 0.5$, we have $\boldsymbol{a}_{i*} \succeq \boldsymbol{a}_{j*}$ from Proposition 5.3. Hence, $I_{a_j}(\boldsymbol{c}) = [f(\boldsymbol{a}_j), 0.5]$. We thus have the contradiction that

$$\begin{aligned} I_A(\boldsymbol{c}) &= I_{a_i}(\boldsymbol{c}) \cap I_{a_j}(\boldsymbol{c}) \\ &= \{f(\boldsymbol{a}_i)\} \cap [f(\boldsymbol{a}_j), 0.5] \\ &= \emptyset \end{aligned}$$

because $f(\boldsymbol{a}_i) < f(\boldsymbol{a}_j)$.

iii. Suppose that $\boldsymbol{a}_{j*} \in V_2^n$ and F contains a simple phrase such that $0.5 \leq f(\boldsymbol{a}_j)\beta_{a_{j*}}(\boldsymbol{c})$ and $f(\boldsymbol{a}_i) < \beta_{a_{j*}}(\boldsymbol{c})$. (Note that β_{a_j} is not a complementary phrase when $\boldsymbol{a}_j \in V_2^n$.) Then, $\boldsymbol{a}_{j*} \in V_2$ immediately follows that $\boldsymbol{a}_j^* = \boldsymbol{a}_{j*} \in V_2$. Since $\beta_{a_j}(\boldsymbol{c}) \geq 0.5$, we have $\boldsymbol{c}' \in C_f(A)$ such that

$$\boldsymbol{c} \succeq \boldsymbol{c}', \quad \boldsymbol{a}_{j*} \succeq \boldsymbol{c}'$$

by Proposition 5.3. Since $\boldsymbol{a}_{j*}$ is an element of V_2^n, $\boldsymbol{c}' = \boldsymbol{a}_{j*}$, and thus $\boldsymbol{c} \succeq \boldsymbol{a}_{j*}$. For $\boldsymbol{c}'$, we have

$$\begin{aligned} I_A(\boldsymbol{c}') &= I_{a_i}(\boldsymbol{c}') \cap I_{a_j}(\boldsymbol{c}') \\ &= \{f(\boldsymbol{a}_i)\} \cap [0, f(\boldsymbol{a}_j)], \end{aligned}$$

which leads to contradiction that $I_A(\boldsymbol{c}') = \emptyset$ because $f(\boldsymbol{a}_j) > f(\boldsymbol{a}_i)$.

As we have shown above, the $F(\boldsymbol{c}) \in I_A(\boldsymbol{c})$ holds for all $\boldsymbol{c} \in C_f(A)$ when $\boldsymbol{c} = \boldsymbol{a}_{i*} = \boldsymbol{a}_i^*$.

(ii). If $\boldsymbol{c} \succeq \boldsymbol{a}_{i*}$ then the possibility can be either $I_{a_i}(\boldsymbol{c}) = [f(\boldsymbol{a}_i), 0.5]$ or $I_{a_i}(\boldsymbol{c}) = [0.5, f(\boldsymbol{a}_i)]$. We shall prove $f(\boldsymbol{c}) = F(\boldsymbol{c})$ by two cases: (a) $f(\boldsymbol{a}_i) > 0.5$; (b) $f(\boldsymbol{a}_i) \leq 0.5$.

(a) Suppose $f(\boldsymbol{a}_i) > 0.5$. There is a simple phrase in F such that $\alpha_{a_i^*}(\boldsymbol{c}) \geq 0.5$, and hence

$$0.5 \leq f(\boldsymbol{a}_i)\alpha_{a_i^*}(\boldsymbol{c}) \leq F(\boldsymbol{c}),$$

while there is no simple phrase such that $f(\boldsymbol{a}_j)\alpha_{a_j^*}(\boldsymbol{c}) > f(\boldsymbol{a}_i)$. Otherwise, from Proposition 5.2, we have $\boldsymbol{a}_j^* \succeq \boldsymbol{c}$, which leads to contradiction as follows:

$$\begin{aligned} I_A(\boldsymbol{c}) &= I_{a_i}(\boldsymbol{c}) \cap I_{a_j} \\ &= [0.5, f(\boldsymbol{a}_i)] \cap [f(\boldsymbol{a}_j), 1] \\ &= \emptyset. \end{aligned}$$

Therefore $F(\boldsymbol{c}) \in I_A(\boldsymbol{c})$ holds whenever $f(\boldsymbol{a}_i) > 0.5$.

(b) Suppose $f(\boldsymbol{a}_i) \leq 0.5$. There is a complementary phrase in F such that $\beta_{a_{i*}}(\boldsymbol{c}) = 0.5$, which satisfies

$$f(\boldsymbol{a}_i) = f(\boldsymbol{a}_i)\beta_{a_{i*}}(\boldsymbol{c}) \leq F(\boldsymbol{c}).$$

To show $F(\boldsymbol{c}) \in I_A(\boldsymbol{c}) = [f(\boldsymbol{a}_i), 0.5]$, we shall prove that there is no other phrase in F such that $f(\boldsymbol{a}_j)\alpha_{a_j^*}(\boldsymbol{c}) > 0.5$. Note that we do not have to show nonexistence of any complementary phrase because such phrase is always less than or equal to 0.5.

For such simple phrase, Proposition 5.3 gives $\boldsymbol{c}' \in C_f(A)$ such that $\boldsymbol{c} = \boldsymbol{a}_{i*} \succeq \boldsymbol{c}'$ and $\boldsymbol{a}_j^* \succeq \boldsymbol{c}'$. Hence, for $\boldsymbol{c}'$, we have contradiction that

$$\begin{aligned} I_A(\boldsymbol{c}') &= I_{a_i}(\boldsymbol{c}') \cap I_{a_j}(\boldsymbol{c}') \\ &= [f(\boldsymbol{a}_i), 0.5] \cap [f(\boldsymbol{a}_j), 1] \\ &= \emptyset \end{aligned}$$

Therefore, we have $F(\boldsymbol{c}) \in I_A(\boldsymbol{c})$ for all $\boldsymbol{c}$ such that $\boldsymbol{c} = \boldsymbol{a}_{i*}$.

(iii). If $\boldsymbol{a}_i^* \succeq \boldsymbol{c}$ then the possibility $I_{a_i}(\boldsymbol{c})$ can be either $[f(\boldsymbol{a}_i), 1]$ or $[0, f(\boldsymbol{a}_i)]$. We prove by two cases: (a) $f(\boldsymbol{a}_i) > 0.5$: (b) $f(\boldsymbol{a}_i) \leq 0.5$.

(a) Suppose $f(\boldsymbol{c}) > 0.5$. Then, there is a simple phrase in F such that $\alpha_{a_i^*}(\boldsymbol{c}) = 1$. Hence, we have

$$f(\boldsymbol{a}_i) = f(\boldsymbol{a}_i)\alpha_{a_i^*}(\boldsymbol{c}) \leq F(\boldsymbol{c}),$$

which shows $F(\boldsymbol{c}) \in I_A(\boldsymbol{c}) = [f(\boldsymbol{a}_i), 1]$.

(b) Suppose $f(\boldsymbol{c}) \leq 0.5$. Then, there is no phrase that is greater than $f(\boldsymbol{a}_i)$. We show this by two cases: (i) $0.5 < f(\boldsymbol{a}_j)$; (ii) $f(\boldsymbol{a}_i) < f(\boldsymbol{a}_j) \leq 0.5$.

i. Suppose there is a simple phrase in F such that $f(\boldsymbol{a}_i) < f(\boldsymbol{a}_j)\alpha_{a_j^*}(\boldsymbol{c})$. Then, by Proposition 5.3, there is an element $\boldsymbol{c}'$ of $C_f(A)$ satisfing $\boldsymbol{a}_i^* \succeq \boldsymbol{c}'$ and $\boldsymbol{a}_j^* \succeq \boldsymbol{c}'$. For $\boldsymbol{c}'$, we have

$$\begin{aligned} I_A(\boldsymbol{c}') &= I_{a_i}(\boldsymbol{c}') \cap I_{a_j}(\boldsymbol{c}') \\ &= [0, f(\boldsymbol{a}_i)] \cap [f(\boldsymbol{a}_j), 1] \\ &= \emptyset, \end{aligned}$$

which conflicts with the hypothesis.

ii. Suppose there is $f(\boldsymbol{a}_j)\beta_{a_j*}(\boldsymbol{c}) > f(\boldsymbol{a}_i)$. If $\boldsymbol{a}_j^* \neq \boldsymbol{a}_{j*}$ then $\beta_{a_{j*}}(\boldsymbol{c}) = 0.5$. By Proposition 5.4, we have $\boldsymbol{c} \succeq \boldsymbol{a}_{j*}$, which leads to

$$\begin{aligned} I_A(\boldsymbol{c}') &= I_{a_i}(\boldsymbol{c}) \cap I_{a_j}(\boldsymbol{c}) \\ &= [0, f(\boldsymbol{a}_i)] \cap [f(\boldsymbol{a}_j), 0.5] \\ &= \emptyset, \end{aligned}$$

since now $f(\boldsymbol{a}_i) < f(\boldsymbol{a}_j)$.
Even if $\boldsymbol{a}_j^* = \boldsymbol{a}_{j*}$, i.e., $\boldsymbol{a}_j^* \in V_2^n$ and $\beta_{a_{j*}}$ becomes a simple phrase, $\boldsymbol{c}'$ involves

$$\begin{aligned} I_A(\boldsymbol{c}') &= [0, f(\boldsymbol{a}_i)] \cap \{f(\boldsymbol{a}_j)\} \\ &= \emptyset. \end{aligned}$$

As we have shown above, the theorem holds for all cases, and we obtain the theorem.

□

Here shows the necessary and sufficient conditions for an existence of Kleenean function that satisfies $F(\boldsymbol{a}) = f(\boldsymbol{a})$ for all $\boldsymbol{a}$ in A.

Theorem 5.7 (K-representability) Let $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) \in V^n$, and f be a mapping $f : A \to V$, respectively.

$$I_A(\boldsymbol{c}) \neq \emptyset \quad \text{for all } \boldsymbol{c} \in C_f(\boldsymbol{a})$$

if and only if there is at least one Kleenean function F such that

$$F(\boldsymbol{a}) = f(\boldsymbol{a}) \quad \text{for all } \boldsymbol{a} \in A.$$

Proof. As we have shown, by the theorems and lemmas, now the followings are equivalent:

$$\begin{aligned} & \forall \boldsymbol{c} \in C_f(A) \quad I_A(\boldsymbol{c}) \neq \emptyset \\ \Leftrightarrow\ & \forall \boldsymbol{c} \in C_f(A) \quad F(\boldsymbol{c}) \in I_A(\boldsymbol{c}) \\ \Leftrightarrow\ & \forall \boldsymbol{a} \in A \quad F(\boldsymbol{a}) = f(\boldsymbol{a}). \end{aligned}$$

□

The following propositions are not used here, but also important properties with respect to K-representablity.

Proposition 5.6 Let $f : \boldsymbol{a} \to B$. For any $\boldsymbol{a}$, $\boldsymbol{b}$ of $C_f(A)$, if $\boldsymbol{a} \succeq \boldsymbol{b}$ then

$$\text{lub}(I_A(\boldsymbol{a})) \succeq \text{lub}(I_A(\boldsymbol{b}))$$

where $\text{lub}(I_A(\boldsymbol{a}))$ is a least upper bound of $I_A(\boldsymbol{a})$.

Proof. The proof is omitted. □

Proposition 5.7 Let $f : A \to B$, $\boldsymbol{a}$, $\boldsymbol{b}$ elements of $C_f(A)$. If $\text{glb}(\{\boldsymbol{a}, \boldsymbol{b}\})$ is not empty set then

$$\text{glb}(\{\text{lub}(I_A(\boldsymbol{a})), \text{lub}(I_A(\boldsymbol{b}))\}) \neq \emptyset,$$

where glb is a greatest lower bound.

Proof. The proof is omitted. □

5.3.4 Example

We consider a subset of V^2

$$A = \{(0.1, 0.6), (0.4, 0.5), (0.1, 0.2), (0.9, 0.2), (0.7, 0.9)\}$$

and a mapping $f : A \to V$ defined by:

$$\begin{aligned} f(\boldsymbol{a}_1) &= f(0.1, 0.6) = 0.3, \\ f(\boldsymbol{a}_2) &= f(0.4, 0.5) = 0.4, \\ f(\boldsymbol{a}_3) &= f(0.1, 0.2) = 0.2, \\ f(\boldsymbol{a}_4) &= f(0.9, 0.2) = 0.8, \\ f(\boldsymbol{a}_5) &= f(0.7, 0.9) = 0.6. \end{aligned}$$

The goal of the identification problem is to find a logic formula F which satisfies the given f for all elements of A.

First, by quantizing each $\boldsymbol{a}_i$ by $f(\boldsymbol{a}_i)$, we have

$$\begin{aligned} \boldsymbol{a}_1^* &= \boldsymbol{a}_{1*} = (0, 0.5), \\ \boldsymbol{a}_2^* &= (0, 0.5), \\ \boldsymbol{a}_{2*} &= (0.5, 0.5), \\ \boldsymbol{a}_3^* &= (0, 0), \\ \boldsymbol{a}_{3*} &= (0, 0.5), \\ \boldsymbol{a}_4^* &= (1, 0), \\ \boldsymbol{a}_{4*} &= (1, 0.5), \\ \boldsymbol{a}_5^* &= \boldsymbol{a}_{5*} = (1, 1), \end{aligned}$$

that give subsets of V_3^n for $\boldsymbol{a}_1 \in A$,

$$\begin{aligned} C_f(\boldsymbol{a}_1) &= \{\boldsymbol{c} \in V_3^n | \boldsymbol{a}_1^* \succeq \boldsymbol{c}\} \cup \{\boldsymbol{a}_{1*}\} \\ &= \{(0, 0), (0, 1), (0, 0.5)\} \cup \{(0, 0.5)\} \end{aligned}$$

similarly,

$$\begin{aligned} C_f(\boldsymbol{a}_2) &= \{(0, 0), (0, 1), (0, 0.5), (0.5, 0.5)\}, \\ C_f(\boldsymbol{a}_3) &= \{(0, 0), (0, 0.5)\}, \\ C_f(\boldsymbol{a}_4) &= \{(1, 0), (1, 0.5)\}, \\ C_f(\boldsymbol{a}_5) &= \{(1, 1)\}. \end{aligned}$$

Next, we have every possibility for each $\boldsymbol{a}_i$ as follows:

$$I_{a_1}(0, 0) = \{x \in V | f(\boldsymbol{a}_1) \succeq x\} = [0, 0.3]$$

$$\begin{aligned}
&= I_{a_1}(0,1),\\
I_{a_1}(0,0.5) &= \{0.3\},\\
I_{a_2}(0,0) &= [0,0.4]\\
&= I_{a_2}(0,1) = I_{a_2}(0,0.5),\\
I_{a_2}(0.5,0.5) &= \{x \in V | x \succeq f(\boldsymbol{a}_2)\} = [0.4,0.5],\\
I_{a_3}(0,0) &= [0,0.3],\\
I_{a_3}(0,0.5) &= [0.3,0.5],\\
I_{a_4}(1,0) &= [0.8,1],\\
I_{a_4}(1,0.5) &= [0.5,0.8],\\
I_{a_5}(1,1) &= \{0.6\}.
\end{aligned}$$

For $\boldsymbol{c} = (0,0.5) \in C_f(A)$, we have

$$\begin{aligned}
I_A(0,0.5) &= \bigcap_{a_i \in A,(0,0.5)=C_f(a_i)} I_{a_i}(0,0.5)\\
&= I_{a_1}(0,0.5) \cap I_{a_2}(0,0.5) \cap I_{a_3}(0,0.5)\\
&= \{0.3\} \cap [0,0.4] \cap [0.2,0.5]\\
&= \{0.3\}.
\end{aligned}$$

As the same way, we obtain the other possibilities and show them on truth table of V_3^2 in Table 5.1.

Table 5.1: Possibilities on V_3^2

$x_2 \backslash x_1$	0	0.5	1
0	$[0,0.2]$	·	$[0.8,1]$
0.5	$\{0.3\}$	$[0.2,0.5]$	$[0.5,0.8]$
1	$[0,0.3]$	·	$\{0.6\}$

Looking at the table, we show there is no possibility of $\emptyset$, thereby, from Theorem 5.7, there must be at least one Kleenean function F.

Finally, we obtain the logic formula F as defined in the proof of Theorem 5.6:

$$\begin{aligned}
F &= \bigvee_{a_i \in A} f(\boldsymbol{a}_i)\gamma_{a_i}\\
&= f(\boldsymbol{a}_1)\beta_{(0,0.5)} \vee f(\boldsymbol{a}_2)\beta_{(0.5,0.5)} \vee f(\boldsymbol{a}_3)\beta_{(0,0.5)}\\
&\quad \vee f(\boldsymbol{a}_4)\alpha_{(1,0.5)} \vee f(\boldsymbol{a}_5)\alpha_{(1,1)}\\
&= 0.3{\sim}x\ y{\sim}y \vee 0.4x{\sim}xy{\sim}y \vee 0.2{\sim}xy{\sim}y\\
&\quad \vee 0.8x{\sim}y \vee 0.6xy\\
&= 0.3{\sim}xy{\sim}y \vee 0.8x{\sim}y \vee 0.6xy,
\end{aligned}$$

which satisfies

$$F(\boldsymbol{a}) = f(\boldsymbol{a}) \text{ for all } \boldsymbol{a} \in A.$$

We show the whole mapping F on Figure 5.4.

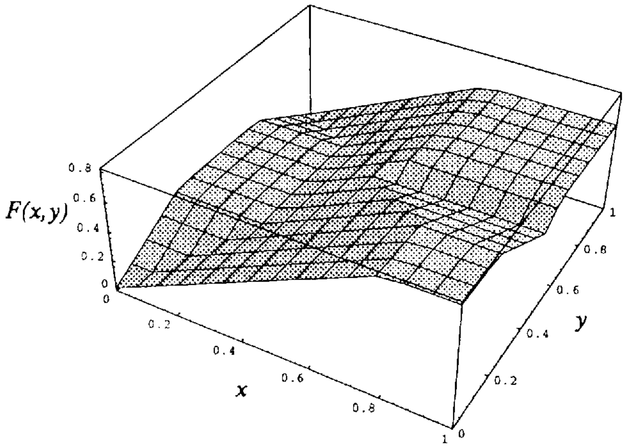

Figure 5.4: Kleenean function F

5.4 Conclusion

We have studied the partially specified Kleenean functions, and have solved the identification problem. The main result is that the necessary and sufficient condition for existence of Kleenean function is non-empty possibilities defined by a given f.

Chapter 6

Fitting Fuzzy Switching Function

In this chapter, we propose an algorithm which takes a piece of uncertain knowledge which is a mapping with restricted domain and outputs the logic formula with the shortest distance to the given mapping. The execution is done in three steps; first, the given mapping is divided into some *Q-equivalent classes*; second, the distances between the mapping and each local fuzzy switching function are calculated by a simplified logic formula; and last, the shortest distance is obtained by a modified graph-theoretic algorithm. After showing example, the total time to execute the algorithm is also estimated.

6.1 Introduction

We have studied some uncertain reasoning methods based on some properties of fuzzy switching functions, P-fuzzy switching function, and Kleenean functions. These models can treat any given response of a human expert with *representability* and *uniqueness*. In this inference scheme, we suppose that uncertain knowledge is a mapping, say f_0, and find the fuzzy switching function f that satisfies the exact value of f_0.

However, these models are too restrictive for human knowledge. Thus, some noise and incompleteness involved by human response could spoil the consistency of fuzzy switching function. Here, we suppose a human expert's response based on a logic formula but with some noise, and then attempt to fit fuzzy switching function to the underlying knowledge. Let us formalize the fitting problem.

Fitting Problem Find a 2-variable fuzzy switching function f^* with the shortest distance to the following mapping f_0:

$$
\begin{aligned}
f_0(0.8, 0.7) &= 0.9, \\
f_0(0.3, 0.1) &= 0.6, \\
f_0(0.4, 0.8) &= 0.3, \\
f_0(0.7, 0.6) &= 0.8, \\
f_0(0.6, 0.9) &= 0.6.
\end{aligned}
$$

Where a mapping f_0 is defined on $A = \{(0.8,0.7),(0.3,0.1),(0.4,0.8),(0.7,0.6),(0.6,0.9)\}$. We should also define a *distance* between two functions. Let f and g be mappings $f : A \to V$ and $g : B \to V$. The *distance* $D(f,g)$ between f and g is defined by

$$D(f,g) = \sum_{a \in A \cap B} |f(\boldsymbol{a}) - g(\boldsymbol{a})|,$$

where $|f(\boldsymbol{a}) - g(\boldsymbol{a})|$ is the absolute value of $f(\boldsymbol{a}) - g(\boldsymbol{a})$.

The first thing we can do with this problem is to examine each fuzzy switching function one by one, which is called the "brute force algorithm." This works well with a low number of variables up to 3, because there are exact 43,918 fuzzy switching functions with 3 variables. According to [15], the number of n-variable fuzzy switching functions increases in time $O(2^{3^n})$.

In this chapter, we will divide a problem into some meaningful small problems, what we call *Q-equivalent classes*, and estimate each distance to the given f_0, and then combine them so as to represent a fuzzy switching function. This will allow us to find the fuzzy switching function f^* with the shortest distance for all possible combinations of the classes. We use a graph-theoretic algorithm to find f^*, which is a variation of Dijkstra's shortest path algorithm.

6.2 Dividing into Q-equivalent Classes

6.2.1 Quantized Sets and Q-equivalent Classes

Definition 6.1 Let $\boldsymbol{a} = (a_1, \dots, a_n)$ in V^n. A quantized set of $\boldsymbol{a}$ is a subset of V_3^n defined by

$$C(\boldsymbol{a}) = \{\overline{\boldsymbol{a}}^\lambda \in V_3^n \mid \lambda \in V\}.$$

For a subset $A = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$ of V^n, we often write $C(A)$ to mean that

$$C(A) = C(\boldsymbol{a}_1) \cup \cdots \cup C(\boldsymbol{a}_m).$$

Example 6.1

$$\begin{aligned} C((0.4,0.8)) &= \{\overline{(0.4,0.8)}^{0.4}, \overline{(0.4,0.8)}^{0.8}, \overline{(0.4,0.8)}^{1}\} \\ &= \{(0,1),(0.5,1),(0.5,0.5)\} \end{aligned}$$

Definition 6.2 Elements $\boldsymbol{a}$ and $\boldsymbol{b}$ of V^n are *Q-equivalent* if and only if the quantized sets satisfy $C(\boldsymbol{a}) = C(\boldsymbol{b})$. We write $\boldsymbol{a} \approx \boldsymbol{b}$ in this case.

Example 6.2

$$(0.8,0.7) \approx (0.7,0.6)$$

$$(0.3,0.8) \approx (0.2,0.9) \approx (0.4,0.9)$$

Definition 6.3 Let A be a subset of V^n and $\boldsymbol{a} \in A$. A *Q-equivalent class* containing $\boldsymbol{a}$ is a subset of A defined by

$$[\boldsymbol{a}] = \{\boldsymbol{b} \in A \mid \boldsymbol{a} \approx \boldsymbol{b}\}.$$

The set of all equivalence classes of A is denoted by $[A]$, i.e., $[A] = \{[\boldsymbol{a}] \subset A | \ \boldsymbol{a} \in A\}$.[1]

A partition of a nonempty set S is a collection of nonempty subsets which are disjoint and whose union is S. Since $\approx$ is an equivalence relation, we have a partition $A/\approx = [A] = A_1, \ldots, A_m$, that is, A_i and A_j are disjoint and the union is A:

$$A_i \cap A_j = \emptyset \quad \text{for every } i \neq j,$$
$$A = A_1 \cup \cdots \cup A_m.$$

Example 6.3 Recall A in the introduction. We have the partition

$$\begin{aligned} A &= \{(0.4,0.8),(0.6,0.9),(0.7,0.6),(0.8,0.7),(0.3,0.1)\} \\ &= \{(0.4,0.8)\} \cup \{(0.6,0.9)\} \cup \{(0.7,0.6),(0.8,0.7)\} \cup \{(0.3,0.1)\} \\ &= A_1 \cup A_2 \cup A_3 \cup A_4. \end{aligned}$$

Figure 6.1 shows how we divide A into Q-equivalence classes. The three small circles indicate all elements of quantized set of $C(0.4,0.8)$ that characterizes Q-equivalent class A_1.

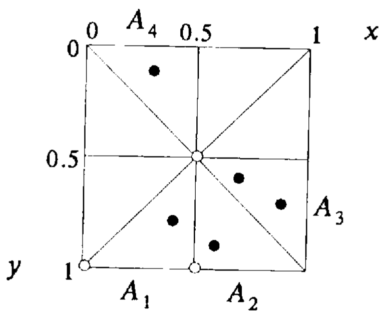

Figure 6.1: Partition $[A]$

6.2.2 Some Properties of Q-equivalent Classes

Proposition 6.1 Let $C(\boldsymbol{a}) = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m\}$ be a Q-equivalent class in V_3^n. Then

$$\boldsymbol{a}_i \succeq \boldsymbol{a}_j \text{ or } \boldsymbol{a}_j \succeq \boldsymbol{a}_i \quad \text{for every } i,j \in \{1,\ldots,m\}.$$

[1] In general, a domain of fuzzy switching functions can be divided into a few subsets in which variables are in order[18]. This is called *cell space* and corresponds to a Q-equivalent class. It should be noted that their boundary conditions are slightly different. For example, both $(0.5,0.2)$ and $(0.6,0.2)$ are in the same cell space, while $C(0.5,0.2) = \{(0.5,0),(0.5,0.5)\} \neq C(0.6,0.2) = \{(1,0),(0.5,0),(0.5,0.5)\}$.

This means that there are subscripts $i_1 < i_2 < \cdots < i_m$ so that

$$\boldsymbol{a}_{i_1} \succeq \boldsymbol{a}_{i_2} \succeq \cdots \succeq \boldsymbol{a}_{i_m},$$

which implies that

$$\alpha^1 \geq \alpha^2 \geq \cdots \geq \alpha^m$$

where α^i corresponds to a quantized set $\boldsymbol{a}_i$.

Example 6.4 For quantized sets $C(\boldsymbol{a}) = \{(0,1),(0.5,1),(0.5,0.5)\}$,

$$\alpha^{(0.5,0.5)} = 1 \geq \alpha^{(0.5,1)} = y \geq \alpha^{(0,1)} = \overline{x}y$$

Proposition 6.2 The number of elements of a quantized set by $\boldsymbol{a} \in V^n$ is at most $n+1$.

Proof. Proof is omitted. □

Proposition 6.3 The number of quantized sets that have $n+1$ elements is $2^n n!$, and of quantized sets that have one element is 3^n.

Proof. Proof is omitted. □

Proposition 6.4 Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in V^n$ such that $a_i \neq a_j$, $a_i \neq 1 - a_j$ for all $i \neq j$ in $1, \ldots, n$. Then, the number of quantized sets that have $n+1-i$ elements is

$$\begin{pmatrix} n \\ i \end{pmatrix} 3^i 2^{n-i}(n-i)!$$

Proof. Proof is omitted. □

The total number of 1-variable quantized sets is 33 and that of 2-variable quantized sets is 33. However, general number of n-variable quantized sets is unknown. Note that a number of quantized sets of A is always less than the number of elements of A.

Proposition 6.5 Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V^n. Then $\boldsymbol{a} \approx \boldsymbol{b}$ if and only if $\overline{a_i}^{a_j} = \overline{b_i}^{b_j}$ for every i, j in $\{1, \ldots, n\}$.

Proof. Suppose $\overline{a_i}^{a_j} \succeq \overline{b_i}^{b_j}$ for some i, j when $\boldsymbol{a} \approx \boldsymbol{b}$. Then, $\overline{\boldsymbol{a}}^j \succeq \overline{\boldsymbol{b}}^{b_j}$. Consider $\lambda \in V$ such that $\overline{b_i}^{\lambda} = \overline{a_i}^{a_j} = 0.5$. Since $b_j \succeq \lambda$, $\overline{b_j}^{\lambda} = 0.5 \neq \overline{a_j}^{a_j} \in \{0,1\}$. Thus, there is no λ that satisfies $\overline{\boldsymbol{a}}^{a_j} = \overline{\boldsymbol{b}}^{\lambda}$, that is, $\overline{\boldsymbol{a}}^{a_j} \notin C(\boldsymbol{b})$. When $\overline{b_i}^{b_j} \succeq \overline{a_i}^{a_j}$, similarly $\overline{\boldsymbol{a}}^{a_j} \notin C(\boldsymbol{b})$. When $\overline{a_i}^{a_j} = 0$ and $\overline{b_i}^{b_j} = 1$, apparently there is no λ such as $\overline{b_i}^{\lambda} = \overline{a_i}^{\lambda}$. Therefore, in any case, $\overline{a_i}^{a_j} = \overline{b_i}^{b_j}$.

Conversely, suppose $\overline{a_i}^{a_j} = \overline{b_i}^{b_j}$ for every i, j. For any $\lambda \in \{0,1\}$, $\overline{\boldsymbol{a}}^{\lambda} = \overline{\boldsymbol{b}}^{\lambda} = (0.5, \ldots, 0.5)$. For any $\lambda \in V - \{0,1\}$, there is k such that $\overline{\boldsymbol{a}}^{\lambda} = \overline{\boldsymbol{a}}^{a_k}$ and $\overline{\boldsymbol{b}}^{\lambda} = \overline{\boldsymbol{b}}^{b_k}$. Thus, we have $\boldsymbol{a} \succeq \boldsymbol{b}$. □

Theorem 6.1 Let f and g be fuzzy switching functions, and $\boldsymbol{d} \in V^n$. Then, $f(\boldsymbol{a}) = g(\boldsymbol{a})$ for all $\boldsymbol{a} \in C(\boldsymbol{d})$ if and only if $f(\boldsymbol{b}) = g(\boldsymbol{b})$ for all $\boldsymbol{b} \in V^n$ such that $\boldsymbol{b} \approx \boldsymbol{d}$.

Proof. Suppose $f(\boldsymbol{b}) \neq g(\boldsymbol{b})$ for some $\boldsymbol{b} \in V^n$ when $f(\boldsymbol{a}) = g(\boldsymbol{a})$ for all $\boldsymbol{a} \in C(\boldsymbol{d})$. By Theorem 2.3,

$$\overline{f(\boldsymbol{b})}^{\lambda} = f(\overline{\boldsymbol{b}}^{\lambda}) \neq \overline{g(\boldsymbol{b})}^{\lambda} = g(\overline{\boldsymbol{b}}^{\lambda})$$

for some $\lambda \in V$. This is contradictory to the hypothesis.

Conversely, suppose $f(\boldsymbol{a}) \neq g(\boldsymbol{a})$ for some $\boldsymbol{a} \approx \boldsymbol{d} \in C(\boldsymbol{d})$ when

$$f(\boldsymbol{b}) = g(\boldsymbol{b}) \quad \text{for every } \boldsymbol{b} \approx \boldsymbol{d}.$$

Then, there is a $\boldsymbol{b} \in V^n$ such as $\overline{\boldsymbol{b}}^{\lambda} = \boldsymbol{a}$, and for $\boldsymbol{b}$,

$$f(\boldsymbol{a}) = f(\overline{\boldsymbol{b}}^{\lambda}) = \overline{f(\boldsymbol{b})}^{\lambda} \neq g(\boldsymbol{a}) = g(\overline{\boldsymbol{b}}^{\lambda}) = \overline{g(\boldsymbol{b})}^{\lambda}.$$

Since $\boldsymbol{b} \approx \boldsymbol{d}$, this contradicts the hypothesis. □

Example 6.5 Given $f(0,1)$, $f(0.5,1)$ and $f(0.5,0.5)$, subset of V^2, $f(0.3,0.8)$ is determined uniquely by them.

Corollary 6.1 Let f be a fuzzy switching function, and $C(\boldsymbol{x})$ be a quantized set of $\boldsymbol{x}$ in $C(\boldsymbol{x})$.

$$F(\boldsymbol{x}) = \bigvee_{a \in C(x)} \left(\bigwedge_{a \succeq b \in C(x)} F(\boldsymbol{b})\alpha^a(\boldsymbol{x}) \vee F(\boldsymbol{a})\beta^a(\boldsymbol{x}) \right)$$

For example, $(0.3, 0.8)$ can be determined from only three values $F(0,1)$, $F(0.5,1)$ and $F(0.5,0.5)$. This means that any fuzzy switching function can be divided with some of quantized sets.

6.3 Estimation of Local Distances

6.3.1 Partially Specified Fuzzy Switching Function

According to Theorem 2.5(representation theorem), any fuzzy switching function can be determined by 3^n parameters $F(\boldsymbol{a})$ for $\boldsymbol{a} \in V_3^n$. However, 3^n is too many to identify a logic formula. In this section, we will reduce the number of parameters.

Definition 6.4 Let $\boldsymbol{a} \in V_3^n$, and $b \in \{0,1\}$. We say a fuzzy switching function f is a *partially specified* or a *restriction* if and only if $f(\boldsymbol{a}) = b$ and $f(\boldsymbol{c}) = 0.5$ for all $\boldsymbol{c} \in V_3^n$ such that $\boldsymbol{c} \succ \boldsymbol{a}$. A partially specified fuzzy switching is denoted by f_a^b. We often write $R(C)$ to mean a set of all partially specified fuzzy switching functions by $C = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_i\}$, that is,

$$R(C) = \{f_a^b | \boldsymbol{a} \in C, b \in \{0,1\}\}.$$

Note that $\boldsymbol{c} \succ \boldsymbol{a}$ does not mean $\boldsymbol{c} = \boldsymbol{a}$.

Also, note that a partially specified fuzzy switching function is, generally, not unique. For example, here are two restrictions $f^1_{(0)}$:

$$f_1(x) = \overline{x}, \quad f_2(x) = x \vee \overline{x}.$$

Both of them satisfy $f(0) = 1$ and $f(0.5) = 0.5$.

Example 6.6 We illustrates six partially specified fuzzy switching functions for $C(\boldsymbol{a}) = \{(0,1),(0.5,1),(0.5,0.5)\}$ in Figure 6.2. Where a small circle for each function indicates $\boldsymbol{a}$ of f^b_a. There are two choices of b for each element of $C(\boldsymbol{a})$, which has three elements. The total of partially specified fuzzy switching functions is thus six.

Next, we will show that any fuzzy switching function can be represented by combing some partially specified fuzzy switching functions.

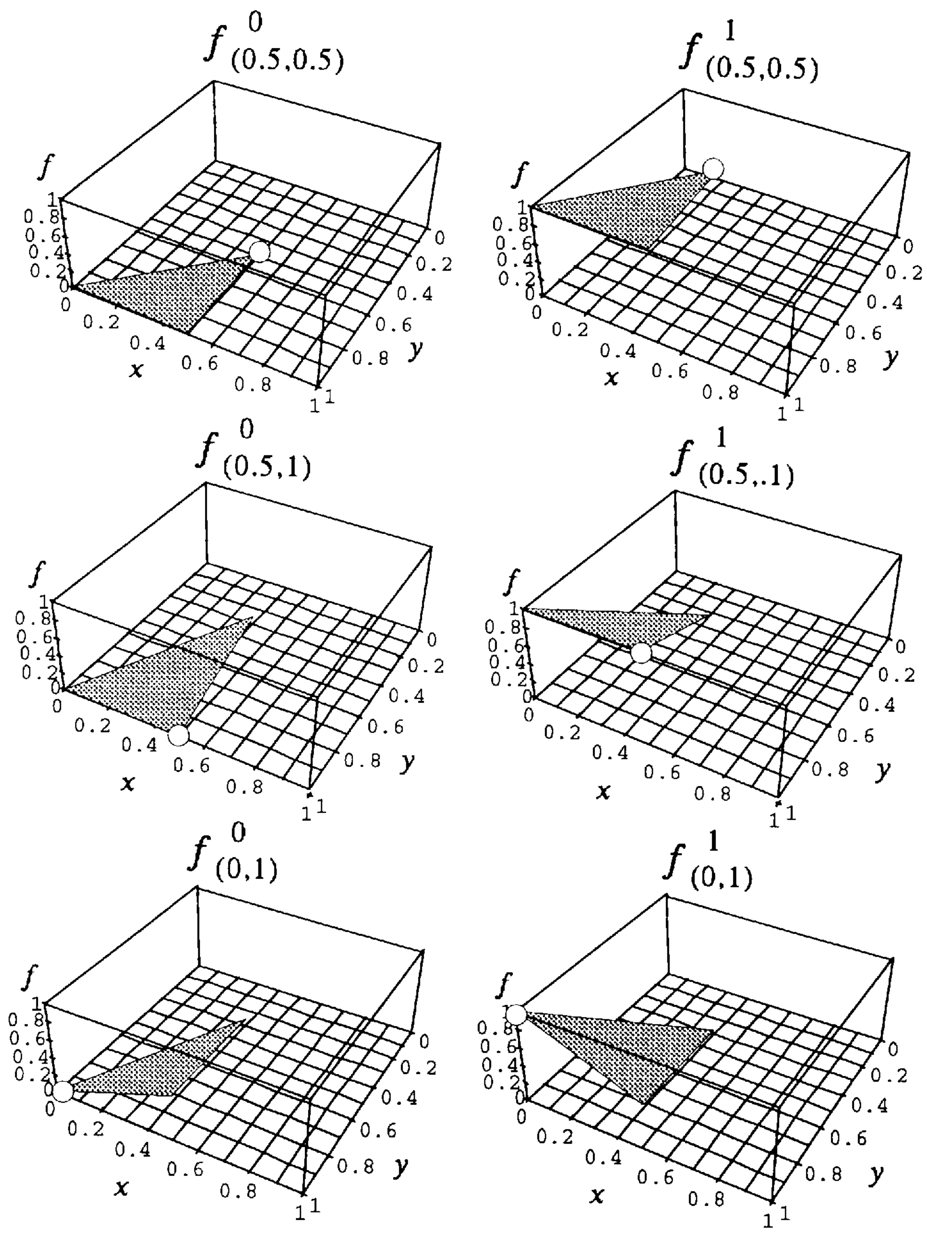

Figure 6.2: Partially specified fuzzy switching functions f_a^b

Lemma 6.1 Let f be a fuzzy switching function and $\boldsymbol{d} \in V^n$. If $f(\boldsymbol{d}) \notin V_3$ then

$$f(\boldsymbol{a}) = 0.5 \quad \text{for all } \boldsymbol{a} \in C(\boldsymbol{d}) \text{ such as } \boldsymbol{a} \succ \boldsymbol{a}^*,$$
$$f(\boldsymbol{a}^*) \in \{0, 1\},$$

where $\boldsymbol{a}^* = \overline{\boldsymbol{d}}^{f(d)}$.

Proof. Since $f(\boldsymbol{d}) \notin V_3$, obviously, $\overline{f(\boldsymbol{d})}^{f(d)} = f(\overline{\boldsymbol{d}}^{f(d)}) = f(a^*) \in \{0, 1\}$. For any $\boldsymbol{b} \succ \boldsymbol{a}^*$, there is a $\lambda \in V$ such that $\boldsymbol{b} = \overline{\boldsymbol{d}}^{\lambda} \succeq \overline{\boldsymbol{d}}^{f(d)}$ and $f(\boldsymbol{d}) \succ \lambda$. Thus $\overline{f(\boldsymbol{d})}^{\lambda} = 0.5$, and thus, $f(\overline{\boldsymbol{d}}^{\lambda}) = f(\boldsymbol{b}) = 0.5$. □

Theorem 6.2 Let $\boldsymbol{d} \in V^n$ and f be a fuzzy switching function. For every $\boldsymbol{c} \in V^n$ with $\boldsymbol{c} \approx \boldsymbol{d}$,

$$f(\boldsymbol{c}) = F_{a^*}^{f(a^*)}(\boldsymbol{c}),$$

where $\boldsymbol{a}^* = \overline{\boldsymbol{d}}^{f(d)}$.

Proof. For any $\boldsymbol{c} = \boldsymbol{a}^*$, obviously $F_{a^*}^{f(a^*)}(\boldsymbol{c}) = f(\boldsymbol{c})$. For any $\boldsymbol{c} \succeq \boldsymbol{a}^*$, $F_{a^*}^{f(a^*)}(\boldsymbol{c}) = 0.5 = f(\boldsymbol{c})$ [Lemma 6.1]. For any $\boldsymbol{a}^* \succeq \boldsymbol{c}$, $F_{a^*}^{f(a^*)}(\boldsymbol{c}) = f(\boldsymbol{a}^*) = f(\boldsymbol{c})$ [Theorem 2.2]. Thus, $F_{a^*}^{f(a^*)}(\boldsymbol{c}) = f(\boldsymbol{a}^*) = f(\boldsymbol{c})$ for all elements of $C(\boldsymbol{d})$, and by Theorem 6.1, $F_{a^*}^{f(\boldsymbol{a}^*)}(\boldsymbol{c}) = f(\boldsymbol{a}^*) = f(\boldsymbol{c})$ for all elements of $[\boldsymbol{d}]$. □

This theorem states that a fuzzy switching function can be determined uniquely within the given Q-equivalence class, and that for any fuzzy switching function F, there must be F_a^b in a given Q-equivalence class. Hence, combinations of $(\boldsymbol{a}, b)$ are enough to represent all fuzzy switching functions.

Corollary 6.2 Let $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ be a subset of V^n, $A_1, \ldots, A_s$ be Q-equivalent classes of A, and f be a function $f : A \to V$. For any fuzzy switching function F, there are $\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_s \in V_3^n$ and $b_1, \ldots, b_s \in \{0, 1\}$ that satisfy

$$D(F, f) = \sum_{i=1}^{m}(f(\boldsymbol{a}_i) - F(\boldsymbol{a}_i))^2 = \sum_{i=1}^{s} D(F_{a'_i}^{b_i} f|_{A_i})$$

where $f|_{A_i}$ is a restriction of f by A_i, that is, $f|_{A_i} : A_i \to V$.

Proof. It is a straightforward from Theorem 6.2 and the definition of D. □

Example 6.7 Let A_i be as in Example 6.3. For $F(x, y) = \overline{y} \vee \overline{x}y$,

$$D(F, f) = D(F_{(0,1)}^1, f|_{A_1}) + D(F_{(1,0)}^0, f|_{A_2}) + D(F_{(1,0)}^0, f|_{A_3}) + D(F_{(0.5,0)}^1, f|_{A_4}).$$

6.3.2 Representation of Restrictions

Theorem 6.3 Let $\boldsymbol{d} \in V^n$, $\boldsymbol{c} \in C(\boldsymbol{d})$, $b \in V_2$, and f be a fuzzy switching function. For $\boldsymbol{a} \in V^n$ such as $\boldsymbol{a} \approx \boldsymbol{d}$,

$$f_c^b(\boldsymbol{a}) = \begin{cases} \alpha^c(\boldsymbol{a}) & \text{if } b = 1 \\ \overline{\alpha^c}(\boldsymbol{a}) & \text{if } b = 0. \end{cases}$$

Proof. For $\boldsymbol{c} \succeq \boldsymbol{a}$, $f_c^1(\boldsymbol{a}) = 1 = \alpha^c(\boldsymbol{a}) = 1$ and $f_c^0(\boldsymbol{a}) = 0 = \overline{\alpha^c}(\boldsymbol{a}) = 0$ [Proposition 2.2]. For $\boldsymbol{a} \succ \boldsymbol{c}$, $f_c^b(\boldsymbol{a}) = 0.5 = \alpha^c(\boldsymbol{a}) = \overline{\alpha^c}(\boldsymbol{a}) = 0.5$ [Proposition 2.4]. Thus, the theorem holds for all elements of $C(\boldsymbol{d})$. Therefore, by Theorem 6.1, we have $f_c^1(\boldsymbol{a}) = \alpha^c(\boldsymbol{a})$ and $f_c^0(\boldsymbol{a}) = \overline{\alpha^c}(\boldsymbol{a})$ for every $\boldsymbol{a} \approx \boldsymbol{d}$. □

Example 6.8 Let $\boldsymbol{d} = (0.3, 0.8)$, and $\boldsymbol{c} = (0,1) \in C(\boldsymbol{d})$. For any $\boldsymbol{a} \in V^n$ such as $\boldsymbol{a} \approx (0.3, 0.8)$,

$$\begin{aligned} f^1_{(0,1)}(\boldsymbol{a}) &= \alpha^{(0,1)}(\boldsymbol{a}) = \overline{x}y(\boldsymbol{a}), \\ f^0_{(0,1)}(\boldsymbol{a}) &= \overline{\alpha^{(0,1)}}(\boldsymbol{a}) = \overline{\overline{x}y}(\boldsymbol{a}) = (x \vee \overline{y})(\boldsymbol{a}). \end{aligned}$$

Since $0 < \overline{y} < x < 0.5 < \overline{x} < y < 1$ [Proposition 6.5], we have

$$f|_{f(0,1)=1}(\boldsymbol{x}) = \overline{x}, \quad f|_{f(0,1)=0}(\boldsymbol{x}) = x.$$

Example 6.9 Let f_0 and A be as for the fitting problem shown in the Introduction. We calculate every distance to f_0 for each Q-equivalent class in $[A]$. For $\boldsymbol{d} = (0.8, 0.7)$ in A, we have the quantized set $C(\boldsymbol{d})$ and the Q-equivalent class $[\boldsymbol{d}]$:

$$\begin{aligned} C(\boldsymbol{d}) &= \{(1,1), (1,0.5), (0.5,0.5)\}, \\ [\boldsymbol{d}] &= \{(0.8,0.7), (0.7,0.6)\}. \end{aligned}$$

By Corollary 6.2, the distances for every fuzzy switching function are given by examining each $\boldsymbol{a}$ in $C(\boldsymbol{d})$, that is, $|C(\boldsymbol{d})| \cdot |V_2| = 3 \cdot 2 = 6$ restrictions in $[\boldsymbol{d}]$. Theorem 6.3 is useful to simplify the calculation of distances. For example,

$$\begin{aligned} D(f^1_{(1,1)}, f_0|_{[d]}) &= \sum_{a \in [d]} |f_0(\boldsymbol{a}) - \alpha^{(1,1)}(\boldsymbol{a})| \\ &= |f_0(0.8, 0.7) - (y)(0.8, 0.7)| + |f_0(0.7, 0.6) - (y)(0.7, 0.6)|^2 \\ &= |0.9 - 0.7| + |0.8 - 0.6| = 0.4 \\ D(f^0_{(1,1)}, f_0|_{[d]}) &= |0.9 - 0.3| + |0.8 - 0.4| = 1.0 \end{aligned}$$

Table 6.1 lists the distances to f_0 for each Q-equivalent class, $A_1, \ldots, A_4$.

6.4 Combining Local Solutions

In the previous section, we have obtained all possible local distances for each Q-equivalence class in the partitions $[A]$. The next problem is how to combine them and find the global solution f^* with the shortest distance. We use a graph-theoretic algorithm, which is a variation of Dijkstra's shortest path algorithm[23].

Table 6.1: Distances $E(f, h)$

$E(f_a^b, f_0|_{A_1})$

$\boldsymbol{a}\backslash b$	1	0
(0,1)	0.3	0.1
(0.5,1)	0.5	0.1
(0.5,0.5)	0.7	0.3

$E(f_a^b, f_0|_{A_2})$

$\boldsymbol{a}\backslash b$	1	0
(1,1)	0	0.2
(0.5,1)	0.3	0.2
(0.5,0.5)	0.4	0.6

$E(f_a^b, f_0|_{A_3})$

$\boldsymbol{a}\backslash b$	1	0
(1,1)	0.4	1.0
(1,0.5)	0.2	1.2
(0.5,0.5)	0.3	1.7

$E(f_a^b, f_0|_{A_4})$

$\boldsymbol{a}\backslash b$	1	0
(0,0)	0.1	0.3
(0.5,0)	0.3	0.5
(0.5,0.5)	0.4	0.6

6.4.1 Condition for Two Functions for Representable

Corollary 6.3 Let A and B be distinct subsets of V_3^n, f and g be restrictions of a fuzzy switching function such that $f : A \to V$ and $g : B \to V$. There is a fuzzy switching function F that satisfies $F(\boldsymbol{c}) = f(\boldsymbol{c})$ for all $\boldsymbol{c} \in A$ and $F(\boldsymbol{c}) = g(\boldsymbol{c})$ for all $\boldsymbol{c} \in B$ if and only if

$$\text{(R) } (C_i^*(f) \cup C_i^*(g)) \cap (C_j^*(f) \cup C_j^*(g)) = \emptyset \quad \text{for every } i \neq j.$$

Proof. The corollary is obvious from Theorem 3.2 (disjoint) and Theorem 3.5 (representability). □

Example 6.10 For the following restrictions of fuzzy switching function f_1, f_2, f_3 :

$$f_1(0,1) = 1, \quad f_1(0.5,1) = 1, \quad f_1(0.5,0.5) = 0.5,$$
$$f_2(1,1) = 0, \quad f_2(1,0.5) = 0, \quad f_2(0.5,0.5) = 0.5,$$

there is no fuzzy switching function that satisfies both. That is because element $(1,1) \in C_1^*(f_1) \cap C_0^*(f_2)$ violates the condition (R) in Corollary 6.3.

6.4.2 Algorithm FFSF

Input: A subset of $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ and a mapping $f_0 : A \to V$.

Step1: For each $\boldsymbol{a}_i$ of A, obtain the quantized set $C(\boldsymbol{a}_i)$.

Step2: Based on $C(\boldsymbol{a}_1), \ldots, C(\boldsymbol{a}_m)$, make A be partitioned into several Q-equivalent classes as follows:

$$A = [\boldsymbol{a}_1'] \cup [\boldsymbol{a}_2'] \cup \cdots \cup [\boldsymbol{a}_l'],$$

where $\boldsymbol{a}_i'$ is the representative element of Q-equivalent class.

Step3: For each Q-equivalent class $[\boldsymbol{a}_i']$, obtain a set of partially specified fuzzy switching functions $R(C(\boldsymbol{a}_i))$ as follows:

$$R(C(\boldsymbol{a}_i)) = \{f_c^b | \boldsymbol{c} \in C(\boldsymbol{a}_i), b \in \{0,1\}\}.$$

Step4: For each element f_c^b of a set $R(C(\boldsymbol{a}_i))$, calculate the distance to the restriction f_0 of $[\boldsymbol{a}'_i]$. For simplification, we write the distance by

$$D^i(f_c^b) \equiv D(f_c^b, f_0|_{[a_i]}).$$

According to Theorem 6.3, $D^i(f_c^b)$ is obtained by,

$$D^i(f_c^b) = \begin{cases} \sum_{a\in[a_i]} |f_0(\boldsymbol{a}) - \alpha^c(\boldsymbol{a})| & \text{if } b = 1, \\ \sum_{a\in[a_i]} |f_0(\boldsymbol{a}) - \overline{\alpha^c(\boldsymbol{a})}| & \text{if } b = 0. \end{cases}$$

Step5: Chose the partially specified fuzzy switching function that has the shortest distance to f_0 for $R(C(\boldsymbol{a}'_i))$, and write it by r_i^0. If there are alternative element, chose any of it. Let $k = 0$.

Step6: Verify that $(r_1^k, \ldots, r_l^k)$ satisfies the condition (R) of Theorem 3.5, that is,

$$\bigcup_{j=1,\ldots,l} C_i^*(r_j^k) \cap \bigcup_{j=1,\ldots,l} C_{i'}^*(r_j^k) = \emptyset$$

for every $i \neq i' \in \{0, 1, 0.5\}$. If this holds, go to Step 8.

Step7: Chose an alternative combination $(r_1^{k+1}, \ldots, r_l^{k+1})$ so that the distance to f_0 is as short as possible. If there are several combinations, chose carefully to avoid infinite loop. Then, letting $k = k + 1$ and back to step 6.

Step8: Let f^* be the fuzzy switching function defined by

$$f^*(\boldsymbol{a}) = r_i^k(\boldsymbol{a})$$

for all $\boldsymbol{a} \in A$ such that $\boldsymbol{a} \in [\boldsymbol{a}'_i]$. Obtain the logic formula F^* as follows:

$$F^* = \bigvee_{a\in C_1^*(f^*)} \alpha^a \bigvee_{b\in C_{0.5}^*(f^*)} \beta^b,$$

where α^a and β^b are a simple phrase and a complementary phrase corresponding to $\boldsymbol{a}$ and $\boldsymbol{b}$.

Output The fuzzy switching function f^* and the logic formula F^*.

Proposition 6.6 A fuzzy switching function f^* computed by Algorithm FFSF has the shortest distance to f_0.

Proof. Consider fuzzy switching function g that has shorter distance than f^* when $k = 0$. By Theorem 6.2, there must exist a partially specified fuzzy switching function f_a^b such that $g(\boldsymbol{a}_i) < r_i^0(\boldsymbol{a}_i) = f_a^b(\boldsymbol{a}_i)$, which contradicts step 5. Hence, there is no g for $k = 0$.

By the mathematical principle of induction and step 6 and 7, no fuzzy switching function g has smaller distance than f^*. □

Proposition 6.7 A logic formula F^* computed by Algorithm FFSF represents the smallest fuzzy switching function for all fuzzy switching functions that satisfy f^*.

Proof. If $f*$ is unique, that is, $C(f^*) = V_3^n$, then the proposition is clearly true. We prove that F^* is the smallest for all $\boldsymbol{a} \in V_3^n - C(f^*)$.

Obviously, there is no $\boldsymbol{a}$ such that $F^*(\boldsymbol{a}) = 1$, because from the definition of F^*, F^* contains a simple phrase α^b such that $\boldsymbol{a} \succeq \boldsymbol{a}$, which conflicts with step 8 and the definition of C^*.

Next, when $F(\boldsymbol{a}) = 0$, the proposition holds. Finally, we show there is no fuzzy switching function g such that $g(\boldsymbol{a}) = 0$, $F^*(\boldsymbol{a}) = 0.5$. Since $F^*(\boldsymbol{a}) = 0.5$, there exists a simple phrase α^b such that $\boldsymbol{b} \in C_1^*(f^*)$ and $\boldsymbol{a} \succeq \boldsymbol{b}$. Then, $g(\boldsymbol{a}) = 0$ and $g(\boldsymbol{b}) = 1$ violates the monotonicity. Therefore, we have that F^* is the smallest. □

Proposition 6.8 Algorithm FFSF always stops.

Proof. It is obviously true because the shortest fuzzy switching function must exist for any f_0. □

6.4.3 Replacement with Undirected Graph G

Algorithm FFSF is general and does not specify any rule for selecting partially fuzzy switching functions. This section provides more concrete algorithm that is suitable for an implementation on computer.

Definition 6.5 Let $D = \{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_m\}$ be a subset of V^n. A (undirected) *graph* G characterized by D consists of two nonempty sets, the set $V(G)$ of vertices of G and the set $Ed(G)$ of edges of G as follows:

$$V(G) = \{f|_{[d]} \mid f \in \mathcal{F}_n, \boldsymbol{d} \in D\} = Ve_1 \cup Ve_2 \cup \cdots \cup Ve_s$$

where $\mathcal{F}$ is a set of all n-variable fuzzy switching functions and Ve_i $(i = 1, \ldots, s)$ is defined by

$$Ve_i = \{f|_{[d_i]} \mid f \in \mathcal{F}_n\} = \{v_{i1}, v_{i2}, \cdots, v_{i(2(n+1))}\}.$$

If vertices $u, v \in V(G)$ satisfy the following conditions:

(R) $(C_i^*(u) \cup C_i^*(v)) \cap (C_j^*(u) \cup C_j^*(v)) = \emptyset$ for every $i \neq j$,
(A) there is no vertex w in $V(G)$ with edges $\{u, w\}$ and $\{w, v\}$,

then edge $\{u, v\}$ is in $Ed(G)$.

A *weight* $W(v)$ of a vertex v in $V(G)$ is defined by

$$W(v) = E(v, h) = \sum_{d \in C(v)} |f_0(\boldsymbol{d}) - v(\boldsymbol{d})|$$

where $E(v, h)$ is a distance between v and f_0, and $C(v)$ is the quantized set of v, which corresponds to the Q-equivalent class.

Example 6.11 Figure 6.3 illustrates the undirected graph G given by D in Example 3.3, where restrictions v_{ij} are indicated on the truth tables for V_3^2.

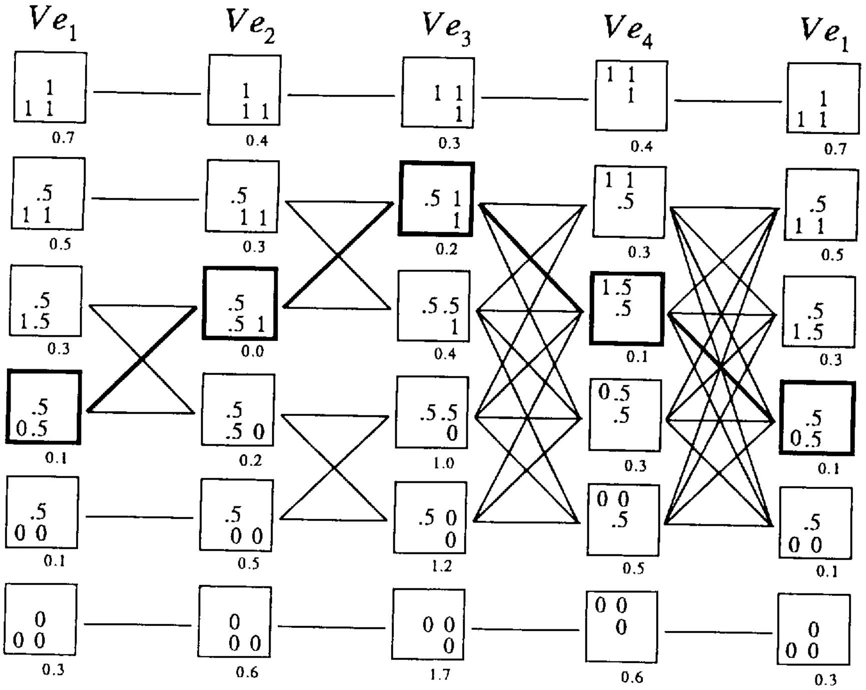

Figure 6.3: Graph G

6.4.4 Shortest Path Algorithm(SPA)

To find f^* with the shortest distance $D(f^*, h)$ is to find sequence $v_1^*, \ldots, v_s^*$ in $V(G)$ such that $\sum_i^s E(v_i^*, h)$ is the lowest value for all others. We prefer a quicker way without checking all possible combinations.

Assume $Ve_i, Ve_{i+1} \in V(G)$ are in order so that edge $\{v_i, v_{i+1}\} \in Ed(G)$ for some $v_i \in Ve_i$ and $v_{i+1} \in Ve_{i+1}$. Figure 6.4 shows how the shortest path algorithm works with min-weight W_i.

Example 6.12 We apply the shortest path algorithm to the graph G in Example 3.9 (in Table 6.2).

Table 6.2: Matrices $W(G)$

$j \backslash i$	1
1	.7
2	.5
3	.3
4	.1
5	.1
6	.3

$W^1(v_{ij})$

$j \backslash i$	1	2
1	.7	1.1
2	.5	.8
3	.3	.1
4	.1	.3
5	.1	.6
6	.3	.9

$W^2(v_{ij})$

$j \backslash i$	1	2	3
1	.7	1.1	1.4
2	.5	.8	.3
3	.3	.1	.5
4	.1	.3	1.3
5	.1	.6	1.5
6	.3	.9	2.3

$W^3(v_{ij})$

$j \backslash i$	1	2	3	4
1	.7	1.1	1.4	2.1
2	.5	.8	**.3**	.6
3	.3	**.1**	.5	**.4**
4	**.1**	.3	1.3	.6
5	.1	.6	1.5	1.0
6	.3	.9	2.3	2.9

$W^4(v_{ij})$

Clearly the shortest path algorithm always stops, and the minimum weight in final

Shortest Path Algorithm(SPA).

For each v in Ve_1

    Set $W^1(v) = E(v,h)$

    Set $P^1(v) = \{v\}$

End for

For $i = 2$ to s

    For each v in Ve_i

        For each u in Ve_{i-1} with $\{u,v\}$ in $Ed(G)$

            If $W^i(v) > W^{i-1}(u) + E(v,h)$

                Set $W^i(v) = W^{i-1}(u) + E(v,h)$

                Set $u^* = u$

        End for

        Set $P^i(v) = P^{i-1}(u^*) \cup \{v\}$

    End for

End for

Figure 6.4: Algorithm SPA

W^s:

$$W^* = \min_{v \in Ve_s} W^s(v) = E(f^*, h)$$

is the intended distance to f^* and $P^* = P^s(v^*) = \{v_1^*, v_2*, \ldots, v_s^*\}$ gives f^*, that is, for any $\boldsymbol{d} \in D$,

$$f^*(\boldsymbol{d}) = v^*(\boldsymbol{d}) \quad \text{for some } v^* \in P^*.$$

Example 6.13 Given W^i and P^i in Example 6.12, we have

$$\begin{aligned} W^* &= \min_{v \in Ve_4} W^4(v) = W^4(v_{43}) = 0.03, \\ P^* &= \{v_{14}, v_{23}, v_{32}, v_{43}\}. \end{aligned}$$

According to Theorem 4.8, we can reconstruct the logic formula representing f^* as follows:

$$\begin{aligned} f^*(x,y) &= xy \vee x \vee x\overline{y} \vee \overline{x}\,\overline{y}x\overline{x}y \vee x\overline{x}\,\overline{y} \vee \overline{x}y\overline{y} \\ &= x \vee \overline{x}\,\overline{y}. \end{aligned}$$

This is the answer f^* and we have solved the fitting problem. Note that $x \vee \overline{x}\,\overline{y} \neq x \vee \overline{y}$ in fuzzy logic.

6.4.5 Estimation of SPA

If we simply examine each fuzzy switching functions one by one, using a brute force algorithm (BFA), it takes as much time as the number of n-variable fuzzy switching functions $|\mathcal{F}_n|$, which is greater than the number of n-variable boolean switching functions $|\mathcal{B}_n| = 2^{2^n}$. The time to execute BFA is thus $O(2^{2^n})$.

For simplifying the estimation, we assume $D \in (V - V_3)^n$ as we have done in the examples. The number of Q-equivalent classes $|[D]|$ is at most $2^n n!$, because by Proposition 6.5, it is equal to the permutation of n variables with two literals x_i and $\overline{x_i}$. For each Q-equivalent class $[\boldsymbol{d}]$, there are $2(n+1)$ restrictions of fuzzy switching functions. The number of vertices $|V(G)|$ is thus less than or equal to $2(n+1) \cdot 2^n n! = 2^{n+1}(n+1)!$. For example, when $n = 2$, $|V(G)| = 2^{2+1} \cdot (2+1)! = 48$. If we examine all of the possible sequences $v_1, \ldots, v_s$ in $V(G)$ so that $v_i \in Ve_i$, then the total time is $O(2(n+1)^{2^n n!})$, which is worse than BFA.

The computation could be speeded up considerably by the shortest path algorithm (SPA). The comparison/replacement step inside the u-loop takes at most some fixed amount of time. The step is done at most $|Ve_{i-1}| = 2(n+1)$ times in the u-loop, which is done $|Ve_i| = 2(n+1)$ times for the v-loop, which is done $s - 1 = 2^n n! - 1$ times for the outside i-loop. The total time to execute the algorithm is $O(2(n+1) \cdot 2(n+1) \cdot 2^n n!) = O(2^{n+2} n!(n+1)^2)$. Table 6.3 is how long each algorithm takes for $n = 1, \ldots, 5$.

Table 6.3: Computational times

$n\backslash$ algorithm	BFA($\lvert\mathcal{F}_n\rvert$) $O(?)$	BFA'($\lvert\mathcal{B}_n\rvert$) $O(2^{2^n})$	SPA $O(2^{n+1}n!(n+1)^2)$
1	6	4	16
2	84	16	144
3	43918	256	1536
4	160297985276	65536	19200
5	–	4294967296	276480

6.5 Conclusion

We have proposed an algorithm to find the fuzzy switching function with the shortest distance to a given partial function. We have proved that the division of the uncertain knowledge into some Q-equivalent classes is sufficient to calculate all possible local distances in Theorem 6.1, and have shown some useful properties with respect to restrictions of fuzzy switching functions. The shortest path algorithm can work with a large value of n on weighted graph G which corresponds to all possible n-variable fuzzy switching functions. We have also estimated the algorithm and shown that it can reduce computation in finding the fuzzy switching function with the shortest distance for all others.

Chapter 7

Practical Example

This section shows how a logic formula is derived from a real estimation made by a person. An example of winning possibilities of several baseball teams is demonstrated.

In the algorithm, an evaluation made by humans is represented by a single logic formula instead of several IF-THEN rules.

7.1 Example: Baseball teams

7.1.1 Introduction

When a person makes some decisions, he/she unconsciously uses a certain logic. For example, a winning possibility of baseball game depends mainly upon the good batter and the good pitcher. Other factors such as the defense or the director are used optionally. Our decision making processes is made based on the logical relationship among the estimation factors. The estimation, however, involves uncertainty and incomplete, which make it difficult to derive the underling logical knowledge.

In this section, we are trying to model such human estimation by means of fuzzy switching functions. The algorithm proposed in preceding section takes input data given by an expert and output the best logic formula that has the least difference of the input data. This approach has the advantages as follows:

- A single logic formula is never inconsistent, while several IF-THEN rules might conflict each other.
- By taking the sum of all input data would cancel the small noise of input data, which is a critical in exact matching.
- The total time required in the algorithm is proportional to number of input, so it can be used in practical large scale knowledge acquisition.
- The total error provides the degree of approximation.

7.1.2 Sample Judgment by Expert

A sample data of evaluation, which will be used as a learning data, is given by questioning an human expert. We consider four factors:

x_1: batting,
x_2: defense,
x_3: pitching,
x_4: manager,

to determine the possibility of winning. The range is 0 to 1, with 0 means "absolutely bad," 1 means "the best," and 0.5 means "unknown." For example, $x_1 = 0.8$ means "the team has good pitcher with degree of 0.8," or "the team has many nice pitchers," and $x_2 = 0.2$ means "the lack of good pitcher is weak point of the team with certainty." Notice that the degree of 0.2 simply shows certainty of $1 - 0.2 = 0.8$ rather than uncertainty. Hence, the most uncertain degree is 0.5.

With the above estimation, the expert answers the possibility of winning by a truth value of $[0, 1]$. We denote the possibility by $f_0(x_1, x_2, x_3, x_4)$ or shortly $f_0(\boldsymbol{x})$.

Table 7.1 shows a result of questioning for twelve baseball teams.

Table 7.1: Sample Estimation (input)

Team	x_1	x_2	x_3	x_4	f_0
T	.5	.6	.8	.6	.5
G	.6	.6	.7	.7	.7
C	.5	.7	.6	.6	.5
S	.6	.5	.7	.8	.7
Y	.4	.5	.5	.6	.4
D	.5	.4	.4	.3	.3
L	.8	.8	.8	.8	.8
B	.8	.6	.6	.6	.6
F	.6	.5	.6	.5	.4
H	.6	.6	.7	.8	.5
M	.6	.4	.5	.6	.4
X	.4	.5	.5	.3	.3

7.1.3 Fitting Steps

Input: Let A and f_0 be a subset of V^4 and a mapping $f_0 : A \to V$ defined by Table 7.1. We denote A by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{12}$ corresponding to symbols $T, G, \ldots$ in the table.

Step1: For each $\boldsymbol{a}_i$ of A, We have the quantized set $C(\boldsymbol{a}_i)$ as displayed in Table 7.2.

Step 2: We wish A be partitioned into several Q-equivalent classes $[\boldsymbol{a}_{i_1}], \ldots, [\boldsymbol{a}_{i_{12}}]$, but there is no two element of A that belong to same Q-equivalent class in this case. So we still remain to use the same notation of $[\boldsymbol{a}_{i_j}] = \{\boldsymbol{a}_j\}$.

Table 7.2: Quantized sets $C(A)$

$\boldsymbol{a}_i$	$C(\boldsymbol{a}_i)$
T	(.5,1,1,1),(.5,.5,1,.5),(.5,.5,.5,.5)
G	(1,1,1,1), (.5,.5,1,1), (.5,.5,.5,.5)
C	(.5,1,1,1), (.5,1,.5,.5), (.5,.5,.5,.5)
S	(1,.5,1,1), (.5,.5,1,1), (.5,.5,.5,1), (.5,.5,.5,.5)
Y	(0,.5,.5,1), (.5,.5,.5,.5)
D	(.5,0,0,0), (.5,.5,.5,0), (.5,.5,.5,.5)
L	(1,1,1,1), (.5,.5,.5,.5)
B	(1,1,1,1), (1,.5,.5,.5), (.5,.5,.5,.5)
F	(1,.5,1,.5), (.5,.5,.5,.5)
H	(1,1,1,1), (.5,.5,1,1), (.5,.5,.5,1), (.5,.5,.5,.5)
M	(1,0,.5,1), (.5,.5,.5,.5)
X	(0,.5,.5,0), (.5,.5,.5,0), (.5,.5,.5,.5)

Step 3,4: For each Q-equivalent class $[\boldsymbol{a}_i]$, we get a set of the restrictions or the partial specified fuzzy switching functions $R(C(\boldsymbol{a}_i))$. For example, team T's quantized set $C(\boldsymbol{a}_1) = \{(.5,1,1,1).(.5,.5,1,.5),(.5,.5,.5,.5)\}$ has the set of restrictions as follows:

$$\begin{aligned} R(C(\boldsymbol{a}_1)) &= \{f^1_{(.5,1,1,1)}, f^0_{(.5,1,1,1)}, f^1_{(.5,.5,1,.5)}, \\ &\quad f^0_{(.5,.5,1,.5)}, f^1_{(.5,.5,.5,.5)}, f^0_{(.5,.5,.5,.5)}\} \\ &= \{x_2x_3x_4,\ \overline{x_2x_3x_4},\ x_3,\ \overline{x_3},\ 1,\ 0\}. \end{aligned}$$

Moreover, we compute the distances from restrictions to f_0 for all Q-equivalent classes $[\boldsymbol{a}_i]$ as follows:

$$\begin{aligned} D(f^1_{(.5,1,1,1)}, f_0(\boldsymbol{a}_1)) &= |x_2x_3x_4(\boldsymbol{a}_1) - f_0(\boldsymbol{a}_1)| \\ &= |\min(.6,.8,.6) - .5| = 0.1, \\ D(f^0_{(.5,1,1,1)}, f_0(\boldsymbol{a}_1)) &= |\overline{x_2x_3x_4}(\boldsymbol{a}_1) - f_0(\boldsymbol{a}_1)| \\ &= |1 - \min(.6,.8,.6) - .5| = 0.1, \end{aligned}$$

Thus, we show the result of Step 3 and 4 on Table 7.3.

Step 6: Let r^0_i be a restriction that has the shortest distance to f_0 in $R(C(\boldsymbol{a}_i))$ for $i = 1, \ldots, 12$.

$$\begin{aligned} r^0_1 &= x_2x_3x_4 \\ r^0_2 &= x_3x_4 \\ r^0_3 &= x_2x_3x_4 \\ r^0_4 &= x_3x_4 \\ r^0_5 &= \overline{\overline{x_1}x_4} = x_1 \vee \overline{x_4} \end{aligned}$$

$$\begin{aligned}
r_6^0 &= x_4 \\
r_7^0 &= x_1 x_2 x_3 x_4 \\
r_8^0 &= x_1 x_2 x_3 x_4 \\
r_9^0 &= \overline{x_1 x_3} = \overline{x_1} \vee \overline{x_3} \\
r_{10}^0 &= x_1 x_2 x_3 x_4 \\
r_{11}^0 &= \overline{x_1 \overline{x_2} x_4} = \overline{x_1} \vee x_2 \vee \overline{x_4} \\
r_{12}^0 &= \overline{x_4}
\end{aligned}$$

We then take a sum of restrictions $r_1^1, \ldots, r_{12}^1$ and see if the sum f^1 is representable by the condition of consistency. Table 7.4 shows the result of sum of the selected restrictions. Looking at the table, unfortunately, we detect a contradiction that element of $(.5, .5, 1, 1)$ takes both 1 and .5, which is denoted by 1/.5. Thereby we know that f^1 obviously is not representable, so we must go to Step 7.

Step 7: We chose an alternative combination of restrictions for each $R(C(\boldsymbol{a}_i))$ that has as short distance to f_0 as possible. In this case we have two choices of restrictions: for $R(C(\boldsymbol{a}_1))$,

$$\begin{aligned}
D(r_1^0, f_0(\boldsymbol{a}_1)) &= D(f^1_{(.5,1,1,1)}, f_0(\boldsymbol{a}_1)) = 0.1 \\
&= D(f^0_{(.5,1,1,1)}, f_0(\boldsymbol{a}_1)) \\
&= D(\overline{x_2\, x_3\, x_4}, f_0(\boldsymbol{a}_1))
\end{aligned}$$

and for $R(C(\boldsymbol{a}_3))$,

$$\begin{aligned}
D(r_3^0, f_0(\boldsymbol{a}_3)) &= D(f^1_{(.5,1,1,1)}, f_0(\boldsymbol{a}_3)) = 0.1 \\
&= D(f^0_{(.5,1,1,1)}, f_0(\boldsymbol{a}_3)) \\
&= D(\overline{x_2 x_3 x_4}, f_0(\boldsymbol{a}_3))
\end{aligned}$$

and for $R(C(\boldsymbol{a}_{10}))$,

$$\begin{aligned}
D(r_{10}^0, f_0(\boldsymbol{a}_3)) &= D(f^1_{(1,1,1,1)}, f_0(\boldsymbol{a}_{10})) = 0.1 \\
&= D(f^0_{(1,1,1,1)}, f_0(\boldsymbol{a}_{10})) \\
&= D(\overline{x_1 x_2 x_3 x_4}, f_0(\boldsymbol{a}_{10}),
\end{aligned}$$

so we have total of $2 \times 2 \times 2 - 1 = 7$ combinations with the same distance as that of f^0. For each combination we go on trying to check the condition for representable in step 6. This trial goes on till the condition of representable is to be satisfied.

As the results, we find the combination of restrictions that can hold the condition for the first time in Table 7.5. On the table, the restrictions of Y and M are fuzzy switching functions that take a value of .5 for the chosen element of $C(\boldsymbol{a}_5)$ and $C(\boldsymbol{a}_{10})$, respectively. The total distance to f_0 is 1.0, that is, the shortest distance.

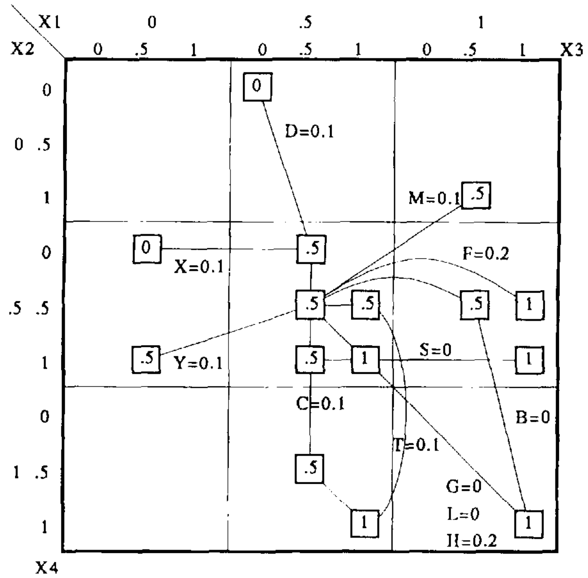

Figure 7.1: Sum of restrictions $f_1^k, \ldots, f_{12}^k$

Step 8: Figure 7.1 illustrates the fuzzy switching function f^k defined by the sum of restrictions chosen in step 7. Where, we denote the restrictions and distances to f_0 by the symbols such as $T = 0.1$.

Taking the expansion $C^*(f^*)$ of the quantized set, now we obtain the fuzzy switching function f^* on Table 7.6.

Since the all of cells of the table are not filled, according to the condition of uniqueness, there could be several fuzzy switching functions with the shortest distance. Hence, we shall define two special fuzzy switching functions by the minimum f_{cd} and the maximum f_{cc} with respects to numerical order of truth value. The minimum fuzzy switching function is defined by filling in all blank cells with as many values of 0 as possible. Note that some of blank cells such as $(0.5, 0, 0.5, 1)$ or $(1, 1, 0.5, 1)$ must be 0.5 because of the monotonicity. We show the result on Table 7.7. Similarly, Table 7.8 shows the result of the maximum fuzzy switching function f_{cc} which is filled in blanks with value of 1.

Finally, we apply the representation rule to each fuzzy switching functions in step 8, and have the logic formulae F_{cd} and F_{cc} as follows:

$$
\begin{aligned}
F_{cd} &= \bigvee_{a \in C_1^*(f^*)} \alpha^a \bigvee_{b \in C_U^*(f^*)} \beta^b \\
&= \alpha_{(1,0,1,0)} \vee \alpha_{(1,0,1,.5)} \vee \alpha_{(1,0,1,1)} \vee \alpha_{(1,.5,1,0)} \vee \alpha_{(1,.5,1,.5)} \vee \\
&\quad \alpha_{(1,.5,1,1)} \vee \alpha_{(1,1,1,0)} \vee \alpha_{(1,1,1,.5)} \vee \alpha_{(1,1,1,1)} \vee \alpha_{(.5,0,1,1)} \vee \\
&\quad \alpha_{(.5,.5,1,1)} \vee \alpha_{(.5,1,1,1)} \vee \alpha_{(0,0,1,1)} \vee \alpha_{(0,.5,1,1)} \vee \alpha_{(0,1,1,1)} \vee \\
&\quad \beta_{(1,0,.5,0)} \vee \beta_{(1,0,.5,.5)} \vee \beta_{(1,0,.5,1)} \vee \beta_{(1,.5,.5,0)} \vee \beta_{(1,.5,.5,.5)} \vee
\end{aligned}
$$

$$\begin{aligned}
&\beta_{(1,.5,.5,1)} \vee \beta_{(1,1,.5,0)} \vee \beta_{(1,1,.5,.5)} \vee \beta_{(1,1,.5,1)} \vee \beta_{(.5,0,.5,0)} \vee \\
&\beta_{(.5,0,.5,.5)} \vee \beta_{(.5,0,.5,1)} \vee \beta_{(.5,.5,.5,0)} \vee \beta_{(.5,.5,.5,.5)} \vee \beta_{(.5,.5,.5,1)} \vee \\
&\beta_{(.5,1,.5,0)} \vee \beta_{(.5,1,.5,.5)} \vee \beta_{(.5,1,.5,1)} \vee \beta_{(.5,0,1,.5)} \vee \beta_{(.5,.5,1,.5)} \vee \\
&\beta_{(.5,1,1,.5)} \vee \beta_{(0,0,.5,.5)} \vee \beta_{(0,0,.5,1)} \vee \beta_{(0,.5,.5,.5)} \vee \beta_{(0,.5,.5,1)} \vee \\
&\beta_{(0,1,.5,.5)} \vee \beta_{(0,1,.5,1)} \vee \beta_{(0,0,1,.5)} \vee \beta_{(0,.5,1,.5)} \vee \beta_{(0,1,1,.5)} \\
= \; & x_1 x_3 \vee x_3 x_4,
\end{aligned}$$

Similarly,

$$F_{cc} = x_1x_3 \vee x_3x_4 \vee x_1x_2 \vee x_2x_4.$$

In Table 7.9, we make sure that the both logic formulae satisfy the given f_0 for all $\boldsymbol{a}_i$ of A, i.e.,

$$F_{cc}(\boldsymbol{a}_i) = F_{cd}(\boldsymbol{a}_i) = f_0(\boldsymbol{a}_i) \quad \text{for all } \boldsymbol{a}_i \text{ of } A.$$

thus

$$D(F_{cc}, f_0) = D(F_{cd}, f_0) = 1.0.$$

We remark that the logic formulae represents P-fuzzy switching functions, i.e., both satisfy the uniformity. Hence, these contain no complementary phrases. Thus, the result F_{cd} can be interpreted as "*the team would win if and only if it has powerful batters (x_1) and pitchers (x_3) or good pitchers (x_3) and great manager (x_4).*" Indeed, the expert who gives the estimation data f_0 mentions that the factor of pitching is the most important, which agrees with the result that can be transformed as

$$F_{cd} = x_3(x_1 \vee x_4).$$

We therefore believe that our fitting algorithm can be used for knowledge acquisition by means of logical expression.

If the expert does not know about baseball, a logic formula could not be unique, or no logic formula could exist because of a contradiction derived from a lack of information. In the sense, the total distance implies the degree of unknown.

7.1.4 Conclusion

We have demonstrated how to work the fitting algorithm of fuzzy switching function by a practical example of a winning possibility of baseball teams, and have shown the logic formulae that approximate the knowledge of an expert.

Table 7.3: Restrictions and distances

Teams	$\boldsymbol{c}$	f_c^1	$D_{f_c^1}$	f_c^0	$D_{f_c^0}$
	$(.5, 1, 1, 1)$	$x_2x_3x_4$	$\lvert .5 - .6 \rvert$	$\overline{x_2x_3x_4}$	$\lvert .5 - .4 \rvert$
T	$(.5, .5, 1, .5)$	x_3	$\lvert .5 - .8 \rvert$	$\overline{x_3}$	$\lvert .5 - .2 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .5 - 1 \rvert$	0	$\lvert .5 - 0 \rvert$
	$(1, 1, 1, 1)$	$x_1x_2x_3x_4$	$\lvert .7 - .6 \rvert$	$\overline{x_1x_2x_3x_4}$	$\lvert .7 - .4 \rvert$
G	$(.5, .5, 1, 1)$	x_3x_4	$\lvert .7 - .7 \rvert$	$\overline{x_3x_4}$	$\lvert .7 - .3 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .7 - 1 \rvert$	0	$\lvert .7 - 0 \rvert$
	$(.5, 1, 1, 1)$	$x_2x_3x_4$	$\lvert .5 - .6 \rvert$	$\overline{x_2x_3x_4}$	$\lvert .5 - .4 \rvert$
C	$(.5, 1, .5, .5)$	x_2	$\lvert .5 - .7 \rvert$	$\overline{x_2}$	$\lvert .5 - .3 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .5 - 1 \rvert$	0	$\lvert .5 - 0 \rvert$
	$(1., 5, 1, 1)$	$x_1x_3x_4$	$\lvert .7 - .6 \rvert$	$\overline{x_1x_3x_4}$	$\lvert .7 - .4 \rvert$
S	$(.5, .5, 1, 1)$	x_3x_4	$\lvert .7 - .7 \rvert$	$\overline{x_3x_4}$	$\lvert .7 - .3 \rvert$
	$(.5, .5, .5, 1)$	x_4	$\lvert .7 - .8 \rvert$	$\overline{x_4}$	$\lvert .7 - .2 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .7 - 1 \rvert$	0	$\lvert .7 - 0 \rvert$
Y	$(0, .5, .5, 1)$	$\overline{x_1}x_4$	$\lvert .4 - .6 \rvert$	$\overline{\overline{x_1}x_4}$	$\lvert .4 - .4 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .4 - 1 \rvert$	0	$\lvert .4 - 0 \rvert$
	$(.5, 0, 0, 0)$	$\overline{x_2}\,\overline{x_3}\,\overline{x_4}$	$\lvert .3 - .6 \rvert$	$\overline{\overline{x_2}\,\overline{x_3}\,\overline{x_4}}$	$\lvert .3 - .4 \rvert$
D	$(.5, .5, .5, 0)$	$\overline{x_4}$	$\lvert .3 - .7 \rvert$	x_4	$\lvert .3 - .3 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .3 - 1 \rvert$	0	$\lvert .3 - 0 \rvert$
L	$(1, 1, 1, 1)$	$x_1x_2x_3x_4$	$\lvert .8 - .8 \rvert$	$\overline{x_1x_2x_3x_4}$	$\lvert .8 - .2 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .8 - 1 \rvert$	0	$\lvert .8 - 0 \rvert$
	$(1, 1, 1, 1)$	$x_1x_2x_3x_4$	$\lvert .6 - .6 \rvert$	$\overline{x_1x_2x_3x_4}$	$\lvert .6 - .4 \rvert$
B	$(1, .5, .5, .5)$	x_1	$\lvert .6 - .8 \rvert$	$\overline{x_1}$	$\lvert .6 - .2 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .6 - 1 \rvert$	0	$\lvert .6 - 0 \rvert$
F	$(1, .5, 1, .5)$	x_1x_3	$\lvert .4 - .6 \rvert$	$\overline{x_1x_3}$	$\lvert .4 - .4 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .4 - 1 \rvert$	0	$\lvert .4 - 0 \rvert$
	$(1, 1, 1, 1)$	$x_1x_2x_3x_4$	$\lvert .5 - .6 \rvert$	$\overline{x_1x_2x_3x_4}$	$\lvert .5 - .4 \rvert$
H	$(.5, .5, 1, 1)$	x_3x_4	$\lvert .5 - .7 \rvert$	$\overline{x_3x_4}$	$\lvert .5 - .3 \rvert$
	$(.5, .5, .5, 1)$	x_4	$\lvert .5 - 8 \rvert$	$\overline{x_4}$	$\lvert .5 - 2 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .5 - 1 \rvert$	0	$\lvert .5 - 0 \rvert$
M	$(1, 0, .5, 1)$	$x_1\overline{x_2}x_4$	$\lvert .4 - .6 \rvert$	$\overline{x_1\overline{x_2}x_4}$	$\lvert .4 - .4 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .4 - 1 \rvert$	0	$\lvert .4 - 0 \rvert$
	$(0, .5, .5, 0)$	$\overline{x_1}\,\overline{x_4}$	$\lvert .3 - 6 \rvert$	$\overline{\overline{x_1}\,\overline{x_4}}$	$\lvert .3 - .4 \rvert$
X	$(.5, .5, .5, 0)$	x_4	$\lvert .3 - .7 \rvert$	$\overline{x_4}$	$\lvert .3 - .3 \rvert$
	$(.5, .5, .5, .5)$	1	$\lvert .3 - 1 \rvert$	0	$\lvert .3 - 0 \rvert$

Table 7.4: Fuzzy switching function f^1

$x_2\backslash$	x_1	0			0.5			1		
	$x_4\backslash x_3$	0	.5	1	0	.5	1	0	.5	1
	0				0					
0	.5									
	1			1			1		0	
	0		0			0				
.5	.5					.5	.5		.5	0
	1		0			.5	1/.5			1
	0									
1	.5					.5				
	1						1			1

Table 7.5: Minimum combination of $r_1^k, \ldots, r_{12}^k$

Teams	r_i^k	r_i^k	$D(r_i^k, f_0)$
T	$f^1_{(.5,1,1,1)}$	$x_2x_3x_4$	0.1
G	$f^1_{(.5,.5,1,1)}$	x_3x_4	0
C	$f^1_{(.5,1,1,1)}$	$x_2x_3x_4$	0.1
S	$f^1_{(.5,.5,1,1)}$	x_3x_4	0
Y	$f^{.5}_{(0,.5,.5,1)}$	$\overline{x_1}x_2\overline{x_2}x_4$	0.2
D	$f^0_{(.5,0,0,0)}$	$x_2 \vee x_3 \vee x_4$	0.1
L	$f^1_{(1,1,1,1)}$	$x_1x_2x_3x_4$	0
B	$f^1_{(1,1,1,1)}$	$x_1x_2x_3x_4$	1
F	$f^1_{(1,.5,1,.5)}$	x_1x_3	0.2
H	$f^1_{(.5,.5,1,1)}$	x_3x_4	0.2
M	$f^{.5}_{(1,0,.5,1)}$	$\overline{x_1} \vee x_2 \vee x_3 \vee \overline{x_3} \vee \overline{x_4}$	0.1
X	$f^0_{(0,.5,.5,0)}$	$x_1 \vee x_4$	0.2
A	f^k	–	1.0

Table 7.6: Fuzzy switching function f^*

$x_2\backslash$	x_1	0			0.5			1		
	$x_4\backslash x_3$	0	.5	1	0	.5	1	0	.5	1
0	0	0	0	0	0			0	.5	1
	.5		.5			.5			.5	1
	1			1			1		.5	1
.5	0	0	0	0		.5				1
	.5		.5			.5	.5		.5	1
	1		.5	1		.5	1			1
1	0	0	0	0						1
	.5		.5	.5		.5				1
	1			1			1			1

Table 7.7: The minimum fuzzy switching function f_{cd}

$x_2\backslash$	x_1	0			0.5			1		
	$x_4\backslash x_3$	0	.5	1	0	.5	1	0	.5	1
0	0	0	0	0	0	.5	0	0	.5	1
	.5	0	.5	.5	0	.5	.5	0	.5	1
	1	0	.5	1	0	.5	1	0	.5	1
.5	0	0	0	0	0	.5	.5	0	.5	1
	.5	0	.5	.5	0	.5	.5	0	.5	1
	1	0	.5	1	0	.5	1	0	.5	1
1	0	0	0	0	0	.5	.5	0	.5	1
	.5	0	.5	.5	0	.5	.5	0	.5	1
	1	0	.5	1	0	.5	1	0	.5	1

Table 7.8: The maximum fuzzy switching function f_{cc}

$x_2 \backslash$	x_1	0			0.5			1		
	$x_4 \backslash x_3$	0	.5	1	0	.5	1	0	.5	1
0	0	0	0	0	0	.5	.5	0	.5	1
	.5	0	.5	.5	0	.5	.5	0	.5	1
	1	0	.5	1	0	.5	1	0	.5	1
.5	0	0	0	0	.5	.5	.5	.5	.5	1
	.5	.5	.5	.5	.5	.5	.5	.5	.5	1
	1	.5	.5	1	.5	.5	1	.5	.5	1
1	0	0	0	0	.5	.5	.5	1	1	1
	.5	.5	.5	.5	.5	.5	.5	1	1	1
	1	1	1	1	1	1	1	1	1	1

Table 7.9: Distances to f_0

	f_{cd}	f_{cc}	f_0	$D_{f_{cd}}$
T	.6	.6	.5	.1
G	.7	.7	.7	0
C	.6	.6	.5	.1
S	.7	.7	.7	0
Y	.5	.5	.4	.1
D	.4	.4	.3	.1
L	.8	.8	.8	0
B	.6	.6	.6	0
F	.6	.6	.4	.2
H	.7	.7	.5	.2
M	.5	.5	.4	.1
X	.4	.4	.3	.1
	total			1.0

Chapter 8

Conclusion

We have studied knowledge acquisition based on fuzzy switching functions and some classes of multiple-valued functions, and investigated partially specified fuzzy switching functions. These results make it possible to extract essential information from incomplete and uncertain knowledge, and to identify a whole mapping with a logic formula. This is the first attempt to consider a fuzzy switching function as a method for approximate reasoning.

The necessary and sufficient condition in order for a restriction to be a fuzzy switching functions (Theorem 3.5) and the necessary and sufficient condition for fuzzy switching functions to be uniquely determined by a restriction (Theorem 3.6) have been clarified. We can see in a finite number of steps whether a given restriction has a solution as a fuzzy switching function, and whether the solution is determined uniquely or not. From the point of view of inference systems, this works much more effectively than conventional approximate methods that involves trial and error.

We have studied the properties of P-fuzzy switching functions, and clarified the necessary and sufficient conditions for restrictions to be P-consistent in Theorem 4.5 and to be P-unique in Theorem 4.6. These conditions are useful for automatically deriving knowledge represented as simple logic formula from any given learning data. We also described a way to represent P-fuzzy switching functions from any P-unique restriction in Theorem 4.8.

We have studied the partially specified Kleenean functions, mappings representable by logic formula that contains any constant truth values of $[0, 1]$, and have solved the identification problem of Kleenean functions. We have clarified the necessary and sufficient condition for existence of Kleenean function is non-empty possibilities defined by a given f in Theorem 5.7. The results provide a robust logic description of knowledge.

We have proposed an algorithm to find the fuzzy switching function with the shortest distance to a given partial function. We have proved that the division of the uncertain knowledge into some Q-equivalent classes is sufficient to calculate all possible local distances in Theorem 6.1, and have shown some useful properties with respect to restrictions of fuzzy switching functions. The shortest path algorithm can work with a large value of n on weighted graph G which corresponds to all possible n-variable fuzzy switching functions. We have also estimated the algorithm and shown that it can reduce computation in finding the fuzzy switching function with the shortest distance for all others.

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