Research on Special Classes of Multiple-Valued Logic Functions Based on Fuzzy Logic

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Chapter 1

Introduction

Throughout the orthodox mainstream of the development of logic in the West, the prevailing view was that every proposition is either true or else false. The thesis is commonly called *Law of Bivalence*, and everyone knows that binary logic (or classical logic) is based on *Law of Bivalence.* Binary logic has led us to the rapid development of modern science. They say that classical logic was proposed by Aristotle in the ancient Greek era, and it is assumed in classical logic that every proposition has to take truth values either true (1) or else false (0).

In contrast to classical logic, there are other logical systems permitted to take truth values besides true (1) and false (0), and such logical systems are commonly called *multiple-valued logics.* For the first time, the requirement of multiple-valued logic was suggested from the philosophical point of view by Aristotle who is also the originator of classical logic. He claimed that propositions about future events are neither actually true nor actually false but are potentially either one of them, hence their truth value is undetermined at least prior to the event. He also considered about propositions containing the concepts of possibility, impossibility, contingency and necessity. This kind of idea is nothing but modal logic of today, however, of course he did not systematize such a logic concretely. Many multiple-valued logics have been proposed and studied by researchers [37], however, the actual inauguration of multiple-valued logic must be dated from the pioneering papers of the Pole Jan Lukasiewicz and the American Emile L. Post, published in the early 1920s, in which the first developed systematization of multiple-valued logic were presented.

In 1920, Lukasiewicz proposed a ternary logic on the set of truth values $\{0, 1/2, 1\}$. In the ternary logic, each truth value is interpreted as follows; 1 means true, 0 means false and 1/2 means indeterminately third truth value besides true and false. He suggested that the third truth value 1/2 should be for propositions of future-contingent events.

In propositional logic, logic formulas are usually defined by finite application of variables and operations, and Lukasiewicz defined some operations in his ternary logic as the extension of binary logic as below.

$$x \wedge y = \min(x, y), \qquad x \vee y = \max(x, y),$$
$$x \Rightarrow y = \min(1, 1 - x + y), \quad \sim x = 1 - x$$

The law of the excluded middle ($x \vee \sim x = 1$) in classical logic is widely different from Lukasiewicz ternary logic, that is, the law of the excluded middle does not generally hold in his logic. Among almost all of multiple-valued logics other than Lukasiewicz's one, the law of the excluded middle usually does not hold neither, and so we might say that the law peculiarly holds in classical logic. It is very interesting to remark that Lukasiewicz ternary logic is closely related with intutionistic logic of Brouwer and modal logic of Lewis.

After the proposal of Lukasiewicz ternary logic, in 1921 Post proposed m-valued logic on the set of truth values $\{1, 2, \ldots, m\}$. He defined three kinds of operations $\wedge$, $\vee$, $\sim$, and two of them ($\wedge$, $\vee$) are same definitions of Lukasiewicz's logic and the operations $\sim$ is defined as below [34].

$$\sim x = \begin{cases} x+1 & \text{if } x \neq m \\ 1 & \text{if } x = m \end{cases}$$

Especially, Post proved that any n-variable m-valued function $F : \{1, 2, \ldots, m\}^n \to \{1, 2, \ldots, m\}$ can always be represented by means of a logic formula of his logic which is constructed by n variables and only two operations $\vee$ and $\sim$, that is, Post's m-valued logic is functionally complete for m-valued functions. Here, *functional completeness* is defined as follows; a set of operations $\{f_1, \cdots, f_s\}$ is said to be functionally complete for m-valued functions $F : \{1, 2, \ldots, m\}^n \to \{1, 2, \ldots, m\}$ if and only if any m-valued function can be defined in terms of these operations $f_1, \cdots, f_s$ and variables taking m values. By the way, it is evident from easy verification that Lukasiewicz ternary logic is not functionally complete for ternary functions $F : \{0, 1/2, 1\}^n \to \{0, 1/2, 1\}$ [28].

Here, we should pay attention to the difference of the motivations between Lukasiewicz and Post. That of Lukasiewicz who was a logician is that multiple-valued logic should have some kind of meaning, for example Lukasiewicz tried to treat propositions of future-contingent in his ternary logic. Therefore, as a natural consequence, binary logic has to be contained in his multiple-valued logic as its special case. That is, in the case, any logic formula has to take 0 or 1 as its truth value if any assignment to variables is restricted to the orthodox truth values 0 and 1. In contrast, mathematician Post studied multiple-valued logic from the point of view of functionally completeness. Classical logic treating only the orthodox truth values 0 and 1 can represent any two-valued function $F : \{0, 1\}^n \to \{0, 1\}$ as its logic formulas, that is, classical logic is functionally complete for two-valued functions. The Post's motivation of multiple-valued logic was based on this concept, that is, his subject was to investigate what kinds of operations of m-valued logic can represent any m-valued function $F : \{1, 2, \ldots, m\}^n \to \{1, 2, \ldots, m\}$.

Table 1.1: Operations of Bochvar's Ternary Logic

p \ q	$p \wedge q$: 0	1/2	1	$p \vee q$: 0	1/2	1	$p \Rightarrow q$: 0	1/2	1	$p \Leftrightarrow q$: 0	1/2	1	$\sim p$
0	0	1/2	0	0	1/2	1	1	1/2	1	1	1/2	0	1
1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
1	0	1/2	1	1	1/2	1	0	1/2	1	0	1/2	1	0

Table 1.2: Operations of Kleene's Ternary Logic

p \ q	$p \wedge q$: 0	1/2	1	$p \vee q$: 0	1/2	1	$p \Rightarrow q$: 0	1/2	1	$p \Leftrightarrow q$: 0	1/2	1	$\sim p$
0	0	0	0	0	1/2	1	1	1	1	1	1/2	0	1
1/2	0	1/2	1/2	1/2	1/2	1	1/2	1/2	1	1/2	1/2	1/2	1/2
1	0	1/2	1	1	1	1	0	1/2	1	0	1/2	1	0

Table 1.3: Operations of Gödel's Ternary Logic

q \ p	$p \wedge q$: 0	1/2	1	$p \vee q$: 0	1/2	1	$p \Rightarrow q$: 0	1/2	1	$p \Leftrightarrow q$: 0	1/2	1	$\sim p$
0	0	0	0	0	1/2	1	1	1	1	1	0	0	1
1/2	0	1/2	1/2	1/2	1/2	1	0	1	1	0	1	1/2	0
1	0	1/2	1	1	1	1	0	1/2	1	0	1/2	1	0

The investigations related with multiple-valued logics can be classified roughly into two types, that is, some of them are in the type of Lukasiewicz and the others are in the type of Post. Of course, we can't clearly classify the studies related with multiple-valued logic. We first describe some of studies based on Lukasiewicz's concept and then show those based on Post's.

Since the proposal of Lukasiewicz ternary logic, many multiple-valued (propositional) logics have been proposed and studied along the line of Lukasiewicz's work [37]. Especially, the studies of Kurt Gödel, D. A. Bochvar and Stephen C. Kleene are of importance. Table 1.1, 1.2 and 1.3 show truth tables of operations of Bochvar, Kleene and Gödel (we already showed the definition of Lukasiewicz's operations).

The studies of Bochvar and Kleene are of particular importance because they developed 3-valued systems of logic which, while fundamentally different from the system of Lukasiewicz, have significant applications in metamathematics. In ternary logic of Bochvar, the third truth value 1/2 is interpreted as paradoxical or meaningless, and that of Kleene is interpreted as unknown.

It is one of the investigations of logic to ask the existence of tautologies in a logical system, and if tautologies exist in the logic then which of logic formulas should be tautologies. Here, a tautology is a logic formula that uniformly takes on the truth value 1 for all assignments of truth values to the variables. If a logical system has a tautology, then generally it has infinite number of tautologies.

Now, let us describe axiomatization of a logical system which enables us to treat such infinite number of tautologies with finite concepts. An axiom system is constructed by two concepts, that is, some logic formulas called axioms which are tautologies of the given logical system, and an inference rule (modus ponens and substitution) that enables us to derive a new logic formula from another logic formulas that have been derived already or are axioms. In an axiom system, every meaning of symbols appearing in logic formulas are ignored, and one only pays attention to the symbols and derives a new logic formula by applying the inference rules. For example, Lukasiewicz ternary logic is axiomatized in the following manner, where α, β and γ imply any logic formulas and the operations $\wedge$ and $\vee$ can be defined as $x \vee y = (x \Rightarrow y) \Rightarrow y$, $x \wedge y = \sim(\sim x \vee \sim y)$ and $x \Leftrightarrow y = (x \Rightarrow y) \wedge (y \Rightarrow x)$ using operations $\Rightarrow$ and $\sim$.

<*Axioms*>

(1) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$

(2) $(\alpha \Rightarrow \beta) \Rightarrow \{(\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma)\}$

(3) $(\sim \alpha \Rightarrow \sim \beta) \Rightarrow (\beta \Rightarrow \alpha)$

(4) $\{(\alpha \Rightarrow \sim \alpha) \Rightarrow \alpha\} \Rightarrow \alpha$

<*Inference Rules*>

Modus Ponens: β is derived from $\alpha \Rightarrow \beta$ and α.

Substitution: Every occurrence of a (propositional) variable can be simultaneously replaced by a logic formula.

Here, the following is important to construct an axiom system from a logical system; it has to be proved that any logic formula derived from axioms and inference rules always has to be a tautology in the given logical system, and conversely any tautology always has to be derived as a logic formula from axioms and inference rules. It is interesting that we can treat infinite number of tautologies of a logical system with finitely concepts as axiom system. Moreover, it is easy to perform the axiom system on computer since the deriving method of new logic formulas only depends on automatic manipulation, and therefore, this concept is important for the application to computer science.

Next, we show studies related with Post's concept.

Historically, the functionally completeness problem was first studied in $\boldsymbol{O}_2$, here $\boldsymbol{O}_2$ means the set of all 2-valued functions, that is, $\boldsymbol{O}_2 = \bigcup_{n=0}^{\infty} \boldsymbol{O}_2^{(n)}$ where $\boldsymbol{O}_2^{(n)} = \left\{ f \mid f : \{0,1\}^n \to \{0,1\} \right\}$. Although several functionally complete systems were known earlier, a general completeness criterion was given by E. Post in 1921 [34]. In the investigations for functional completeness, the most natural problem is the characterization of subsets $X \subseteq \boldsymbol{O}_k$ such that $[X] = \boldsymbol{O}_k$, where $\boldsymbol{O}_k = \bigcup_{n=0}^{\infty} \boldsymbol{O}_k^{(n)}$ and $\boldsymbol{O}_k^{(n)} = \left\{ f \mid f : \{0,1,\ldots k\}^n \to \{0,1,\ldots,k\} \right\}$, and $[X]$ is set of all functions from $\boldsymbol{O}_k$ which can be obtained as finitely compositions of functions from X. Post's criterion, which has been rediscovered many times, is most naturally expressed in terms of maximal classes. A closed class $M \subsetneq \boldsymbol{O}_k$ is said to be maximal in $\boldsymbol{O}_k$ if M can't be properly extended to a closed proper subclass of $\boldsymbol{O}_k$. Then, we have the following simple but basic fact [38].

> *A set $X \subseteq \boldsymbol{O}_k$ is functionally complete in $\boldsymbol{O}_k$ if and only if X is a subset of no maximal class in $\boldsymbol{O}_k$, or in other words, X is functionally complete if and only if to every maximal class M in $\boldsymbol{O}_k$ the set X contains f not belonging to M.*

Also, we know that which closed class can be a maximal class in $\boldsymbol{O}_k$, and the number of maximal classes is clarified by [39].

Great importance attaches to the use of multiple-valued logics for the formal analysis of electronic circuitry and in switching theory. Here the pioneering work is by Claude E. Shannon in 1938. Unfortunately, in those days, it was very hard to realize multiple-valued signal as the stable and simple electronic circuits by utilization devices, and therefore, it was difficult to show the ability of multiple-valued logic circuits more than two-valued one. However, since 1970s in which many investigations have been started for electronic circuits based on semiconductor technology, it has been cleared that multiple-valued logic is effective for digital information processing. The following are some of reasons why multiple-valued signal processing is interested in.

(i) Possibility of new hardware implementations beyond the limitation of digital IC technology. Especially, interconnection problem has been recognized to be a basic limitation in present-day VLSI system [42], [12].

(ii) Expectation for new devices suitable for multiple-valued signal processing. Especially, a model of molecular switching devices based on enzyme-substrate reaction is recently proposed to construct a new type of logic network [1]. The reaction of enzyme-substrate reaction is a good example of new devices suitable for multiple-valued signal processing.

In investigations of multiple-valued signal processing, each truth value merely represents a symbol carrying on some information, that is, no meaning attaches to each truth value. Therefore, adopted operations in multiple-valued signal processing should be functionally complete since they enable us to represent any function. The above investigations seem to be related with Post's concept.

Multiple-valued logics and mathematics, especially, algebra and topology, are closely related with each other. Various logicians have attempted to construct algebraic or topological models of multiple-valued logics. The relationships between multiple-valued logics and lattices have been studied by A. Rose, who has been among the most active contributors to the field generally, as has G. C. Moisil. Moreover, in multiple-valued logic, Post's m-valued logic was systematized as Post algebras in the lattice theory by G. Epstein in 1960 as same as binary logic was systematized as Boolean algebras in the lattice theory. Therefore, multiple-valued logic closely connects with lattice theory when they are regarded as algebraic systems.

By the way, since the concept of fuzzy sets, which is a means of handling fuzziness as quantity, is proposed by L. A. Zadeh in 1965 [52], infinite-valued logic functions permitted to take truth values 0 and 1 have been investigated again in the field of engineering, in the name of fuzzy logic functions. Fuzzy logic functions are extension of Kleene's ternary logic [16] into the closed interval $[0,1]]$, however, it should mention specially that such kind of concept arose in the field of engineering independently of the field of logic. By the way, fuzzy set theory is recently axiomatized in terms of intuitionistic logic [43], [44].

The investigations of fuzzy logic functions in the field of engineering are as follows; applications of fuzzy logic functions to logical circuits [19], the studies of fundamental properties of them, especially validity, consistency and inconsistency of fuzzy logic formulas and prime implicant expressions of them [17], resolution principle for fuzzy logic [18], [32], [40], minimization of fuzzy logic functions [41], [13], [14] [26], [30], a necessary and sufficient condition for fuzzy logic functions [27], the studies about an algebraic system of which the set of fuzzy logic functions is a typical example [29], and so on.

Ternary functions considering ambiguity have been proposed and studied as the name of B(Binaritic)-ternary logic functions by M. Mukaidono in 1972 [24], and B-ternary logic functions are essentially equivalent to Kleene's ternary logic. It is known that the set of fuzzy logic functions and that of B-ternary logic functions form lattices isomorphic to each other [25]. The properties of B-ternary logic functions can be applicable to detecting and correcting hazard in combinational circuits [19], to fail-safe theory which can be treated by a special subclass of B-ternary logic functions called C-type logic functions [23], to the prime implicant expressions of Boolean functions [22] (One can see the same result for the prime implicant expressions of Boolean functions in [22] and [33]). It is interesting that C-type logic functions are equivalent to Bochcar's ternary logic.

Ternary logic functions besides B-ternary logic functions, called regular ternary logic functions [31], have been studied in which the third truth value $1/2$ is interpreted as an indeterminate state, whereas $1/2$ of B-ternary logic functions is interpreted as the neutral state between 0 and 1.

The studies of this paper are along a series of investigations beginning from fuzzy logic functions in the field of engineering. The paper proposes and discusses some different kinds of multiple-valued logic functions, especially in mathematical aspects.

Chapter 3 is for preliminary of mathematical concepts concerning with the paper. Chapter 4 describes about fuzzy logic functions, B-ternary logic functions and regular ternary logic functions as the results obtained until now which directly relate with the investigations of the paper. Multiple-valued Kleenean functions are described in Chapter 4. They are defined as functions extended from regular ternary logic functions into m-valued ($m \geq 4$). In multiple-valued Kleenean functions, the set of truth values $\{0, 1/2, 1\}$ of regular ternary logic functions is expanded into the set of m-truth-value $\{0, 1/(m-1), \ldots, (m-2)/(m-1), 1\}$. The name of the functions originates a Kleene algebra of which the set of them takes the form. The unary operation $\sim$ of fuzzy logic functions is defined as $\sim x = 1 - x$, and in contrast, Chapter 5 shows multiple-valued logic functions given from fuzzy logic functions by replacing the unary operation $\sim$ with $\neg$ defined as follows:

$$\neg x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

The functions discussed in Chapter 5 are called Stone logic functions, and the set of Stone logic functions forms an algebraic system called a Stone algebra. Fuzzy logic functions with the unary operation $\neg$ are discussed in Chapter 6. As looking at an algebraic system, the functions in Chapter 6 form a Kleene-Stone algebra, and so we call them Kleene-Stone logic functions. Finally in Chapter 7, we describe fuzzy logic functions with the unary operation $\neg_\alpha$ defined below.

$$\neg_\alpha x = \begin{cases} 1 & \text{if } x < \alpha \\ 0 & \text{if } x \geq \alpha \end{cases}$$

where α is an element of the open set $(0, 1/2)$. Unfortunately, there has been no discussion about an algebraic system forming the set of functions.

Kleene algebras, Stone algebras and Kleene-Stone algebras are different kinds of concepts to each other, and their axioms will appear in each chapter, respectively.

The following subjects will be described in each Chapter 4 $\sim$ 7.

(1) A partial order relation and each kinds of multiple-valued logic functions are closely connected to each other, and their fundamental properties will be cleared in terms of partial order relations. Note that there are two different kinds of relations in each Chapter 4 $\sim$ 7, one of them is for operating among truth values (that is, the linear order) and the other one is for describing properties of the functions.

(2) Each kind of functions is defined as a function $F : [0, 1]^n \to [0, 1]$ represented by a logic formula. An algorithm deriving a special kind of disjunctive form, called canonical disjunctive form, is presented, and the disjunctive form enables us to uniquely determine a give function.

(3) A necessary and sufficient condition for each kind of functions will be shown in term of a partial order relation. A necessary and sufficient condition is initially obtained in the form of some conditions, and therefore, we describe the relationship among the conditions. However, this discussion is not applied to Chapter 4.

(4) An algorithm that derives a minimal form for a given function is discussed. The definition of minimal form is motivated by Boolean functions [35], [36], [15], [20]. However, this argument is not applied to Chapter 7.

(5) The number of n-variable functions is discussed, except for Chapter 7. The number of Kleenean functions described in Chapter 4 has been discussed in [9] and [10].

In [52], L. A. Zadeh originally adopted *min* and *max* as intersection and union of fuzzy sets, and in a series of investigations for fuzzy set theory, various pairs of operations have been proposed as what they correspond to intersection and union of fuzzy sets. The following list shows some of such pairs of operations. Triangular norms (t-norms for short) and triangular conorms (t-conorms)

Table 1.4: Some Examples of T-Norms and T-Conorms

t-norms		t-conorms	
(1)	(logical product) $x \wedge_L y = \min(x, y)$	(1')	(logical sum) $x \vee_L y = \max(x, y)$
(2)	(Hamacher product) $x \wedge_H y = \dfrac{xy}{x+y-xy}$	(2')	(Hamacher sum) $x \vee_H y = \dfrac{x+y-2xy}{1-xy}$
(3)	(algebraic product) $x \wedge_A y = xy$	(3')	(algebraic sum) $x \vee_A y = x+y-xy$
(4)	(Einstein product) $x \wedge_E y = \dfrac{xy}{1+(1-x)(1-y)}$	(4')	(Einstein sum) $x \vee_E y = \dfrac{x+y}{1+xy}$
(5)	(bounded product) $x \wedge_B y = \max(0, x+y-1)$	(5')	(bounded sum) $x \vee_B y = \min(1, x+y)$

were introduced by Menge, and studied extensively by Schweizer and Sklar in the context of statistical metric spaces. There has recently been consensus to admit the concept of t-norms and t-conorms to represent pointwise fuzzy set-theoretical intersection and union, respectively. The following are definitions of t-norms and t-conorms. The pairs of operations listed above exactly satisfy the definitions of t-norms and t-conorms. Of course, there are many t-norms and t-conorms other than the above operations, however, we follow this kind discussions with the papers [45], [21].

Defintion of Triangular Norms

A function $t : [0,1] \times [0,1] \to [0,1]$ is called *t-norm* if and only if for any $a, b, c \in [0,1]$:

(t_1) $t(0,0) = 0$, $t(a,1) = t(1,a) = a$ (boundary condition)

(t_2) $t(a,b) \leq t(c,d)$ whenever $a \leq c$ and $b \leq d$ (monotonicity)

(t_3) $t(a,b) = t(b,a)$ (commutativity)

(t_4) $t(a,t(b,c)) = t(t(a,b),c)$ (associativity)

Definition of Triangular Conorms

A function $s : [0,1] \times [0,1] \to [0,1]$ is called *t-conorm* if and only if for any $a, b, c \in [0,1]$:

(s_1) $s(1,1) = 1$, $s(0,a) = s(a,0) = a$

(s_2) $s(a,b) \leq s(c,d)$ whenever $a \leq c$ and $b \leq d$

(s_3) $s(a,b) = s(b,a)$

(s_4) $s(a,s(b,c)) = s(s(a,b),c)$

Now, as mentioned before, multiple-valued logics and algebraic systems, especially lattice theory are closely related with each other. It is easy to verify that there is nothing to form a lattice but a pair of *min* and *max* among any pair of operations satisfying the definitions of t-norms and t-conorms [45]. In this paper, from the standpoint of respecting such elegant relationships between multiple-valued logics and lattice theory, *min* and *max* will be adopted as operations corresponding to AND and OR, respectively, throughout a series of proposed multiple-valued logic functions.

Note that all concepts and symbols defined in Chapter 2 are available througtout the paper, and any concepts and symbols defined in the remaining chapter, that is, from Chapter 3 to Chapter 7 are available only in itself.

Chapter 2

Preliminaries for Mathematical Concepts

The chapter is for preliminary of some of mathematical concepts concerning with the paper. Especially, the following concepts are denoted concisely: sets, relations (especially, equivalence relations and partial order relations) and lattices.

2.1 Sets

A collection of objects is called a *set*. A member of this collection is also called an *element* (or a *member*) of the set. Instead of saying that x is an element of a set A, we shall also write $x \in A$.

If A and B are sets, and if every element of B is an element of A, then we say that B is a *subset* of A. Observe that our definition of a subset does not exclude the possibility that $A = B$. If B is a subset of A, but $B \neq A$, then we shall say that B is a *proper subset* of A. To denote that the fact that B is a subset of A, we write $B \subseteq A$ or $A \supseteq B$.

If a set has no element, it is called the *empty set*, and denoted by $\emptyset$. Note that $\emptyset \subseteq A$ for every set A.

The set theoretic operations $\cup$, $\cap$, $-$ (they are called *union*, *intersection* and *difference*, respectively) have their usual meaning. If a set A is fixed, then for subsets B of A the *complement* B' of A is defined by $B - A$. By definition $B \cup (A - B) = A$ and $B \cap (A - B) = \emptyset$. Note that $B \subseteq A$ is equivalent to $B = B \cup A$, which, in turn, is equivalent to $A = B \cap A$. If $A \cap B = \emptyset$, we say that A and B are *disjoint*.

If a set A has finite elements, then we shall say it as a *finite set*, while it has infinite elements, then it is called an *infinite set*. Especially, the number of a finite set A is denoted by $|A|$.

If A is a set, then $\mathcal{P}(A)$ (called a *power set* of A) denotes the set of all subsets of A.

A partition π of A is a subset of $\mathcal{P}(A)$ not containing $\emptyset$, satisfying the following property: every $a \in A$ is an element of exactly one $B \in \pi$. The members of π are called *blocks* of the partition π. We use Part(A) for the set of all partition of A. If π_0 and π_1 are partition of A, we will write $\pi_0 \trianglelefteq \pi_1$, if for every block B of π_0, there exists a block C of π_1, with $B \subseteq C$. In this case π_0 is a *refinement* of π_1.

If A and B are sets, the *Cartesian product* or *direct product* $A \times B$ of A and B is defined as the set of all ordered pairs (a, b), with $a \in A$ and $b \in B$. In symbols, $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$, where $\mid$ reads "for those which satisfy". In general, if $A_1, A_2, \ldots, A_n$ are sets, then

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

Here, $(a_1, \ldots, a_n)$ is called an *n-tuple*. If $A_1 = A_2 = \ldots = A_n = A$, then we set

$$A^n = \underbrace{A \times \ldots \times A}_{n \text{ times}}.$$

2.2 Relations

For a positive integer n and for a set A, we define an n-ary relation r on A as a subset of A^n. If r is a relation on A and $A \subseteq B$, then we can consider r as a relation on B, since $r \subseteq A^n \subseteq B^n$.

We shall be particularly interested in binary relations. For a binary relation r, $(a, b) \in r$ will also be denoted by arb. If $(a, b) \notin r$, then we write it by $a \not r b$. If r is a binary relation on A, then the *inverse* r^{-1} of r is defined by the rule; $ar^{-1}b$ if and only if bra.

2.2.1 Equivalence Relations

A binary relation ε on A is defined to be an *equivalence relation* if the following three conditions hold for all $a, b, c \in A$.

(e_1) $a\varepsilon a$ (ε is *reflexive*),

(e_2) $a\varepsilon b$ implies $b\varepsilon a$ (ε is *symmetric*),

(e_3) $a\varepsilon b$ and $b\varepsilon c$ imply $a\varepsilon c$ (ε is *transitive*).

The set of all equivalence relations on A will be denoted by $E(A)$. For $\varepsilon_0, \varepsilon_1 \in E(A)$ we agree to write $\varepsilon_0 \trianglelefteq \varepsilon_1$ for $\varepsilon_0 \subseteq \varepsilon_1$.

We shall now state a theorem which relates partitions and equivalence relations.

Theorem 1 *(1) Let π be a partition of the set A and define a binary relation ε_π on A by $a\varepsilon_\pi b$ if and only if a and b are in the same block of the partition π. Then ε_π is an equivalence relation on A.*

(2) Let ε be an equivalence relation on A. For $a \in A$ set

$$A_a = \{b \mid b \in A \text{ and } a\varepsilon b\}.$$

Let π_ε be the set of all $B \subseteq A$ which are of the form A_a. Then, π_ε is a partition of A.

(3) If $\pi_0 \trianglelefteq \pi_1$, then $\varepsilon_{\pi_0} \trianglelefteq \varepsilon_{\pi_1}$. If $\varepsilon_0 \trianglelefteq \varepsilon_1$, then $\pi_{\varepsilon_0} \trianglelefteq \pi_{\varepsilon_1}$.

(4) $\pi = \pi_{(\varepsilon_\pi)}$ and $\varepsilon = \varepsilon_{(\pi_\varepsilon)}$.

If r is an n-ary relation on A and $B \subseteq A$, then $r|B = r \cap B^n$ is an n-ary relation on B. The relation $r|B$ is called the *restriction* of r to B.

Note that a restriction of an equivalence relation is always an equivalence relation.

Let ε be an equivalence relation on A and let π_ε be the corresponding partition (see Theorem 1.1). For $H \subseteq A$ set

$$[H]_\varepsilon = \{a \mid a \in A \text{ and } h\varepsilon a \text{ for some } h \in H\}.$$

This set is called the *closure* of H under ε. If $H = \{x\}$, we will write $[x]_\varepsilon$ for $[\{x\}]_\varepsilon$. By Theorem 1.1, for every $x \in A$, $[x]_\varepsilon \in \pi_\varepsilon$, the block $[x]_\varepsilon$ is called the *equivalence class* containing x. Thus $[H]_\varepsilon$ is the union of all blocks of π_ε which contain at least one element of H. The set constructing all equivalence classes is called the *quotient set* of A under ε, and it is usually denoted by A/ε, that is,

$$A/\varepsilon = \{[x]_\varepsilon \mid x \in A\}.$$

Of course, π_ε and A/ε are identical.

Example 1 *Let $\boldsymbol{Q}$ be the set of all rational numbers. Then, the relation $\equiv_k$, defined on $\boldsymbol{Q}$ in the following manner, is an equivalence relation.*

Let $k \in \boldsymbol{Q}$ and $m, n \in \boldsymbol{Q}$, then $m \equiv_k n$ if and only if $x \cdot k = m - n$ for some $x \in \boldsymbol{Q}$.

Indeed, for any $m \in \boldsymbol{Q}$, $m - n = 0 \cdot k$. This means $\equiv_k$ is reflexive. $m - n = d \cdot k$ implies $n - m = (-d) \cdot k$, and this means $\equiv_k$ is symmetric. The transitivity of $\equiv_k$ is shown as follows: if $l - m = d \cdot k$ and $m - n = d' \cdot k$, then $l - n = (l - m) + (m - n) = (d + d') \cdot k$.

When $k = 2$, we obtain two equivalence classes $[0]_2 = \{2n \mid n \in \boldsymbol{Q}\}$ and $[1]_2 = \{2n+1 \mid n \in \boldsymbol{Q}\}$.

2.2.2 Partial Order Relations

A binary relation $\leq_P$ defined of a set A is said to be a *partial order* on A if the following three conditions hold for every elements $a, b, c \in A$.

(o_1) $a \leq_P a$ ($\leq_P$ is *reflexive*)

(o_2) $a \leq_P b$ and $b \leq_P a$ imply $a = b$ ($\leq_P$ is *antisymmetric*)

(o_3) $a \leq_P b$ and $b \leq_P c$ imply $a \leq_P c$ ($\leq_P$ is *transitive*)

An *ordered set* is a pair $< A, \leq_P >$ where A is a non-empty set and $\leq_P$ is a partial order on A. We often denote the ordered set $< A, \leq_P >$ as A, if it is clear that the set A has the partial order relation $\leq_P$.

Clearly, if $\leq_P$ is a partial order on A and $A_0 \subseteq A$, then $\leq_P |A_0$ (i.e. the relation $\leq_P$ restricted to A_0) is a partial order on A_0.

An element a of an ordered set $< A, \leq_P >$ is said to be *maximal* (*minimal*) if there is no element b in A such that $a \leq_P b$ ($b \leq_P a$) and $a \neq b$. An ordered set can have several maximal or minimal elements.

An element a of an ordered set $< A, \leq_P >$ is said to be *maximum* (*minimum*) if for every $x \in A$, $x \leq_P a$ ($a \leq_P x$). It follows from this definition and (o_3) that an ordered set can have at most one maximum element and at most one minimum element. The maximum (minimum) element of an ordered set, if it exists, will be denoted by 1 (by 0).

An element a of an ordered set $< A, \leq_P >$ is said to be *an upper* (*a lower*) *bound* of a non-empty subset A_0 of A if $b \leq_P a$ ($a \leq_P b$) for every $b \in A_0$. If the set of all upper (lower) bounds of A_0 contains a minimum (maximum) element, then this element is said to be the *supremum* of A_0 (the *infimum* of A_0) and is denoted by $\sup A_0$ ($\inf A_0$). It follows from this definition that $\sup A_0 = a$ ($\inf A_0 = a$) if and only if the following conditions are satisfied

(1) $b \leq_P a$ ($a \leq_P b$) for every $b \in A_0$,

(2) if $c \in A$ and $b \leq_P c$ ($c \leq_P b$) for ever $b \in A_0$, then $a \leq_P c$ ($c \leq_P a$).

A partial order $\leq_P$ on a set A is called a *linear order* if the following condition is satisfied

(o_4) $x \leq_P y$ or $y \leq_P x$ for every $x, y \in A$.

A finite ordered set $< A, \leq_P >$ can be represented by a diagram called *Hasse diagram.* By "x covers y" in an ordered set $< A, \leq_P >$ it implies that $x \leq_P y$ but that for no $t \in A$ we can have $x \leq_P t$ and $t \leq_P y$. Each element of A is represented by a small circle, placing the circle representing x higher than the one representing y whenever $x \leq_P y$. A line segment will be drawn from x to y when x covers y.

Example 2 *Let M be a set of positive natural numbers. Consider the relation $|$, where $m|n$ if m divides n. It is clear that $|$ is reflexive and transitive. To verify the antisymmetry assume that $m|n$ and $n|m$. In this case we shall have $p, q \in \{0, 1, \ldots, n, \ldots\}$ such that $m = pn$ and $n = qm$. Since $m, n \neq 0$ we have $pq = 1$, which gives $p = q = 1$. Therefore, $m = n$ and this proves that $|$ is indeed a partial order.*

If $M = \{1, 2, \ldots, 12\}$, then the Hasse diagram of the ordered set $< M, | >$ is represented in Figure 2.1.

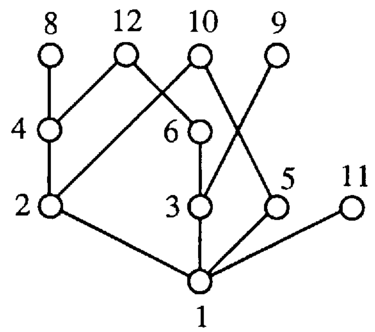

Figure 2.1: Hasse Diagram of $< M, | >$

2.3 Mappings and Operations

Given two sets A and B and a binary relation φ on $A \times B$, we call φ a *mapping* (or a *function*) of A into B if $(a, b) \in \varphi$ only if $a \in A$ and $b \in B$, and for every $a \in A$, there exits exactly one $b \in B$ satisfying $(a, b) \in \varphi$. This element b is called the *image* of the element a under the mapping φ and a is called an *inverse-image* of b under φ. For a mapping φ we introduce the notations $\varphi(a) = b$, and we write $\varphi : A \rightarrow B$ to indicate that φ is a mapping of A into B. A is called the *domain* of φ, in notation $D(\varphi) = A$. If the inverse relation is also a mapping, we will denote it by φ^{-1}. We set

$$\varphi(A) = \{b \in B \mid \text{there exists an } a \in A \text{ with } \varphi(a) = b\}$$

and we call $\varphi(A)$ the *image* of A under φ.

It is easily seen that if φ is a mapping of A into B and $C \subseteq A$, then $\varphi \cap (C \times B)$ is a mapping of C into B, denoted by $\varphi|C$, and called the *restriction* of φ to C.

The mapping $\varphi : A \rightarrow B$ is called *onto* if $\varphi(A) = B$, while it called *one-to-one* if every element of B has at most one inverse-image, that is, if $\varphi(a_0) = \varphi(a_1)$ $(a_0, a_1 \in A)$ implies $a_0 = a_1$.

Let A be a set and n a nonnative integer. An n-ary operation on the set A is a mapping f of A^n into A. Thus an n-ary operation assigns to every n-tuple $(a_1, \ldots, a_n)$ of elements of A^n a unique element of A, which will be denoted by $f(a_1, \ldots, a_n)$.

An n-ary operation f on A can also be described by an $(n+1)$-ary relation r defined by

$$(a_1, \ldots, a_n, a) \in r \text{ if and only if } f(a_1, \ldots, a_n) = a$$

We observe that a 0-ary (nullary) operation is a mapping $f : \{\emptyset\} \rightarrow A$, which is determined by the single image element of $\emptyset$, $f(\emptyset) \in A$.

An *algebra* is a pair $< A, F >$, where A is a non-empty set and F is a family of *finitely* operations on A. When F is finite, $F = \{f_0, \ldots, f_{n-1}\}$, we denote the algebra $< A, F >$ by $< A, f_0, \ldots, f_{n-1} >$.

2.4 Lattices

An algebra $< L, \vee >$ with one binary operation is said to be a *semilattice* if the following equations are satisfied for all $a, b, c \in L$.

(s_1) (Commutative) $a \vee b = b \vee a$,

(s_2) (Associative) $a \vee (b \vee c) = (a \vee b) \vee c$,

(s_3) (Idempotent) $a \vee a = a$.

A partial order can be attached to any semilattice $< L, \vee >$ by defining $x \leq_\vee y$ if and only if $x \vee y = y$. It is easy to verify that $\leq_\vee$ is reflexive, antisymmetric and transitive.

Moreover, under this partial order any set of two elements of L and, in general, any finite subset of L will have a least upper bound. Indeed, it is clear that $x \leq_\vee x \vee y$ and $y \leq_\vee x \vee y$ since $x \vee (x \vee y) = y \vee (x \vee y) = x \vee y$. On the other hand, if $x \leq_\vee z$ and $y \leq_\vee z$ we shall have $x \vee z = y \vee z = z$ and this gives that $(x \vee y) \vee z = z$, as it can be easily verified. This, in turn, gives $x \vee y \leq_\vee z$ showing that $x \vee y$ is $\sup\{x, y\}$.

An algebra $< L, \vee, \wedge >$ with two binary operations is said to be a *lattice* provided the following equations are satisfied for all $a, b, c \in L$.

(l_1) (Commutative) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,

(l_2) (Associative) $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

(l_3) (Idempotent) $a \vee a = a$, $a \wedge a = a$,

(l_4) (Absorption) $(a \wedge b) \vee b = b$, $a \wedge (a \vee b) = a$.

A lattice can be regarded as consisting from two semilattice operations linked together by the absorption laws and each such operation is defining a partial order. Namely, we have $x \leq_\vee y$ if $x \vee y = y$ and $x \leq_\wedge y$ if $x \wedge y = y$. Using the absorption laws, if $x \leq_\vee y$, that is, if $x \vee y = y$ then $x \wedge y = x \wedge (x \vee y) = x$ and this shows that $y \leq_\wedge x$. Similarly, one can prove that $x \leq_\wedge y$ implies $y \leq_\vee x$. This shows that $\leq_\vee$ and $\leq_\wedge$ are *dual* partial orders and therefore we will choose $\leq_\vee$ to represent the partial order induced by the lattice structure.

We quote without proof the following theorem.

Theorem 2 *If $< L, \vee, \wedge >$ is a lattice, then for all $a, b \in L$*

(1) $a \vee b = b$ if and only if $a \wedge b = a$.

The relation $\leq_\vee$ on L, defined as follows

(2) $a \leq_\vee b$ if and only if one of the equations (1) holds,

is a partial order on L, called the lattice order on L. Moreover,

(3) $a \vee b = \sup\{a,b\}$, $a \wedge b = \inf\{a,b\}$,

where $\sup\{a,b\}$ and $\inf\{a,b\}$ are, respectively, the supremum of $\{a,b\}$ and the infimum of $\{a,b\}$ in the ordered set $< L, \leq_\vee >$.

A lattice $< L, \vee, \wedge >$ is said to be distributive if for all $a, b, c \in L$ the following equations hold

(d) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

If one of the distributive laws (d) is satisfied for all $a, b, c \in L$, then the remaining law is also satisfied. Indeed, if we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, then we also have

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c)\\ &= a \vee (c \wedge (a \vee b))\\ &= a \vee ((c \wedge a) \vee (c \wedge b))\\ &= (a \vee (c \wedge a)) \vee (b \wedge c)\\ &= a \vee (b \wedge c).\end{aligned}$$

It also can be proved $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ from $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

The notion of complement can be considered in any lattice having a minimum and a maximum element 0 and 1, respectively. If $\mathcal{L} =< L, \vee, \wedge, 0, 1 >$ is such a lattice then the complement of $x \in L$ is the element $\overline{x}$ such that $x \vee \overline{x} = 1$ and $x \wedge \overline{x} = 0$.

In general, an element of a lattice may have more than one complement or no complement at all. The lattice $< L, \vee, \wedge, 0, 1 >$ is said to be complemented if all its elements have a complement.

For example, consider the lattices given in Figure 2.2. The element c of M_5 has both a and b as its complements. The same holds c in the lattice N_5. However, in the lattice of Figure 2.3 the elements a, b and c have no complement.

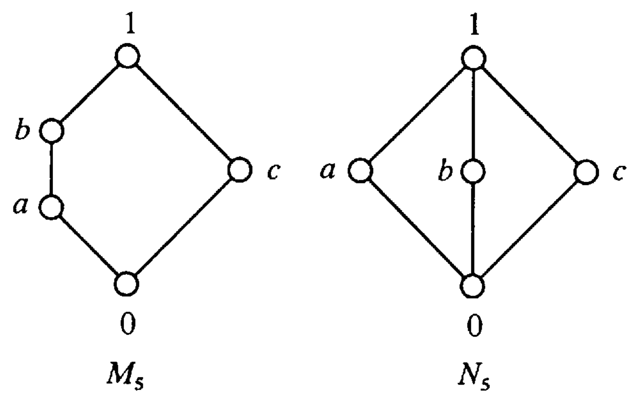

Figure 2.2: Examples of Non-Distributive Lattices

In a distributive lattice, if $c \wedge x = c \wedge y$ and $c \vee x = c \vee y$ then $x = y$ since we have $x = x \wedge (c \vee x) = x \wedge (c \vee y) = (x \wedge c) \vee (x \wedge y) = (c \wedge y) \vee (x \wedge y) = (c \vee x) \wedge y = (c \vee y) \wedge y = y$. As

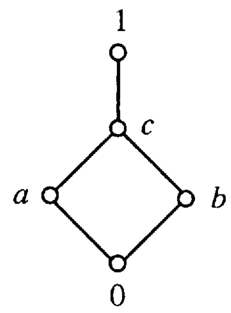

Figure 2.3: Non-Complemented Lattice

a consequence, if an element of a distributive lattice has a complement then this complement is unique.

Boolean algebras play important role for switching theory. A Boolean algebra is an algebra with $\mathcal{B} =< B, \vee, \wedge, h, 0, 1 >$ two binary operations $\vee$, $\wedge$, one unary operation h and two nullary operations 0, 1 such that the following conditions are satisfies.

(b_1) $< B, \vee, \wedge >$ is a distributive lattice.

(b_2) $0 \vee x = x$ and $x \wedge 1 = x$ for all $x \in B$.

(b_3) $x \vee h(x) = 1$ and $x \wedge h(x) = 0$ for all $x \in B$.

The valued of $h(x)$ will be usually denoted by $\overline{x}$. Of course, in a Boolean algebra, every x has a complement, that is, a Boolean algebra is a complemented lattice.

Example 3 *Consider the algebra $< \mathcal{P}(S), \cup, \cap, ', \emptyset, S >$ defined on the set S, where $T' = S - T$ for $T \in \mathcal{P}(S)$. The properties of set operations show that $< \mathcal{P}(S), \cup, \cap >$ is a distributive lattice. Also, $\emptyset \cup T = T$ and $T \cap S = T$ for all $T \in \mathcal{P}(S)$. Finally, we have $T \cup T' = S$ and $T \cap T' = \emptyset$.*

Chapter 3

Previous Works

As the previous works directly relating with our studies, the chapter describes some properties of B-ternary logic functions, regular ternary logic functions and fuzzy logic functions. The results described here are mainly obtained by the papers [24], [31], [25], [25] and [27] except for the discussions of "Necessary and Sufficient Condition (Part I)" in Section 3.3.2. All proofs of theorems, lemmas and corollaries are omitted except for Theorem 21 and Lemma 14.

3.1 B-Ternary Logic Functions

3.1.1 Definition of B-Ternary Logic Functions and Partial Order Relation $\succ$

Let V_3 be a set of truth values such as $V_3 = \{0,\ 1/2,\ 1\}$. Then, an n-variable ternary function is a mapping from V_3^n to V_3. Here, the logic operations AND($\cdot$), OR($\vee$) and NOT($\sim$) of binary logic can be expanded into the set V_3^n as follows.

Table 3.1: Operations AND($\cdot$), OR($\vee$) and NOT($\sim$)

b	$a \cdot b$			$a \vee b$			$\sim a$
a	0	1/2	1	0	1/2	1	
0	0	0	0	0	1/2	1	1
1/2	0	1/2	1/2	1/2	1/2	1	1/2
1	0	1/2	1	1	1	1	0

An *n-variable B-ternary logic function* is defined to be a mapping from V_3^n to V_3, which is represented by a B-ternary logic formula defined below.

Definition 1 *A B-ternary logic formula on variables $x_1, \ldots, x_n$ is defined inductively as follows.*

(1) Constants 0, 1 and variables $x_1, \ldots, x_n$ are B-ternary logic formulas.

(2) If G and H are B-ternary logic formulas, then $(G \cdot H)$, $(G \vee H)$ and $(\sim G)$ are also B-ternary logic formulas.

(3) The only B-ternary logic formulas are given by (1) and (2).

Definition 2 *An n-variable ternary function represented by a B-ternary logic formula is called an n-variable B-ternary logic function.*

Hereafter, we will identify an n-variable B-ternary logic function with the B-ternary logic formula which represents it and call an n-variable B-ternary logic function a B-ternary logic function, if it no confusion arises. The notation $\cdot$ may be often omitted.

The following equations are properties holding in B-ternary logic functions, and almost of them are equivalent to that of Boolean functions except for 10. In the following, $\sim$ is stronger than $\cdot$, and $\cdot$ is stronger than $\vee$ for omitting parentheses.

1. $a \cdot b = b \cdot a$, $a \vee b = b \vee a$ (the commutative laws)
2. $a \cdot a = a$, $a \vee a = a$ (the idempotent laws)
3. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, $a \vee (b \vee c) = (a \vee b) \vee c$ (the associative laws)
4. $a \vee a \cdot b = a$, $a \cdot (a \vee b) = a$ (the absorption laws)
5. $a \cdot (b \vee c) = a \cdot b \vee a \cdot c$, $a \vee b \cdot c = (a \vee b) \cdot (a \vee c)$ (the distributive laws)
6. $\sim (a \cdot b) = \sim a \vee \sim b$, $\sim (a \vee b) = \sim a \cdot \sim b$ (De Morgan's laws)
7. $\sim (\sim a) = a$ (the double negation law)
8. $0 \cdot a = 0$, $0 \vee a = a$ (the least element)
9. $1 \cdot a = a$, $1 \vee a = 1$ (the greatest element)
10. $a \cdot \sim a \cdot (b \vee \sim b) = a \cdot \sim a$, $a \cdot \sim a \vee b \vee \sim b = b \vee \sim b$ (Kleene's laws)

The above 10 equations are available to regular ternary logic functions and fuzzy logic functions, and it is said that an algebraic system $< \mathcal{K}; \cdot, \vee, \sim, 0, 1 >$, where $\mathcal{K}$ is a set and $\cdot$, $\vee$ are two-ary operations, $\sim$ is a unary operation and 0, 1 are null-ary operations on $\mathcal{K}$, having the above 10 equations as its axioms is called a Kleene algebra (or a fuzzy algebra) [2], [29]. Therefore, the set of B-ternary logic functions, regular ternary logic functions and fuzzy logic functions each forms different kinds of models of Kleene algebras.

Next, let us define a partial order relation $\succ$ in the set V_3 follows.

Definition 3

$$1/2 \succ 0, \quad 1/2 \succ 1, \quad i \succ i,$$

where $i \in V_3$ (Note that we also denote the relation $\succ$ as $\prec$, that is, $a \succ b$ is sometimes denoted as $b \prec a$).

Figure 3.1 shows a Hasse diagram of this partial order relation $\succ$. The relation $\succ$ can be expanded among V_3^n as follows. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V_3^n, then $\boldsymbol{a} \succ \boldsymbol{b}$ if and only if $a_i \succ b_i$ for any i $(i = 1, \ldots, n)$. Here, $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be comparable to each other if $\boldsymbol{a} \succ \boldsymbol{b}$ or $\boldsymbol{b} \succ \boldsymbol{a}$ holds, otherwise not comparable.

In the relation $\succ$, 1/2 can be interpreted as a truth values expressing an ambiguous state whether true (1) or false (0). It is sure to exist the least upper bound (supremum) of any element a and b of V_3. If a and b are comparable to each other, then there is the greatest lower bound (infimum) of a and b, otherwise there is not. We will write the infimum of a and b as $a \Delta b$, and

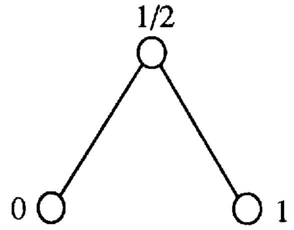

Figure 3.1: Partial Order Relation $\succ$

if the infimum of a and b does not exist, then we will write it as $a\Delta b = \emptyset$. This can be expanded among V_3^n as follows. For two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of V_3^n, we will define $\boldsymbol{a}\Delta\boldsymbol{b}$ as $(a_1\Delta b_1, \ldots, a_n\Delta b_n)$, and if $a_i\Delta b_i = \emptyset$ for some i $(i = 1, \ldots, n)$, then we will define it as $\boldsymbol{a}\Delta\boldsymbol{b} = \emptyset$.

Theorem 1 *Let F be a B-ternary logic function and $\boldsymbol{a}$, $\boldsymbol{b}$ elements of V_3^n. Then, F satisfies the following, where V_2 denotes the set $\{0, 1\}$.*

(1) $\boldsymbol{a} \in V_2^n$ implies $F(\boldsymbol{a}) \in V_2$,

(2) $\boldsymbol{a} \succ \boldsymbol{b}$ implies $F(\boldsymbol{a}) \succ F(\boldsymbol{b})$.

It is shown from Theorem 1 that any B-ternary logic function satisfies monotonicity for the relation $\succ$ with respect to ambiguities.

3.1.2 Canonical Disjunctive Form of B-Ternary Logic Functions

Since rules of Kleene algebras as shown in Section 3.1, an arbitrary B-ternary logic function can be expanded into a disjunctive form $F = \gamma_1 \vee \gamma_2 \vee \ldots \vee \gamma_s$, where $\gamma_i \not\sqsupseteq \gamma_j$ (the symbol $\sqsubseteq$ will be defined just below) for all i, j $(i \neq j)$. However, because the complementary laws are not valid, some terms may contain both a variable and its negation simultaneously. A definition of two kinds of terms follows.

Definition 4 *Let G and H be ternary functions. Then H includes G (or G is included in H) if and only if $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V_3^n, and we denote it as $G \sqsubseteq H$ (or $H \sqsupseteq G$).*

Definition 5 *A variable x or its negation $\sim x$ is called a literal, and the conjunction (AND) of some literals is said to be a product term, where any repeated literals are removed from the product terms. Among product terms, those which do not contain both a variable and its negation at the same time are called simple product terms, while those which do are complementary product terms. A simple product term and a complementary product term in which all variables appear are called a minterm and a complementary minterm, respectively.*

Some properties of simple product terms and complementary minterms will be discussed.

Definition 6 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V_3^n. Then, $\boldsymbol{a}$ and a simple product term $\alpha = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ correspond to each other if one of the following conditions holds for every $i = 1, \ldots, n$.*

$$x_i^{a_i} = \begin{cases} \sim x_i & \text{if} \quad a_i = 0, \\ 1 & \text{if} \quad a_i = 1/2, \\ x_i & \text{if} \quad a_i = 1 \end{cases}$$

Definition 7 *Let* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *be an element of* $V_3^n - V_2^n$*. Then,* $\boldsymbol{a}$ *and a complementary minterm* $\beta = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ *correspond to each other if one of the following conditions holds for every* $i = 1, \ldots, n$.

$$x_i^{a_i} = \begin{cases} \sim x_i & \text{if} \quad a_i = 0, \\ x_i \sim x_i & \text{if} \quad a_i = 1/2, \\ x_i & \text{if} \quad a_i = 1 \end{cases}$$

From the above definitions, the correspondence between elements of V_3^n and the simple product terms is clearly one-to-one, and also between elements of $V_3^n - V_2^n$ and the complementary minterm.

Lemma 1 *Let* $\boldsymbol{a} \in V_3^n$ *and* α *be the corresponding simple product term to* $\boldsymbol{a}$.

(1) $\alpha(\boldsymbol{b}) = 1$, *if and only if* $\boldsymbol{a} \succ \boldsymbol{b}$

(2) $\alpha(\boldsymbol{b}) = 1/2$, *if and only if* $\boldsymbol{a} \not\succ \boldsymbol{b}$ *and* $\boldsymbol{a} \Delta \boldsymbol{b} \neq \emptyset$

(3) $\alpha(\boldsymbol{b}) = 0$, *if and only if* $\boldsymbol{a} \Delta \boldsymbol{b} = \emptyset$

where $\boldsymbol{b} \in V_3^n$.

Lemma 2 *Let* $\boldsymbol{a} \in V_3^n - V_2^n$ *and* α *be the corresponding complementary minterm to* $\boldsymbol{a}$.

(1) $\alpha(\boldsymbol{b}) = 1/2$, *if and only if* $\boldsymbol{b} \succ \boldsymbol{a}$

(2) $\alpha(\boldsymbol{b}) = 0$, *if and only if* $\boldsymbol{b} \not\succ \boldsymbol{a}$

where $\boldsymbol{b} \in V_3^n$.

As stated in the beginning of the section, any B-ternary logic function F can be expanded into disjunctive form $F = \alpha_1 \vee \ldots \vee \alpha_s$, and each α_i can be classified into one of simple product terms or complementary product terms.

Now, suppose that there is not a variable x_i in a complementary minterm α. Then it holds that $\alpha = \alpha(x_i \vee \sim x_i) = \alpha x_i \vee \alpha \sim x_i$ since there is a product $x_j \sim x_j$ for some variable x_j in α and $x_j \sim x_j \leq 1/2 \leq x_i \vee \sim x_i$ stands always true. Accordingly, any complementary minterm can be expanded into disjunction of complementary minterms.

Lemma 3 *Let* α *and* α' *be product terms. Then,* $\alpha' \sqsubseteq \alpha$ *(i.e.* $\alpha \vee \alpha' = \alpha$*) if and only if all of the literals in* α *exist in* α' *as well.*

Let α and β be a simple product term and a complementary minterm, respectively. Then, it is clear from the definition of simple product terms and complementary minterms that $\beta \sqsubseteq \alpha$ never holds.

Lemma 4 *Let* α *and* β *be product terms, and let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be elements of* V_3^n *that correspond to them, respectively. Then* $\beta \sqsubseteq \alpha$ *if and only if*

(1) $\boldsymbol{a} \succ \boldsymbol{b}$, *when* α *and* β *are both simple product terms.*

(2) $\boldsymbol{b} \succ \boldsymbol{a}$, *when* α *and* β *are both complementary minterms.*

(3) $\boldsymbol{a} \Delta \boldsymbol{b} \neq \emptyset$, *when* α *is a simple product term and* β *is a complementary minterm.*

Definition 8 *A B-ternary logic function F represented as $F = \alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_s$ is said to be in canonical disjunctive form when each product term α_i $(i = 1, \ldots, s)$ is a simple product term or a complementary minterm and $\alpha_i \not\sqsubseteq \alpha_j$ for all i and j $(i \neq j)$.*

Example 1 *Let F be a 3-variable B-ternary logic function represented by the B-ternary logic formula $x_1(x_2 \vee x_3 \sim x_3) \vee x_2(x_1x_3 \vee \sim x_2)$. Then the canonical disjunctive form of F is obtained follows.*

$$
\begin{aligned}
F &= x_1(x_2 \vee x_3 \sim x_3) \vee x_2(x_1x_3 \vee \sim x_2)\\
&= x_1x_2 \vee x_1x_2x_3 \vee x_1(x_2 \vee \sim x_2)x_3 \sim x_3 \vee (x_1 \vee \sim x_1)x_2 \sim x_2(x_3 \vee \sim x_3)\\
&= x_1x_2 \vee x_1x_2x_3 \vee x_1x_2x_3 \sim x_3 \vee x_1 \sim x_2x_3 \sim x_3 \vee x_1x_2 \sim x_2x_3 \vee x_1x_2 \sim x_2 \sim x_3 \vee\\
&\quad \sim x_1x_2 \sim x_2x_3 \vee \sim x_1x_2 \sim x_2 \sim x_3
\end{aligned}
$$

Here (1, 1, 1/2) *and* (1, 1, 1) *correspond to simple product terms x_1x_2 and $x_1x_2x_3$, respectively, and therefore by Lemma 4.1 $x_1x_2x_3 \sqsubseteq x_1x_2$, that is, $x_1x_2x_3$ is omitted by x_1x_2. Moreover,* (1, 1, 1/2), (1, 0, 1/2), (1, 1/2, 1), (1, 1/2, 0), (0, 1/2, 1) *and* (0, 1/2, 0) *correspond to complementary minterms $x_1x_2x_3 \sim x_3$, $x_1 \sim x_2x_3 \sim x_3$, $x_1x_2 \sim x_2x_3$, $x_1x_2 \sim x_2x_3$, $x_1x_2 \sim x_2 \sim x_3$, $\sim x_1x_2 \sim x_2x_3$ and $\sim x_1x_2 \sim x_2 \sim x_3$, respectively, and since* $(1, 1, 1/2)\Delta(1, 1, 1/2) \neq \emptyset$, $(1, 1, 1/2)\Delta(1, 1/2, 1) \neq \emptyset$ *and* $(1, 1, 1/2)\Delta(1, 1/2, 0) \neq \emptyset$, *$x_1x_2x_3 \sim x_3$, $x_1x_2 \sim x_2x_3$ and $x_1x_2 \sim x_2 \sim x_3$ are, respectively, omitted by x_1x_2. Therefore, the canonical disjunctive form of F is*

$$x_1x_2 \vee x_1 \sim x_2x_3 \sim x_3 \vee \sim x_1x_2 \sim x_2x_3 \vee \sim x_1x_2 \sim x_2 \sim x_3.$$

Theorem 2 *Any B-ternary logic function can be represented uniquely by canonical disjunctive form (ignoring order of product terms).*

3.1.3 A Characterization of B-ternary Logic Functions

Obviously, by Theorem 1 it is not true that every ternary function can obtain by means of a B-ternary logic formula, that is, the set of B-ternary logic functions is not functionally complete. Therefore, in the section, we discuss a necessary and sufficient condition for a ternary function to be a B-ternary logic function.

The following set of two conditions is a necessary and sufficient condition for a ternary function to be a B-ternary logic function.

(B1) [Normality] $\boldsymbol{a} \in V_2^n$ implies $F(\boldsymbol{a}) \in V_2$

(B2) [Monotonicity for Ambiguity] $\boldsymbol{a} \succ \boldsymbol{b}$ implies $F(\boldsymbol{a}) \succ F(\boldsymbol{b})$

We have been shown in Theorem 1 that Conditions (B1) and (B2) is a necessary condition for B-ternary logic functions, and therefore we will show that a set of the conditions (B1) and (B2) is sufficient for B-ternary logic functions.

Let F be a ternary function satisfying Conditions (B1) and (B2). Then we consider specific three subsets $F^{-1}(0)$, $F^{-1}(1/2)$ and $F^{-1}(1)$ of V_3^n as follows.

$$F^{-1}(i) = \{\boldsymbol{a} \in V_3^n \mid F(\boldsymbol{a}) = i\}$$

where $i \in V_3$. It is clear that $F^{-1}(i) \cap F^{-1}(j) = \emptyset$ and $\bigcup_{i \in V_3} F^{-1}(i) = V_3^n$. Suppose $\boldsymbol{a}$ be an element of $F^{-1}(1)$, then by Condition (B2) any element $\boldsymbol{b}$ such that $\boldsymbol{a} \succ \boldsymbol{b}$ is also element of $F^{-1}(1)$. Similarly if $\boldsymbol{a}$ is an element of $F^{-1}(1/2)$, then any element $\boldsymbol{b}$ such that $\boldsymbol{b} \succ \boldsymbol{a}$ is also element of $F^{-1}(1/2)$. Moreover an element $\boldsymbol{a}$ of $F^{-1}(0)$ implies that any element $\boldsymbol{b}$ such that $\boldsymbol{a} \succ \boldsymbol{b}$ is also element of $F^{-1}(0)$. Therefore, $F^{-1}(1)$, $F^{-1}(1/2)$ and $F^{-1}(0)$ each form a partial order finite set concerning with the relation $\succ$. Thus, for any given B-ternary logic function F the set of maximal elements of $F^{-1}(1/2)$ and the sets of minimum elements of $F^{-1}(1)$ and $F^{-1}(0)$ are uniquely determined, and they are denoted by $\partial F^{-1}(1/2)$, $\partial F^{-1}(1)$ and $\partial F^{-1}(0)$, respectively.

If $\boldsymbol{a}$ is an element of $F^{-1}(1/2)$, then $\boldsymbol{a}$ is also the element of $V_3^n - V_2^n$. Because, if $\boldsymbol{a}$ is the element of V_2^n, then by Condition (B1) it has to be held that $F(\boldsymbol{a}) \in V_2$, and this is a contradiction. Therefore, $F^{-1}(1/2) \subseteq V_3^n - V_2^n$. Accordingly, we can always construct each B-ternary logic formula F_1 and $F_{1/2}$ which is a disjunction of all simple product terms and all complementary minterms corresponding to all elements of $\partial F^{-1}(1)$ and $\partial F^{-1}(1/2)$, respectively. Then we can show the following theorem.

Theorem 3 *If F is a ternary function satisfying Condition (B1) and (B2), then $F_f = F_1 \vee F_{1/2}$ is a B-ternary logic function such that $F(\boldsymbol{a}) = F_f(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_3^n.*

Example 2 *It is easy to verify that 2-variable ternary function F given in Table 3.2 satisfies Condition (B1) and (B2). Then $\partial F^{-1}(1) = \{(0,1)\}$ and $\partial F^{-1}(1/2) = \{(0,1/2),(1/2,0),(1,1/2)\}$, and we obtain the B-ternary logic formula*

$$F_f = x_1 \sim x_2 \vee \sim x_1 x_2 \sim x_2 \vee x_1 \sim x_1 \sim x_2 \vee x_1 x_2 \sim x_2,$$

which represents F, from all elements of $\partial F^{-1}(1)$ and $\partial F^{-1}(0)$.

Table 3.2: Truth Table of Ternary Function F of Example 2

x_1 \ x_2	0	1/2	1
0	0	1/2	0
1/2	1/2	1/2	0
1	1	1/2	0

Conditions (B1) and (B2) are independent to each other. Because the ternary function F_1 in Table 3.3 satisfies Condition (B1), but not (B2). Therefore, F_1 is an example can not derive (B2) form (B1). F_2 in Table 3.3 is an example can not derive (B1) form (B2), since F_2 satisfies (B2), but not (B1).

3.2 Regular Ternary Logic Functions

3.2.1 Definitions and a Characterization of Regular Ternary Logic Functions

Condition 1 *The ternary functions which can be represented by well-formed logic formulas, called regular ternary logic formulas, consisting of variables $x_1, \ldots, x_n$, constants 0, 1/2, 1 and logic operations $AND(\cdot)$, $OR(\vee)$, and $NOT(\sim)$. defined in Table 3.1.*

Table 3.3: Truth Tables of F_1 and F_2

x	0	1/2	1
F_1	0	1	1
F_2	1/2	1/2	1/2

Hereafter, we call a ternary function satisfying the above condition a ternary function representable by a regular ternary logic formula. It is clear by the definition of B-ternary logic functions that any B-ternary logic function is special case of ternary functions representable regular ternary logic formulas.

As a condition for a ternary function F to be significant when the truth value 1/2 is assumed to be represent an ambiguous state, it will be postulated that if the value of $F(\boldsymbol{a})$ is definite, that is, 0 or 1, then $F(\boldsymbol{a}')$ takes an equal value for every element $\boldsymbol{a}'$ which is less ambiguous than or equal to $\boldsymbol{a}$, that is,

Condition 2 [Regularity] *if $F(\boldsymbol{a}) \in V_2$, then $F(\boldsymbol{a}') = F(\boldsymbol{a})$ for every $\boldsymbol{a}'$ such that $\boldsymbol{a} \succ \boldsymbol{a}'$.*

Definition 9 *A ternary function F is called a regular ternary logic function if and only if F satisfies the regularity Condition 2.*

Note that the condition of regularity defined above is an extension of Kleene's definition to n-variable ternary functions where Kleene's original definition [?] of regularity for a truth table is as follows. The truth table never takes 0 or 1 as entry in the "1/2 row (or column)" unless this entry 0 or 1 occurs uniformly throughout its entire column (or row, respectively).

Condition 3 [Monotonicity for Ambiguity] *If $\boldsymbol{a} \succ \boldsymbol{a}'$, then $F(\boldsymbol{a}) \succ F(\boldsymbol{a}')$, is called an A-ternary logic function.*

As stated in the precious section, an A-ternary function F satisfying the condition of normality, that is, $\boldsymbol{a} \in V_2^n$ implies $F(\boldsymbol{a}) \in V_2$ is a B-ternary logic function.

Thus far, three different conditions (1, 2 and 3) have been defined for ternary functions. In the following, we will prove that these three conditions are equivalent to each other.

Theorem 4 *F is a regular ternary logic function if and only if F is an A-ternary logic function.*

Theorem 5 *If F is a ternary function representable by a regular ternary logic formula, the F is a regular ternary logic function.*

3.2.2 Representation of Regular Ternary Logic Functions

Definition 10 *A disjunction of one or more literals is called a simple sum term if it does not contain a literal and its negation $x_i \vee \sim x_i$ simultaneously for at least one variable x_i, and is called a complementary sum term otherwise. In the definition, it is assumed that any repeated literals are removed.*

Definition 11 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V_3^n. Then $\boldsymbol{a}$ and a simple sum term $\alpha = x_1^{a_1} \vee \ldots \vee x_n^{a_n}$ correspond to each other if one of the following relations holds for every $i = 1, \ldots, n$.*

$$x_i^{a_i} = \begin{cases} x_i & \text{if} \quad a_i = 0, \\ 1 & \text{if} \quad a_i = 1/2, \\ \sim x_i & \text{if} \quad a_i = 1 \end{cases}$$

Lemma 5 *Let F be a regular ternary logic function and $\boldsymbol{a}$ be an element of V_3^n. Then,*

(1) if $F(\boldsymbol{a}) = 1$, then $F(\boldsymbol{a}') = 1$ for every $\boldsymbol{a}'$ such that $\boldsymbol{a} \succ \boldsymbol{a}'$,

(2) if $F(\boldsymbol{a}) = 0$, then $F(\boldsymbol{a}') = 0$ for every $\boldsymbol{a}'$ such that $\boldsymbol{a} \succ \boldsymbol{a}'$,

(3) if $F(\boldsymbol{a}) = 1/2$, then $F(\boldsymbol{a}') = 1/2$ for every $\boldsymbol{a}'$ such that $\boldsymbol{a}' \succ \boldsymbol{a}$.

Let F be an n-variable regular ternary logic function. Then we consider three subsets $F^{-1}(1)$, $F^{-1}(0)$ and $F^{-1}(1/2)$ of V_3^n follows.

$$F^{-1}(i) = \{\boldsymbol{a} \in V_3^n | F(\boldsymbol{a}) = i\}$$

where $i \in V_3$. It is clear that $F^{-1}(i) \cap F^{-1}(i) = \emptyset$ and $\bigcup_{i \in V_3} F^{-1}(i) = V_3^n$. Lemma 5 indicates that $F^{-1}(1)$, $F^{-1}(0)$ and $F^{-1}(1/2)$ are partial ordered finite sets in regard to the relation $\succ$ and that the sets $F^{-1}(1)$ and $F^{-1}(0)$ are determined uniquely by their maximal elements while $F^{-1}(1/2)$ is determined uniquely by its minimal elements.

Theorem 6 *Any regular ternary logic function F can be represented by the regular ternary logic formula*

$$F = F^1 \vee 1/2 \cdot F^0$$

where F^1 is the disjunction of simple product terms corresponding to all the maximal elements of $F^{-1}(1)$ and F^0 is the conjunction of simple sum terms corresponding to all the maximal elements of $F^{-1}(0)$.

Example 3 *It is easy to verify that 2-variable ternary function F given in Table 3.4 is a regular ternary logic function, that is, satisfies Condition 2. Then $\partial F^{-1}(1) = \{(1/2,\ 1)\}$ and $\partial F^{-1}(0) = \{(1,\ 0)\}$, and we obtain the regular ternary logic formula*

$$F_f = x_2 \vee 1/2(\sim x_1 \vee x_2),$$

which represents F, from all of the elements of $\partial F^{-1}(1)$ and $\partial F^{-1}(0)$.

3.2.3 Canonical Disjunctive Forms of Regular Ternary Logic Functions

In the section, we shall introduce a canonical form for regular ternary logic functions, which is different from that of Theorem 6. We will also discuss the methods to obtain such a canonical form. Any regular ternary logic formula representing a regular ternary logic function F can be expanded into a disjunctive form

$$F = \alpha_1 \vee \ldots \vee \alpha_s$$

Table 3.4: Regular Ternary Logic Function F of Example 3

x_1 \ x_2	0	1/2	1
0	1/2	1/2	1
1/2	1/2	1/2	1
1	0	1/2	1

where each α_i $(i = 1, \ldots, n)$ is a product term, because the distributive, absorption, De Morgan's idempotent, and other laws stand valid as stated in Section 3.1. Here, each product term α_i is one of the following three types.

type 1: A simple product term α

type 2': $1/2 \cdot \alpha$, where α is a simple product term

type 3': A complementary product term β

If a constant 1/2 exists in a complementary product term β, that is, it forms $1/2 \cdot \beta$, then we can omit 1/2 and it is equal to type 3' because $x_i \sim x_i \leq 1/2$ stands always true. If a variable x_i does not exist in a product term $1/2\alpha$ of type 2', then the following relation holds. $1/2\alpha = 1/2(x_i \vee \sim x_i)\alpha = 1/2\alpha x_i \vee 1/2\alpha \sim x_i$ since $1/2 \leq x_i \sim x_i$ is always valid. As discussed before, any complementary product term can always be expanded into a disjunction of complementary minterms. From the above, we can expand α of type 2' and β of type 3' into disjunction of product terms in which all variables exist, respectively.

Consequently, any regular ternary logic function can always be expanded into the disjunction of the following three types of product terms.

type 1: A simple product term α

type 2: $1/2\alpha$, where α is a minterm

type 3: A complementary minterm β

Definition 12 *If a regular ternary logic function F is represented by a regular ternary logic formula $F = \alpha_1 \vee \ldots \vee \alpha_s$, then it is said that F is in the canonical disjunctive form where each α_i $(i = 1, \ldots, n)$ is one of type 1, type 2 or type 3 and $\alpha_i \not\sqsubseteq \alpha_j$ for any i and j $(i \neq j)$.*

Example 4 *The canonical disjunctive form of 2-variable regular ternary logic function $F = \sim x_1 \vee 1/2(\sim x_1 \vee x_2)$ is obtained as follows.*

$$\begin{aligned} F &= \sim x_1 \vee 1/2 \sim x_1 \vee 1/2x_2 \\ &= \sim x_1 \vee 1/2 \sim x_1x_2 \vee 1/2 \sim x_1 \sim x_2 \vee 1/2x_1x_2 \vee 1/2 \sim x_1x_2 \\ &= \sim x_1 \vee 1/2x_1x_2 \end{aligned}$$

Theorem 7 *Any regular ternary logic function can be represented uniquely (ignoring the order of the product terms) by the canonical disjunctive form.*

3.3 Fuzzy Logic Functions

3.3.1 Definition of Fuzzy Logic Functions and Their Properties

Let V be closed interval $[0, 1]$. Then, an n-variable infinite-valued function is defined as a mapping from V^n into V. Here, the logic operations AND($\cdot$), OR($\vee$) and NOT($\sim$) of Table 3.1 can be expanded into the set V as follows (We permit to use the same symbols $\cdot$, $\vee$ and $\sim$ with that of B-ternary logic functions and regular ternary logic functions since it seems that no confusion arises).

$$a \cdot b = \min(a, b), \quad a \vee b = \max(a, b), \quad \sim a = 1 - a$$

where $a, b \in V$.

An *n-variable fuzzy logic function* is defined to be a mapping from V^n into V represented by a fuzzy logic formula below.

Definition 13 *A logic formula on variables $x_1, \ldots, x_n$ is defined inductively as follows.*

(1) Constants 0, 1 and variables $x_1, \ldots, x_n$ are fuzzy logic formulas.

(2) If G and H are fuzzy logic formulas, then $(G \cdot H)$, $(G \vee H)$ and $(\sim G)$ are also fuzzy logic formulas.

(3) The only fuzzy logic formulas are given by (1) and (2)

Definition 14 *An n-variable infinite-valued function represented by a fuzzy logic formula is called an n-variable fuzzy logic function.*

Hereafter, we will identify an n-variable fuzzy logic function with the fuzzy logic formula which represents it and call an n-variable fuzzy logic function a fuzzy logic function, if it no confusion arises. The notation $\cdot$ may be often omitted.

Next, let us also expand the definition of the partial order relation $\succ$ into the set V follows.

Definition 15 *Let a and b be element of [0, 1]. Then $a \succ b$ if and only if one of the following relations holds.*

$$0 \leq b \leq a \leq 1/2 \quad or \quad 1/2 \leq a \leq b \leq 1$$

Moreover, the relation can be expanded into V^n as follows. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V^n, then $\boldsymbol{a} \succ \boldsymbol{b}$ if and only if $a_i \succ b_i$ for any i ($i = 1, \ldots, n$).

Here, $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be comparable to each other if $\boldsymbol{a} \succ \boldsymbol{b}$ holds, otherwise not comparable. It is sure to exist the least upper bound (supremum) of any element a and b of V. If a and b are comparable to each other, then there is the greatest lower bound (infimum) of a and b, otherwise there is not. We will write the infimum of a and b as $a\Delta b$, and if the infimum of a and b does not exist, then we will write it as $a\Delta b = \emptyset$. This can be expanded among V^n as follows. For two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of V^n, we will define $\boldsymbol{a}\Delta\boldsymbol{b}$ as $(a_1\Delta b_1, \ldots, a_n\Delta b_n)$, and if $a_i\Delta b_i = \emptyset$ for some i ($i = 1, \ldots, n$), then we will define it as $\boldsymbol{a}\Delta\boldsymbol{b} = \emptyset$.

In the above definitions for the symbols $\succ$, Δ and $\emptyset$, it is evident that they are different concepts to the same symbols defined in Section 3.1. However, we allow to use same symbols $\succ$, Δ and $\emptyset$ in the section as long as no confusion arises.

Theorem 8 *Let F be a fuzzy logic function and $\boldsymbol{a}$, $\boldsymbol{b}$ elements of V^n. Then, F satisfies the following.*

(1) $\boldsymbol{a} \in V_2^n$ *implies* $F(\boldsymbol{a}) \in V_2$,

(2) $\boldsymbol{a} \succ \boldsymbol{b}$ *implies* $F(\boldsymbol{a}) \succ F(\boldsymbol{b})$.

Next, we define a mapping from V into V_3.

Definition 16 *Let a be an element of V. Then $\overline{a}^{\varepsilon}$ is defined as follows, where $0 < \varepsilon \leq 1/2$ (see Figure 3.2).*

$$\overline{a}^{\varepsilon} = \begin{cases} 0 & \text{if } 0 \leq a < \varepsilon, \\ 1/2 & \text{if } \varepsilon \leq a \leq 1-\varepsilon, \\ 1 & \text{if } 1-\varepsilon < a \leq 1 \end{cases}$$

It can be expanded into the set V^n as follows. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V^n then $\overline{\boldsymbol{a}}^{\varepsilon}$ is defined by $(\overline{a_1}^{\varepsilon}, \ldots, \overline{a_n}^{\varepsilon})$ of V_3^n.

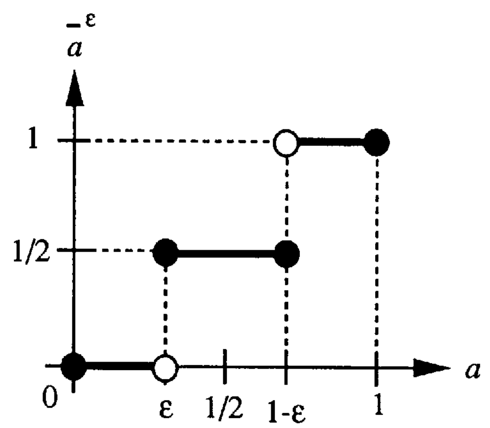

Figure 3.2: A Mapping $\overline{a}^{\varepsilon}$ $(0 < \varepsilon \leq 1/2)$

Theorem 9 *Let F be a fuzzy logic function and $\boldsymbol{a}$ an element of V^n. Then, $\overline{F(\boldsymbol{a})}^{\varepsilon} = F(\overline{\boldsymbol{a}}^{\varepsilon})$ for any ε $(0 < \varepsilon \leq 1/2)$.*

By using the above theorem, we can easily prove the following theorems.

Theorem 10 *Let G and H be fuzzy logic functions. $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V^n if and only if $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_3^n.*

From the above theorem, we can conclude that there is a one-to-one and onto mapping from the set of all n-variable fuzzy logic functions and the set of all B-ternary logic functions, and it is evident there is at least an isomorphism among such mappings.

Corollary 1 *Let G and H be fuzzy logic functions. $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V^n if and only if $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_3^n.*

Corollary 2 *Let G and H be fuzzy logic functions.* $G(\boldsymbol{a}) \succ H(\boldsymbol{a})$ *for any* $\boldsymbol{a}$ *of* V^n *if and only if* $G(\boldsymbol{a}) \succ H(\boldsymbol{a})$ *for any* $\boldsymbol{a}$ *of* V_3^n.

Definition 17 *Let G and H be fuzzy logic functions. Then, H includes G (or G is included in H) if and only if* $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ *for any element* $\boldsymbol{a}$ *of* V^n*, and we denote if as* $G \sqsubseteq H$ *(or* $H \sqsupseteq G$*).*

Here, we also permit to use same symbol $\sqsubseteq$ (or $\sqsupseteq$) with that of Section 3.1.
In accordance with Corollary 1, $G \sqsubseteq H$ if and only if $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_3^n.

3.3.2 A Characterization of Fuzzy Logic Functions

In this section, we show two different kinds of necessary and sufficient conditions for fuzzy logic functions.

(A): Necessary and Sufficient Condition (Part I)

(Fa) $\boldsymbol{a} \in V_2^n$ implies $F(\boldsymbol{a}) \in V_2$

(Fb) $\boldsymbol{a} \succ \boldsymbol{b}$ implies $F(\boldsymbol{a}) \succ F(\boldsymbol{b})$

(Fc) $\overline{F(\boldsymbol{a})}^{\varepsilon} = F(\overline{\boldsymbol{a}}^{\varepsilon})$ for any ε such that $0 < \varepsilon \leq 1/2$

The set of the above three conditions (Fa), (Fb) and (Fc) is a necessary and sufficient condition for fuzzy logic functions, that is, it can be proved the following Lemma 6 and Theorems 11 and 12.

Lemma 6 *Let F be an infinite-valued function satisfying Conditions (Fa), (Fb) and (Fc). Then,* $\boldsymbol{a} \in V_3^n$ *implies* $F(\boldsymbol{a}) \in V_3$.

Theorem 11 *If F is a fuzzy logic function, then F satisfies Conditions (Fa), (Fb) and (Fc).*

Theorem 12 *If F is an infinite-valued function satisfying Conditions (Fa), (Fb) and (Fc), then F is a fuzzy logic function.*

Here, Conditions (Fa), (Fb) and (Fc) are not independent to each other, since (Fb) is derived from (Fc) by the following Theorem 13. Lemma 7 $\sim$ 9 are required for proving Theorem 13.

Lemma 7 *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be elements of* V_3^n *such that* $\boldsymbol{a} \succ \boldsymbol{b}$*. Then, there are an element* $\boldsymbol{t}$ *of* V^n *and constants* ε_1 *and* ε_2 *such that* $\boldsymbol{a} = \overline{\boldsymbol{t}}^{\varepsilon_1}$, $\boldsymbol{b} = \overline{\boldsymbol{t}}^{\varepsilon_2}$ *and* $0 < \varepsilon_1 < \varepsilon_2 \leq 1/2$.

Lemma 8 *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be elements of* V^n*. Then,* $\boldsymbol{a} \succ \boldsymbol{b}$ *if and only if* $\overline{\boldsymbol{a}}^{\varepsilon} \succ \overline{\boldsymbol{b}}^{\varepsilon}$ *for any* ε *such that* $0 < \varepsilon \leq 1/2$.

Lemma 9 *Let* $\boldsymbol{a}$ *be an element of* V^n *and* ε_1, ε_2 *any constants such that* $0 < \varepsilon_1 \leq 1/2$ *and* $0 < \varepsilon_2 \leq 1/2$*. Then* $\varepsilon_1 \succ \varepsilon_2$ *implies* $\overline{\boldsymbol{a}}^{\varepsilon_2} \succ \overline{\boldsymbol{a}}^{\varepsilon_1}$.

From the above three lemmas, we can prove the following theorem.

Theorem 13 *Let F be an infinite-valued function satisfying Condition (Fc). Then, F also satisfies Condition (Fb).*

Conditions (Fa) and (Fc) are independent to each other. Because, F_a of Figure 3.3 satisfies (Fa) while not (Fc). Therefore, f_a is an example that (Fc) is never derived from (Fa). Moreover, f_c of Figure 3.4 satisfies (Fc) while not (Fa).

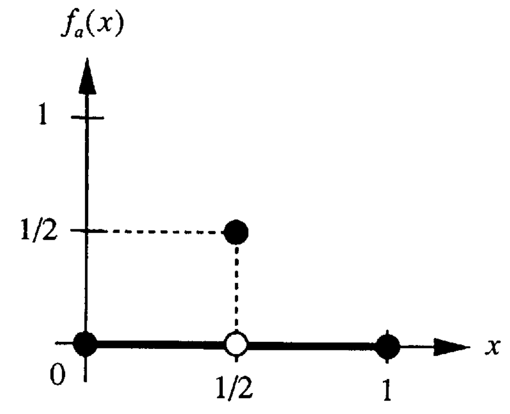

Figure 3.3: Function f_a

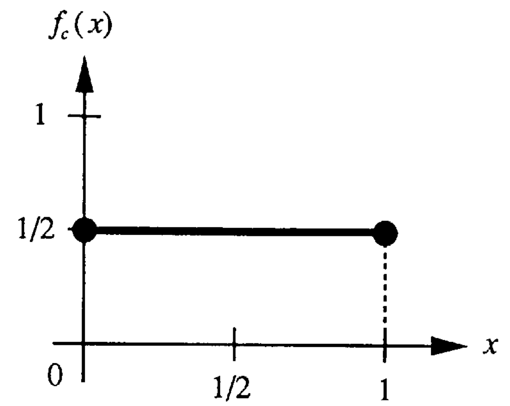

Figure 3.4: Function f_c

(B): Necessary and Sufficient Condition (Part II)

In the section, we denote another necessary and sufficient condition for fuzzy logic functions.

First, let us consider the set V^n which is the domain of fuzzy logic functions. In case of n variables, there are $2n$'s different literals, that is, $x_1, x_2, \ldots, x_n, \sim x_1, \sim x_2, \ldots, \sim x_n$. Hereafter, let x'_i denote either x_i or $\sim x_i$, where $\sim x'_i$ means $\sim x_i$ if $x'_i = x_i$ and means x_i if $x'_i = \sim x_i$. Then, by choosing different $2n$'s literals, in order, we can construct an inequality

$$x'_{i_1} \le x'_{i_2} \le \ldots \le x'_{i_n} \le \sim x'_{i_n} \le \sim x'_{i_{n-1}} \le \ldots \le \sim x'_{i_2} \le \sim x'_{i_1}, \tag{3.1}$$

where $i_1, i_2, \ldots, i_n$ are elements of the set $\{1, 2, \ldots, n\}$ respectively and are all different from each other. The first half $x'_{i_1} \le x'_{i_2} \le \ldots \le x'_{i_n}$ implies necessarily the later half $\sim x'_{i_n} \le \sim x'_{i_{n-1}} \le \cdots \le x'_{i_1}$. Further, the relation $x'_i \le 1/2 \le \sim x'_i$ always holds. Therefore, for the sake of the simplicity, instead of the inequality (1), the first half of which we use, hereafter, as follows.

$$x'_{i_1} \le x'_{i_2} \le \ldots \le x'_{i_n} \tag{3.2}$$

The inequality (2) determines a subset, which is a closed set of V^n. We will call the subset a *cell space*. Now, count the number of distinct cell spaces. There are $2n$ ways of how to choose

x'_{i_1} since we can choose x'_{i_1} or $\sim x'_{i_1}$. Similarly, we can choose x'_{i_2} in $2(n-1)$ ways and so on. Therefore, the number of different cell spaces is

$$2 \times n \times 2 \times (n-1) \times \ldots \times 2 \times 2 \times 2 \times 1 = 2^n \times n!.$$

The sum of all these cell spaces is evidently equal to V^n.

Example 5 *In case of $n = 2$, there are $2^2 \times 2! = 8$ cell spaces, that is,*

(1) $x_1 \leq x_2 \leq 1/2$, *(2)* $x_1 \leq\sim x_2 \leq 1/2$, *(3)* $\sim x_1 \leq x_2 \leq 1/2$, *(4)* $\sim x_1 \leq\sim x_2 \leq 1/2$,
(5) $x_2 \leq x_1 \leq 1/2$, *(6)* $x_2 \leq\sim x_1 \leq 1/2$, *(7)* $\sim x_2 \leq x_1 \leq 1/2$, *(8)* $\sim x_2 \leq\sim x_1 \leq 1/2$.

As illustrated in Figure 3.5, V^2 is represented by the sum of these cell spaces.

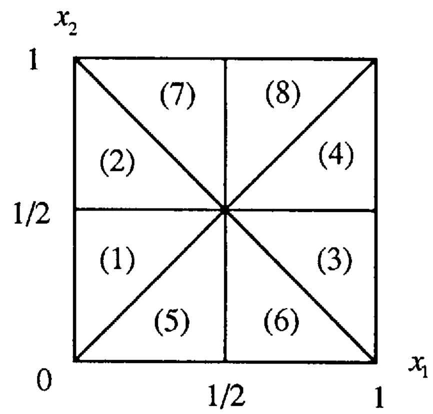

Figure 3.5: Cell Spaces of V^2

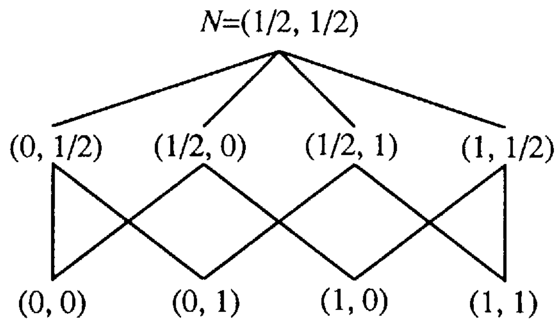

Figure 3.6: Partial Order Set of V_3^2

Next, let us consider the domain V_3^n of B-ternary logic functions. V_3^n is a partial order set concerning the partial order relation $\succ$, where $N = (1/2, 1/2, \ldots, 1/2)$ is the maximum element and V_2^n is the set of all minimal elements. For example, Figure 3.6 illustrates the partial order set for $n = 2$.

Let us consider a sequence composed of $n+1$ elements of V_3^n from the maximum element N to a minimal element follows.

$$N = A_0 - A_1 - A_2 - \ldots - A_n \in V_2^n \tag{3.3}$$

In the above sequence A_i $(i = 1, \ldots, n)$ is assumed to be an element that has just i pieces of 0 or 1 and $(n-i)$ pieces of 1/2 as its components, and further it is assumed that A_i $(i = 1, \ldots, n)$ is different from A_{i-1} in only one place, that is, A_i is given by replacing any one 1/2 of A_{i-1} with 0 or 1. A sequence defined by (3) is called *a descending sequence*. For example, for $n = 2$, $(1/2, 1/2) - (1/2, 0) - (1, 0)$ and $(1/2, 1/2) - (1, 1/2) - (1, 1)$ etc. are descending sequences, respectively. Let us count the number of distinct descending sequences. There are $2n$ ways of how to choose A_1 from A_0 since we can choose 0 or 1 in n places. Likewise, we can choose A_2 from A_1 by $2(n-1)$ ways and so on. Therefore, the number of descending sequences is

$$2 \times n \times 2 \times (n-1) \times \ldots \times 2 \times 2 \times 2 \times 1 = 2^n \times n!.$$

Example 6 *For* $n = 2$, *there are* $2^2 \times 2! = 8$ *descending sequences, that is,*

(1) $(1/2, 1/2) - (0, 1/2) - (0, 0)$, *(2)* $(1/2, 1/2) - (0, 1/2) - (0, 1)$,
(3) $(1/2, 1/2) - (1, 1/2) - (1, 0)$, *(4)* $(1/2, 1/2) - (1, 1/2) - (1, 1)$,
(5) $(1/2, 1/2) - (1/2, 0) - (0, 0)$, *(6)* $(1/2, 1/2) - (1/2, 0) - (1, 0)$,
(7) $(1/2, 1/2) - (1/2, 1) - (0, 1)$, *(8)* $(1/2, 1/2) - (1/2, 1) - (1, 1)$.

These are different sequences from N *to* V_2^2 *as shown in Figure 3.6.*

There is a one-to-one correspondence between cell spaces and descending sequences as follows.

Suppose that a cell space is given by $x'_{i_1} \le x'_{i_2} \le \ldots \le x'_{i_n} \le 1/2$. Then, we can construct a descending sequence corresponding to the cell space by following ways. If x'_{i_1} is x_{i_1} $(\sim x_{i_1})$, then A_1 is given by replacing i_1-th place of A_0 with 0 (1). If x'_{i_2} is x_{i_2} $(\sim x_{i_2})$, then A_2 is given by replacing i_2-th place of A_1 with 0 (1). Similarly, if x'_{i_k} is x_{i_k} $(\sim x_{i_k})$, then A_k is given by replacing i_k-th place of A_k with 0 (1). Since $i_1, i_2, \ldots, i_n$ are all different from each other, a descending sequence is finally obtained by the above manner. Conversely, for any given descending sequence we can construct a cell space corresponding to it in a similar manner.

Definition 18 *A cell space* $x'_{i_1} \le x'_{i_2} \le \ldots \le x'_{i_n} \le 1/2$ *and a descending sequence* $N = A_0 - A_1 - \ldots - A_n$ *are corresponding to each other if the following condition holds.*

A_k *is given by replacing* i_k*-th place of* A_{k-1} *with 0 (1) if and only if* x_{i_k} *is* x_{i_k} $(\sim x_{i_k})$ $(k = 1, \ldots, n)$.

The above correspondence is evidently one-to-one, that is, for any given cell space (descending sequence) there is the unique descending sequence (cell space) corresponding to it.

Example 7 *The cell spaces illustrated in Example 5 and the descending sequences in Example 6 are corresponding to each other in the same number. for example, as shown in Figure 3.7, the cell space (7)* $\sim x_2 \le x_1 \le 1/2$ *corresponds to the descending sequence* $(1/2, 1/2) - (1/2, 1) - (0, 1)$ *as well as (8)* $\sim x_2 \le \sim x_1 \le 1/2$ *corresponds to* $(1/2, 1/2) - (1/2, 1) - (1, 1)$.

Lemma 10 *If a cell space corresponds to a descending sequence, then the cell space includes all elements of the descending sequence.*

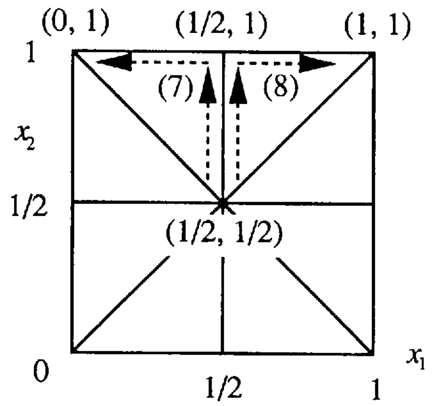

Figure 3.7: Cell Spaces of V^2

Lemma 11 *If a cell space includes all elements of a descending sequence, then the cell space corresponds to the descending sequence.*

From the above two lemmas it was shown that any given descending sequence there exists uniquely the cell space including all elements of it and vice versa.

Theorem 14 *If F is a fuzzy logic function, then the value of F is equal to one of constant (0 or 1), x_i or $\sim x_i$ in each cell space.*

Theorem 15 *Let F be an infinite-valued function which satisfies that the value of F is equal to one of constant (0 or 1), x_i or $\sim x_i$ in each cell space. Then, F is a fuzzy logic function.*

3.3.3 Minimization and Irredundant Form of Fuzzy Logic Functions

Definition 19 *Let F be a disjunctive form of a fuzzy logic function. Then, F is said to be a minimal form of F if and only if no other equivalent disjunctive form involving a smaller total number of literals and constant.*

Note that a minimal form of a fuzzy logic function F does not determined uniquely as well as Boolean functions. Let α be a product term, then α is said to be a implicant of F if and only if $F \sqsupseteq \alpha$. Moreover, a implicant α of F is said to be prime if and only if there is no term α' such that $F \sqsupseteq \alpha' \sqsupseteq \alpha$ and $\alpha \neq \alpha'$.

Theorem 16 *Any minimal form of any fuzzy logic function F is represented by disjunction of prime implicants of F.*

From the above theorem, in order to find a minimal form of F, we have to get all prime implicants of F.

Theorem 17 *Let F be canonical disjunctive form of a fuzzy logic function. Then, every simple product term of F is a prime implicant of F. Conversely, if a simple product term is a prime implicant of F, then it is sure to appear in F.*

From the above theorem, we can easily find all prime implicants of F which are simple product terms. Namely each simple product term existing in canonical disjunctive form of F is prime implicant of F. Therefore, it comes into question that how to find prime implicants of F which are complementary product terms.

Definition 20 *Let α and β be product terms on $x_1, \ldots, x_n$. The consensus of α and β is a complementary term defined as follows (The symbol $C_{\alpha\beta}$ means a set of all consensus of α and β). When there is a variable x_i such that $\alpha = x_i^*\alpha_0$ ($\sim x_i^* \not\sqsupseteq \alpha_0$) and $\beta = \sim x_i^*\beta_0$ ($x_i^* \not\sqsupseteq \beta_0$), where x_i^* means one of x_i or $\sim x_i$*

(1) If $\alpha_0 \cdot \beta_0$ is a complementary product term, then $\alpha_0 \cdot \beta_0 \in C_{\alpha\beta}$.

(2) If $\alpha_0 \cdot \beta_0$ is a simple product term, then $\alpha_0 \cdot \beta_0 \cdot x_j \sim x_j$ for any j such that $i \neq j$.

(3) $C_{\alpha\beta}$ is given only by (1) and (2).

Any repeated literals are removed from the consensus of α and β.

Example 8 *Let α and β be product terms on x, y and z.*

(1) When $\alpha = x \sim y$ and $\beta = \sim xy$, $C_{\alpha\beta} = \{x \sim x, y \sim y\}$,

(2) When $\alpha = xy$ and $\beta = x \sim y$, $C_{\alpha\beta} = \{x \sim x, xz \sim z\}$,

(3) When $\alpha = x$ and $\beta = \sim x$, $C_{\alpha\beta} = \{y \sim y, z \sim z\}$,

Lemma 12 *Let α and β be product terms. If $\gamma \in C_{\alpha\beta}$, then $\gamma \sqsubseteq \alpha \vee \beta$, that is, $\alpha \vee \beta \vee \gamma = \alpha \vee \beta$.*

Theorem 18 *Let $F = \alpha_1 \vee \cdots \vee \alpha_s$ be a disjunctive form of a fuzzy logic function. Then, F is the disjunction of all prime implicants of F if and only if*

(1) no term includes any other term, that is, $\alpha_i \not\sqsupseteq \alpha_j$ for all i, j ($i, j = 1, \ldots, s$ and $i \neq j$) and

(2) the consensus of any two terms dose not exist or is included in any other term α_i ($i = 1, \ldots, s$).

From Theorem 8, we have the following algorithm to find all prime implicants of a fuzzy logic function F which is a disjunctive form.

Algorithm A

Step 1: Remove any term that are included in another term, and let $\alpha_1, \ldots, \alpha_s$ be remaining terms.

Step 2: Find all of the consensus of any two terms α_i and α_j ($i, j = 1, \ldots, s$) If no consensus exists, then in Step 4, otherwise in Step 3.

Step 3: Construct the disjunctive form from $\alpha_1, \ldots, \alpha_s$ and all of the consensus getting in Step 2. If any consensus is included in one of $\alpha_1, \ldots, \alpha_s$, then in Step 4, otherwise in Step 1.

Step 4: The remaining terms are all of the prime implicants of F.

Definition 21 *Let F be a fuzzy logic function and α be an prime implicant of F. Then, α is said to be an essential if it appears all of the minimal form of F, while it is said to be an unessential if it never appear any minimal form of F.*

Theorem 19 *If a simple product term is a prime implicant of a fuzzy logic function, then it is an essential one.*

Theorem 20 *Let simple product terms α and β, and a complementary product termγ be prime implicants of a fuzzy logic function. If γ is a consensus of α and β then γ is an unessential one.*

Lemma 13 *Let α be a complementary minterm and β be arbitrary product term. Then, $\alpha \sqsubseteq \beta$ if and only if $\beta(\boldsymbol{a}) \geq 1/2$ for the corresponding element $\boldsymbol{a} = (a_1, \ldots, a_n)$ to α.*

Proof: Let $\beta = x_1^\beta \cdot \ldots \cdot x_n^\beta$ where each x_i^β is one of x_i, $\sim x_i$, $x_i \sim x_i$ or 1 $(i = 1, \ldots, n)$. Then, $\beta(\boldsymbol{a}) \geq 1/2$ if and only if the following relation holds for every $i = 1, \ldots, n$.

$$x_i^\beta = \begin{cases} \sim x_i \text{ or } 1 & \text{when } a_i = 0, \\ x_i, \sim x_i, x_i \sim x_i \text{ or } 1 & \text{when } a_i = 1/2, \\ x_i \text{ or } 1 & \text{when } a_i = 1 \end{cases}$$

This implies $\alpha \sqsubseteq \beta$, and we have been shown the proof of the lemma. ■

Theorem 21 *Let $\alpha_1 \vee \ldots \vee \alpha_s$ be the canonical disjunctive form of a fuzzy logic function F, and let $\beta_1 \vee \ldots \vee \beta_t$ be a minimal form of F. Then each product term α_i $(i = 1, \ldots, s)$ is included in some product term β_j $(j = 1, \ldots, t)$.*

Proof: First suppose α_i is a simple product term, and let $\boldsymbol{a}$ be the corresponding element to α_i. Then $F(\boldsymbol{a}) = 1$ since $\alpha(\boldsymbol{a}) = 1$, and this implies $(\beta_1 \vee \ldots \vee \beta_t)(\boldsymbol{a}) = 1$. That is, there is a simple product term β_j such that $\beta_j(\boldsymbol{a}) = 1$. Accordingly by Lemma 13 we have $\boldsymbol{a} \succ \boldsymbol{b}$ for the corresponding element $\boldsymbol{b}$ to β_j, and thus $\alpha_i \sqsubseteq \beta_j$.

Next suppose α_i is a complementary minterm, and let $\boldsymbol{a}$ be the corresponding element to α_i. Then $F(\boldsymbol{a}) \geq 1/2$ since $\alpha_i(\boldsymbol{a}) = 1/2$, and this implies there is a product term β_j such that $\beta_j(\boldsymbol{a}) \geq 1/2$. Accordingly by Lemma 13 we have $\alpha_i \sqsubseteq \beta_j$.

This completes the proof of the theorem. ■

Any simple product term of canonical disjunctive form of a fuzzy logic function F is an essential prime implicant of F from Theorem 8 and 9. Moreover, in order to find a minimal form of a fuzzy logic function F, it is not necessary to make a consensus of a pair of simple product terms from Theorem 10. Therefore, we have the following algorithm to find a minimal form of any fuzzy logic function F which is a disjunctive form.

Algorithm B

Step 1: By generating consensus of any two terms which are not both simple product terms, find all prime implicants $\alpha_1, \ldots, \alpha_s$, which are complementary product terms, of F.

Step 2: Expand to canonical disjunctive form, and let $\beta_1, \ldots, \beta_j$ be simple product terms and $\gamma_1, \ldots, \gamma_k$ complementary product terms.

Step 3: Find a minimal group $\alpha'_1, \ldots, \alpha'_t$ from $\alpha_1, \ldots, \alpha_s$ such that $\alpha'_1 \vee \cdots \vee \alpha'_t \sqsupseteq \gamma_1 \vee \cdots \vee \gamma_k$.

Step 4: $F = \beta_1 \vee \cdots \vee \beta_j \vee \alpha'_1 \vee \cdots \vee \alpha'_t$ is a minimal form of the given fuzzy logic function F.

Step 3 of Algorithm B corresponds to the minimum covering problem for Boolean functions. Therefore, Step 3 is solved by using tables like Boolean functions.

Example 9 *Let a 4-variable fuzzy logic function F, which is the canonical disjunctive form, be*

$$\begin{aligned} F &= \sim x_2 \sim x_4 \vee x_1x_2 \sim x_3 \vee \sim x_1x_2x_4 \vee x_1 \sim x_2x_3x_4 \vee \sim x_1x_2x_3 \sim x_3 \sim x_4 \vee \\ & \quad x_1 \sim x_1 \sim x_2 \sim x_3x_4 \vee x_1x_2x_3x_4 \sim x_4 \end{aligned}$$

A minimal form of F is found as follows. By Step 1 of Algorithm B, we have prime implicants of complementary product terms denoted below.

$$\begin{array}{llll} x_4 \sim x_4, & x_2x_3 \sim x_3, & x_1 \sim x_1 \sim x_3, & x_1 \sim x_1 \sim x_2 \\ x_3 \sim x_3 \sim x_4, & x_1x_3 \sim x_3 & x_1 \sim x_1x_4 & \end{array}$$

We get the following 6 minimal forms of F by finding a minimal group from Table 3.

$$\begin{aligned} F &= \sim x_2 \sim x_4 \vee x_1x_2 \sim x_3 \vee \sim x_1x_2x_4 \vee x_1 \sim x_2x_3x_4 \vee x_4 \sim x_4 \\ & \left\{ \begin{array}{c} x_2x_3 \sim x_3 \\ x_3 \sim x_3 \sim x_4 \end{array} \right\} \vee \left\{ \begin{array}{c} x_1 \sim x_1 \sim x_3 \\ x_1 \sim x_1 \sim x_2 \\ x_1 \sim x_1 \sim x_4 \end{array} \right\} \end{aligned}$$

Here, $x_1x_3 \sim x_3$ is an unessential prime implicant.

Table 3.5: Prime Implicants Table of Example 11

	$\sim x_1x_2x_3 \sim x_3$	$x_1 \sim x_1 \sim x_2 \sim x_3x_4$	$x_1x_2x_3x_4 \sim x_4$
$x_4 \sim x_4$			√
$x_2x_3 \sim x_3$	√		
$x_1 \sim x_1 \sim x_3$		√	
$x_1 \sim x_1 \sim x_2$		√	
$x_3 \sim x_3 \sim x_4$	√		
$x_1x_3 \sim x_3$			
$x_1 \sim x_1x_4$		√	

Next, we consider a heuristic method to obtain a minimal form of a given fuzzy logic function F or a similar form to it even if we do not have all of the prime implicants of F of the canonical disjunctive form.

Definition 22 *Let F be a disjunctive form of a fuzzy logic function. Then, F is said to be an irredundant form if we can not omit any product term and any literal appearing in F.*

Theorem 22 *Any irredundant form of a fuzzy logic function F is represented by disjunction of some of the prime implicants of F.*

Theorem 23 *Let F be a minimal form of a fuzzy logic function. Then, F is also an irredundant form.*

From the above, an irredundant form of a fuzzy logic function is considered that it is similar to that of a minimal form.

Next, we discuss a method to find a product term which is able to omit from the given disjunctive form of a fuzzy logic function.

Lemma 14 *Let F be a disjunctive form of a fuzzy logic function, and let α be a simple product term or a complementary product term. Then, $\alpha \sqsubseteq F$ if and only if there exists a product term α' in F such that $\alpha \sqsubseteq \alpha'$.*

Lemma 15 *Let F be a fuzzy logic function and α be a complementary product term. Then, $\alpha \sqsubseteq F$ if and only if $\alpha' \sqsubseteq F$ is valid for any complementary minterm α' such that $\alpha' \sqsubseteq \alpha$.*

Let $F = \alpha_1 \vee \ldots \vee \alpha_s$ be a disjunctive form of a fuzzy logic function. Then by Lemma 14 a simple product term α_i is omitted by another product term if and only if there is a product term α_j $(i \neq j)$ such that $\alpha_i \sqsubseteq \alpha_j$. Also by Lemma 15 a complementary product term α_i is omitted by another product term if and only if there is a product term α_j $(i \neq j)$ such that $\alpha' \sqsubseteq \alpha_j$ for any complementary minterm α' obtained by expanding α_i.

The discussions about which literal can be omitted from a product term follows. However, a simple product term α is a prime implicant if it can not included in any other product term, that is, we never omit any literal appearing in α. Therefore, the target of the following discussions is complementary product terms.

Lemma 16 *Let F be a disjunctive form of a fuzzy logic function, and let β be a complementary product term appearing in F. If a negation $\sim x_i'$ of a literal x_i' appearing in β never exists in β, then the literal is omitted from β if and only if $\beta' \sqsubseteq F$ is valid where β' is a complementary product term obtained by replacing the literal x_i' with its negation $\sim x_i'$.*

Lemma 17 *Let F be a disjunctive form of a fuzzy logic function, and let β be a complementary product term appearing in F. If β is never included in any other product term of F, then any literal x_i' of β is never omitted whenever its negation $\sim x_i'$ exists in β.*

From the above, we have an algorithm to obtained an irredundant form from any given disjunctive form of a fuzzy logic function.

Step 1: Omit any product term including another product term.

Step 2: Omit any literal x_i' of any complementary product term in which its negation $\sim x_i'$ never appears if possible, and then in Step 1. If no literal can be omitted from any product term, then in Step 3.

Step 3: Obtained disjunctive form is an irredundant form.

Example 10 *We again use the canonical disjunctive form in Example ?. The literal $\sim x_1$ of $\sim x_1x_2x_3 \sim x_3 \sim x_4$ can be omitted, and then we have $x_2x_3 \sim x_3 \sim x_4$. Because, $x_1x_2x_3 \sim x_3 \sim x_4 \sqsubseteq x_1x_2 \sim x_3$. Moreover, $\sim x_4$ of the obtained product term $x_2x_3 \sim x_3 \sim x_4$ can be omitted, since by $x_2x_3 \sim x_3x_4 = \sim x_1x_2x_3 \sim x_3x_4 \vee x_1x_2x_3 \sim x_3x_4$ we have $\sim x_1x_2x_3 \sim x_3x_4 \sqsubseteq \sim x_1x_2x_4$ and $x_1x_2x_3 \sim x_3x_4 \sqsubseteq x_1x_2 \sim x_3$. Therefore, we obtain the prime implicant $x_2x_3 \sim x_3$. In the similar manner, $\sim x_2$ and $\sim x_3$ can be omitted from $x_1 \sim x_1 \sim x_2 \sim x_3x_4$, and also x_1, x_2 and x_3 can be omitted from $x_1x_2x_3x_4 \sim x_4$. Finally, we have an irredundant form of F follows.*

$$\sim x_2 \sim x_4 \vee x_1x_2 \sim x_3 \vee \sim x_1x_2x_4 \vee x_1 \sim x_2x_3x_4 \vee x_2x_3 \sim x_3 \vee x_1 \sim x_1x_4 \vee x_4 \sim x_4$$

The above form corresponds to one of minimal forms of F.

3.4 Conclusions

As the previous works directly relating with our studies, the chapter denoted some properties of B-ternary logic functions, regular ternary logic functions and fuzzy logic functions. It is clear by Theorem 10 that the discussions of Section 3.1.2 "Canonical Disjunctive Form of B-Ternary Logic Functions" are available to fuzzy logic functions, and also those of Section 3.3.3 "Minimization and Irredundant Form of Fuzzy Logic Functions" are available to B-ternary logic functions.

Fuzzy logic functions are not functionally complete, that is, they forms a special subset of all functions $F : [0,1]^n \rightarrow [0,1]$. It is very interesting problem to enumerate the number of n-variable fuzzy logic functions to recognize the power of their representations. However, the problem is very hard one, and therefore, the number of them is only enumerating until $n \leq 4$ by the paper [3]. Table 3.6 shows the number of n-variable fuzzy logic functions until $n \leq 4$. Also, by Theorem 10, the number of n-variable B-ternary logic functions is equivalent to that of n-variable fuzzy logic functions.

Table 3.6: The Number of Fuzzy Logic Functions

n	The number of B-ternary logic functions
0	2
1	6
2	84
3	43,918
4	160,297,985,276

Chapter 4

Multiple-Valued Kleenean Functions

4.1 Introduction

In the chapter, we shall describe multiple-valued Kleenean functions which are an effective means to treat ambiguities.

In Chapter 3, we denoted some properties of three different kinds of multiple-valued logic functions, that is, B-ternary logic functions, regular ternary logic functions and fuzzy logic functions. Especially, on regular ternary logic functions, three conditions, i.e., Condition (1): Representation of a regular ternary logic formula, Condition (2): Regularity and Condition (3): Monotonicity for Ambiguity, are equivalent to each other. Therefore, there may be three different kinds of expansions into m-valued logic functions ($m \geq 4$) of regular ternary logic functions. One of them is the expansion of Condition (1). That is, first of all, the operations AND($\cdot$), OR($\vee$) and NOT($\sim$) of Table 3.1 in Chapter 3 are expanded into the set of m truth values $V_m = \{0, 1/(m-1), \ldots, (m-2)/(m-1), 1\}$, which has the property that $x \in V_m$ if and only if $1 - x \in V_m$. Then a expanded n-variable multiple-valued logic function is defined as a function $V_m^n \to V_m$ represented by a logic formula, which is obtained by finitely application of n variables, and three operations AND($\cdot$), OR($\vee$) and ($\sim$). We describe some properties of such functions called multiple-valued Kleenean functions. By the way, some properties [48], [51] and applications [50] of P-type logic functions, which are special case of B-ternary logic functions, have been studied, and they are determined uniquely for only binary inputs $\{0, 1\}$, and are capable of correcting input failures. Majority functions given in [46] are special case of P-type logic functions, and the relationship of these functions also have been studied together with ternary threshold functions [50].

Some properties of multiple-valued Kleenean functions have been studied in [7] $\sim$ [9]. In particular, the paper [7] describes a necessary and sufficient condition for multiple-valued Kleenean functions and a derivation rule of their logic formulas. In [8] relationships between multiple-valued Kleenean functions and ternary functions are discussed, and in [9] it is shown the number of n-variable m-valued Kleenean functions until $n \leq 3$ and $m \leq 8$. On the other hand, the chapter describes the following properties of multiple-valued Kleenean functions. First, in Section 4.2, we will define a partial order relation on the set of truth values $\{0, 1/(m-1), \ldots, (m-2)/(m-2), 1\}$, then show that any multiple-valued Kleenean function is monotone for the relation. Also in Section 4.2, it is shown any multiple-valued Kleenean function is determined uniquely for all assignments of only three kinds truth values 0, 1/2 and 1 to the variables appearing in the logic formula. Next, two different ways representing logic formulas for multiple-valued Kleenean functions are introduced in Section 4.3. The first enables us to compose a logic formula for such a function from its given truth table. The second is a canonical disjunctive form which allows for the unique representation of any multiple-valued Kleenean function. In Section 4.4, minimization for multiple-valued Kleenean functions is described. Finally, in Section 4.5 P-type logic functions, which are multiple-valued Kleenean functions capable of correcting input failures, will be defined, and we will discuss about representation of P-type logic functions, and also show that they are determined uniquely for all assignments of only two kinds truth values 0 and 1 to the variable appearing in the logic formula.

4.2 Multiple-Valued Kleenean Functions and Their Properties

4.2.1 Definition of Multiple-Valued Kleenean Functions

Let V_m be a set of truth values such as $V_m = \{0, 1/(m-1), \ldots, (m-2)/(m-1), 1\}$. Then, an n-variable m-valued function (multiple-valued function, for short) is a mapping from V_m^n to V_m. Here, the logic operations AND($\cdot$), OR($\vee$) and NOT($\sim$) of binary logic can be expanded into multiple-valued logic as follows.

$$a \cdot b = \min(a, b), \quad a \vee b = \max(a, b) \quad \sim a = 1 - a,$$

where a, $b \in V_m$.

An n-variable multiple-valued Kleenean function is defined to be a mapping from V_m^n to V_m, which is represented by a logic formula defined below.

Definition 1 *A logic formula on variables $x_1, \ldots, x_n$ is defined inductively as follows.*

(1) Constants 0, $1/(m-1), \ldots, (m-2)/(m-1)$, 1, and variables $x_1, \ldots, x_n$ are logic formulas.

(2) If G and H are logic formulas, then $(G \cdot H)$, $(G \vee H)$ and $(\sim G)$ are also logic formulas.

(3) The only logic formulas are given by (1) and (2).

Definition 2 *An n-variable multiple-valued function represented by a logic formula is called an n-variable multiple-valued Kleenean function (Kleenean functions below, for short).*

Example 1 $F = (x_1 \cdot x_2 \cdot (\sim x_2)) \vee (1/2 \cdot x_1 \cdot (\sim x_3)) \vee (3/4 \cdot x_2)$ *is 3-variable 5-valued Kleenean function. For the element $\boldsymbol{a} = (1/4,\ 1/2,\ 1)$, $F(\boldsymbol{a}) = 1/2$.*

Hereafter, we will identify a Kleenean function with the logic formula which represents it, if it no confusion arises. The notation $\cdot$ may be often omitted.

Let $\mathcal{K}$ be a set of all n-variable Kleenean functions, then an algebraic system $< \mathcal{K}; \cdot, \vee, \sim, 0, 1 >$ satisfies almost the equations holding Boolean algebras. The following equations are properties holding $< \mathcal{K}; \cdot, \vee, \sim, 0, 1 >$. In the following, $\sim$ is stronger than $\cdot$ and $\vee$, and $\cdot$ is stronger than $\vee$ for omitting parentheses.

1. $a \cdot b = b \cdot a$, $a \vee b = b \vee a$ (the commutative laws)

2. $a \cdot a = a$, $a \vee a = a$ (the idempotent laws)

3. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, $a \vee (b \vee c) = (a \vee b) \vee c$ (the associative laws)

4. $a \vee a \cdot b = a$, $a \cdot (a \vee b) = a$ (the absorption laws)

5. $a \cdot (b \vee c) = a \cdot b \vee a \cdot c$, $a \vee b \cdot c = (a \vee b) \cdot (a \vee c)$ (the distributive laws)

6. $\sim (a \cdot b) = \sim a \vee \sim b$, $\sim (a \vee b) = \sim a \cdot \sim b$ (De Morgan's laws)

7. $\sim (\sim a) = a$ (the double negation law)

8. $0 \cdot a = 0$, $0 \vee a = a$ (the least element)

9. $1 \cdot a = a$, $1 \vee a = 1$ (the greatest element)

10. $a \cdot \sim a \cdot (b \vee \sim b) = a \cdot \sim a$, $a \cdot \sim a \vee b \vee \sim b = b \vee \sim b$ (Kleene's laws)

where a, b, $c \in \mathcal{K}$

It is characteristic feature of the above equations that the complementary laws ($a \cdot \sim a = 0$, $a \vee \sim a = 1$), which hold in Boolean algebras, do not hold. In stead of the complementary laws, it holds Kleene's laws which are weaker conditions than complementary laws. The above equations 1 $\sim$ 10 are identical with the axioms of Kleene algebras given in [2] and [29], that is, a set of Kleenean functions is one of the models of Kleene algebras. This is the reason why multiple-valued functions represented by logic formulas are called Kleenean functions.

In the following sections, we will assume that there exists a truth value i such as $\sim i = i$ (i.e. i=1/2) in the set of truth values V_m, that is, m is assumed to be odd number. Because the results of Kleenean functions obtained in this paper except for Theorem 2 are independent to whether m is odd or not, and we can show them more easily when m is odd than even.

4.2.2 Partial Order Relation $\succ$ for Ambiguity

Definition 3 *Let a and b be elements of V_m. Then, $a \succ b$ (or $b \prec a$) holds if and only if $1/2 \geq a \geq b \geq 0$ or $1/2 \leq a \leq b \leq 1$ holds.*

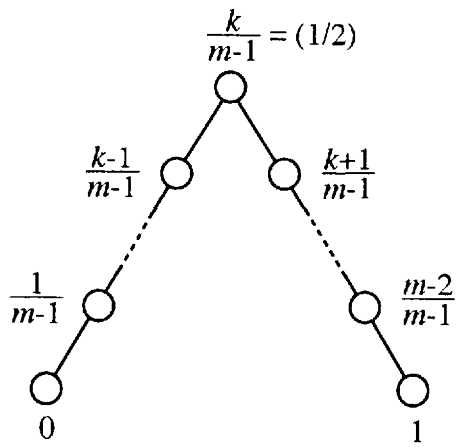

Figure 4.1: Partial Order Relation $\succ$

Figure 1 shows a Hasse diagram of this partial order relation $\succ$, where $k = (m-1)/2$. Moreover, this relation can be expanded among V_m^n as follows. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V_m^n, then $\boldsymbol{a} \succ \boldsymbol{b}$ (or $\boldsymbol{b} \prec \boldsymbol{a}$) holds if and only if $a_i \succ b_i$ (or $b_i \prec a_i$) holds for all i ($i = 1, \ldots, n$). Here, $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be comparable to each other if $\boldsymbol{a} \succ \boldsymbol{b}$ or $\boldsymbol{b} \succ \boldsymbol{a}$ holds, otherwise not comparable.

In the relation $\succ$, 1/2 is interpreted as a truth value expressing an ambiguous state whether true (1) or false (0), and moreover, it can express a ratio of truth to falsehood. It is sure to exist the least upper bound (supremum) of any element a, $b \in V_m$. If a and b are comparable to each other, then there is the greatest lower bound (infimum) of a and b, otherwise there is not. We will write the infimum of a and b as $a\Delta b$, and if the infimum of a and b does not exist, then we will write it as $a\Delta b = \emptyset$. This can be expanded among V_m^n as follows. For two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of V_m^n, we will define $\boldsymbol{a}\Delta\boldsymbol{b}$ as $(a_1\Delta b_1, \ldots, a_n\Delta b_n)$, and if $a_i\Delta b_i = \emptyset$ for some i $(i = 1, \ldots, n)$, then we will define it as $\boldsymbol{a}\Delta\boldsymbol{b} = \emptyset$.

Example 2 *Let* $V_5^3 = \{0, 1/4, 1/2, 3/4, 1\}^3$, $\boldsymbol{a} = (0,\ 1/4,\ 3/4)$, $\boldsymbol{b} = (1/2,\ 1/2,\ 3/4)$ *and* $\boldsymbol{c} = (0,\ 1/2,\ 1) \in V_5^3$, *then* $\boldsymbol{b} \succ \boldsymbol{a}$, $\boldsymbol{b} \succ \boldsymbol{c}$ *and* $\boldsymbol{a}$ *and* $\boldsymbol{c}$ *are not comparable to each other, and* $\boldsymbol{a}\Delta\boldsymbol{b} = \boldsymbol{a}$, $\boldsymbol{b}\Delta\boldsymbol{c} = \boldsymbol{c}$, $\boldsymbol{a}\Delta\boldsymbol{c} = (0,\ 1/4,\ 1)$.

Theorem 1 *Let* F *be a Kleenean function and* $\boldsymbol{a}$, $\boldsymbol{b}$ *elements of* V_m^n. *If* $\boldsymbol{a} \succ \boldsymbol{b}$, *then* $F(\boldsymbol{a}) \succ F(\boldsymbol{b})$.

Proof: It will be shown by induction concerning the number of logic operations. It is clear that the constants $0,\ 1/(m-1), \ldots, (m-2)/(m-1),\ 1$, and each variable $x_1, \ldots, x_n$ satisfy this theorem. Suppose G and H satisfy the theorem. If $\boldsymbol{a} \succ \boldsymbol{b}$ then $G(\boldsymbol{a}) \succ G(\boldsymbol{b})$, that is, $1/2 \geq G(\boldsymbol{a}) \geq G(\boldsymbol{b})$ or $1/2 \leq G(\boldsymbol{a}) \leq G(\boldsymbol{b})$. If $1/2 \geq G(\boldsymbol{a}) \geq G(\boldsymbol{b})$ then $1/2 \leq 1 - G(\boldsymbol{a}) \leq 1 - G(\boldsymbol{b})$, if $1/2 \leq G(\boldsymbol{a}) \leq G(\boldsymbol{b})$ then $1/2 \geq 1 - G(\boldsymbol{a}) \geq 1 - G(\boldsymbol{b})$. Therefore, $(\sim G)$ holds this theorem. Next, we prove that $(G \cdot H)$ and $(G \vee H)$ also satisfy this theorem. $G(\boldsymbol{a}) \succ G(\boldsymbol{b})$ and $H(\boldsymbol{a}) \succ H(\boldsymbol{b})$ hold when $\boldsymbol{a} \succ \boldsymbol{b}$ from the assumption, that is, $1/2 \geq G(\boldsymbol{a}) \geq G(\boldsymbol{b})$ or $1/2 \leq G(\boldsymbol{a}) \leq G(\boldsymbol{b})$, $1/2 \geq H(\boldsymbol{a}) \geq H(\boldsymbol{b})$ or $1/2 \leq H(\boldsymbol{a}) \leq H(\boldsymbol{b})$ hold. First, assume that $1/2 \geq G(\boldsymbol{a}) \geq G(\boldsymbol{b})$ and $1/2 \geq H(\boldsymbol{a}) \geq H(\boldsymbol{b})$ hold, then $1/2 \geq G(\boldsymbol{a}) \cdot H(\boldsymbol{a}) \geq G(\boldsymbol{b}) \cdot H(\boldsymbol{b})$ hold. Therefore, $G(\boldsymbol{a}) \cdot H(a) \succ G(\boldsymbol{b}) \cdot H(\boldsymbol{b})$. For the remaining cases, we can also obtain that $G(\boldsymbol{a}) \cdot H(\boldsymbol{a}) \succ G(\boldsymbol{b}) \cdot H(\boldsymbol{b})$ in the similar manner. Therefore, $(G \cdot H)$ holds this theorem. $(G \vee H)$ is evident, since $(G \vee H) = \sim(\sim G \vee \sim H)$. Thus, this theorem is valid. ∎

It is shown from Theorem 1 that any Kleenean function satisfies monotonicity for the relation $\succ$ with respect to ambiguities.

Example 3 *Let* F *be a Kleenean function such as* $F = x_1x_2 \sim x_2 \vee 1/2 \sim x_1 \sim x_3$. $F(\boldsymbol{a}) = 1/4 \succ 1/4 = F(\boldsymbol{b})$ *when* $\boldsymbol{a} = (1/4,\ 1/2,\ 3/4)$ *and* $\boldsymbol{b} = (1/4,\ 1/4,\ 3/4)$, *and* $F(\boldsymbol{a}) = 1/4 \succ 0 = F(\boldsymbol{b})$ *when* $\boldsymbol{a} = (1/4,\ 1/4,\ 3/4)$ *and* $\boldsymbol{b} = (3/4,\ 0,\ 1)$.

4.2.3 Some Properties

Definition 4 *Let* a *be an element of* V_m. *Then,* $\overline{a}^{\varepsilon}$ *is defined as follows, where* ε *is any value* $0 < \varepsilon \leq 1/2$ *(refer to Fig. 2).*

$$\overline{a}^{\varepsilon} = \begin{cases} 0 & \text{if} \quad 0 \leq a < \varepsilon, \\ 1/2 & \text{if} \quad \varepsilon \leq a \leq 1 - \varepsilon, \\ 1 & \text{if} \quad 1 - \varepsilon < a \leq 1 \end{cases}$$

Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ *be an element of* V_m^n, *then an element* $\overline{\boldsymbol{a}}^{\varepsilon}$ *of* V_3^n *is defined as* $(\overline{a_1}^{\varepsilon}, \ldots, \overline{a_n}^{\varepsilon})$.

Example 4 *Let* $\boldsymbol{a} = (1/2,\ 1/4,\ 1) \in V_5^3$ *and* $\varepsilon = 1/4$. *Then* $\overline{\boldsymbol{a}}^{\varepsilon} = (1/2,\ 1/2,\ 1) \in V_3^3$.

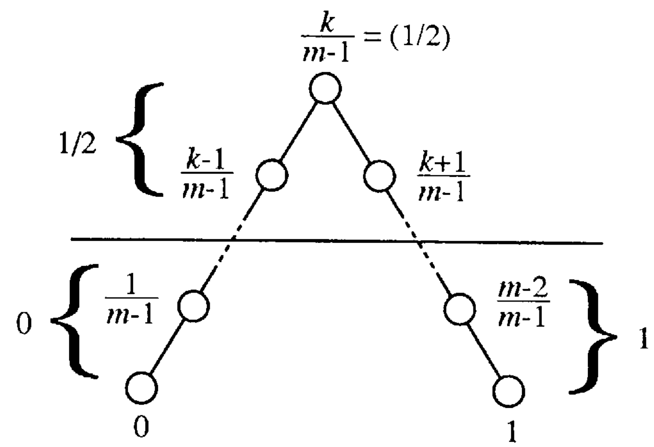

Figure 4.2: Mapping $\overline{a}^{\varepsilon}$ $(0 < \varepsilon \leq 1/2)$

Theorem 2 *Let F be a Kleenean function, $\boldsymbol{a}$ an element of V_m^n. Then, $F_{\varepsilon}(\overline{\boldsymbol{a}}^{\varepsilon}) = \overline{F(\boldsymbol{a})}^{\varepsilon}$ for any value ε $(0 < \varepsilon \leq 1/2)$, where F_{ε} is a Kleenean function obtained by replacing any constant c of the logic formula, which represents F, to $\overline{c}^{\varepsilon}$.*

Proof: It will be shown by induction concerning the number of logic operations. It is clear that the constants 0, $1/(m-1), \dots, (m-2)/(m-1)$, 1, and each variable $x_1, \dots, x_n$ satisfy this theorem. Suppose G and H satisfy this theorem. $\overline{\sim G(\boldsymbol{a})}^{\varepsilon} = \overline{1 - G(\boldsymbol{a})}^{\varepsilon} = 1 - \overline{G(\boldsymbol{a})}^{\varepsilon} = 1 - G_{\varepsilon}(\overline{\boldsymbol{a}}^{\varepsilon}) = \sim G_{\varepsilon}(\overline{\boldsymbol{a}}^{\varepsilon})$. Therefore, $(\sim G)$ satisfies this theorem. Next suppose $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$, then $\overline{G(\boldsymbol{a})}^{\varepsilon} \leq \overline{H(\boldsymbol{a})}^{\varepsilon}$, that is, $G_{\varepsilon}(\overline{\boldsymbol{a}}^{\varepsilon}) \leq H_{\varepsilon}(\overline{\boldsymbol{a}}^{\varepsilon})$. Therefore, we have $\overline{G(\boldsymbol{a}) \cdot H(\boldsymbol{a})}^{\varepsilon} = \overline{G(\boldsymbol{a})}^{\varepsilon} = G_{\varepsilon}(\overline{\boldsymbol{a}}^{\varepsilon}) = G_{\varepsilon}(\overline{\boldsymbol{a}}^{\varepsilon}) \cdot H_{\varepsilon}(\overline{\boldsymbol{a}}^{\varepsilon})$. It is same when $G(\boldsymbol{a}) \geq H(\boldsymbol{a})$. $(G \vee H)$ is evident, since $(G \vee H) = \sim(\sim G \cdot \sim H)$. Thus, this theorem is valid.

Example 5 *Let F be a Kleenean function such as $F = 1/4x_1 \vee x_1 \sim x_2$, and $\varepsilon = 1/5$, $\boldsymbol{a} = (3/4,\ 1/4)$. Then, $F_{\varepsilon} = 1/2x_1 \vee x_1 \sim x_2$. $\overline{F(\boldsymbol{a})}^{\varepsilon} = \overline{1/4 \cdot 3/4 \vee 3/4 \cdot \sim 1/4}^{\varepsilon} = \overline{3/4}^{\varepsilon} = 1/2$ and $F_{\varepsilon}(\overline{\boldsymbol{a}}^{\varepsilon}) = 1/2 \cdot \overline{3/4}^{\varepsilon} \vee \overline{3/4}^{\varepsilon} = 1/2$.*

Lemma 1 *Let G and H be Kleenean functions. If $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n, then $G_{\varepsilon}(\boldsymbol{a}) = H_{\varepsilon}(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n.*

Proof: It is evident from Theorem 2. ∎

Theorem 3 *Let G and H be Kleenean functions. $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n if and only if $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_m^n.*

Proof: Let us suppose that $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n and there is at least an element $\boldsymbol{a}$ of V_m^n such that $G(\boldsymbol{a}) \neq H(\boldsymbol{a})$. This fact means either $G(\boldsymbol{a}) > H(\boldsymbol{a})$ or $G(\boldsymbol{a}) < H(\boldsymbol{a})$. First, suppose $G(\boldsymbol{a}) > H(\boldsymbol{a})$. Then, we can obtain $\overline{G(\boldsymbol{a})}^{\varepsilon} = G_{\varepsilon}(\overline{\boldsymbol{a}}^{\varepsilon}) > H_{\varepsilon}(\overline{\boldsymbol{a}}^{\varepsilon}) = \overline{H(\boldsymbol{a})}^{\varepsilon}$ from Theorem 2, where $\varepsilon = \min(\varepsilon', 1 - \varepsilon')$ and $\varepsilon' = (G(\boldsymbol{a}) + H(\boldsymbol{a}))/2$. This contradicts to the assumption and Lemma 1 since $\overline{\boldsymbol{a}}^{\varepsilon} \in V_3^n$ Therefore, we have been shown the first half of this theorem. The later half is trivial. ∎

Corollary 1 *Let G and H be Kleenean functions. $G(\boldsymbol{a}) \geq H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n if and only if $G(\boldsymbol{a}) \geq H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_m^n.*
(Proof is omitted)

Corollary 2 *Let G and H be Kleenean functions. $G(\boldsymbol{a}) \succ H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n if and only if $G(\boldsymbol{a}) \succ H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_m^n.*
(Proof is omitted)

Definition 5 *Let G and H be Kleenean functions. Then, G includes H (or H is included in G) if and only if $G(\boldsymbol{a}) \geq H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_m^n, and we denote it as $G \sqsupseteq H$ or $H \sqsubseteq G$.*

In accordance with Corollary 1, $G \sqsupseteq H$ if and only if $G(\boldsymbol{a}) \geq H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n.

4.3 Representation for Kleenean Functions

4.3.1 Representation Theorem

In this Section it is shown how a Kleenean function given as a truth table can always be represented in the form of a logic formula.

Since rules of Kleene algebras as shown in section 2, an arbitrary Kleenean function can be expanded into a disjunctive form $F = \gamma_1 \vee \gamma_2 \vee \ldots \vee \gamma_s$, where $\gamma_i \not\sqsupseteq \gamma_j$ for all i, j $(i \neq j)$. However, because the complementary laws are not valid, some terms may contain both a variable and its negation simultaneously. A definition of two kinds of terms follows.

Definition 6 *A variable x or its negation $\sim x$ is called a literal, and the conjunction (AND) of a constant and some literals is said to be a product term (called a term below), where any repeated literals are removed from the terms. Among terms made up of literals only, those which do not contain both a variable and its negation at the same time are called simple terms, while those which do are complementary terms. A simple term and a complementary term in which all variables appear are called a minterm and a complementary minterm, respectively.*

Example 6 *The logic formulas $x_1 \sim x_2x_3$ and $x_1 \sim x_1 \sim x_2x_3$ are simple and complementary terms, respectively. If we consider exactly three variables, then $x_1 \sim x_1 \sim x_2x_3$ is a complementary minterm.*

Some properties of simple terms and complementary minterms will be discussed. In the following, symbol V_2 denotes the set $\{0, 1\}$.

Definition 7 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V_3^n. Then, $\boldsymbol{a}$ and a simple term $\alpha = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ correspond to each other if the following conditions hold.*

$$x_i^{a_i} = \begin{cases} \sim x_i & \text{if} \quad a_i = 0, \\ 1 & \text{if} \quad a_i = 1/2, \\ x_i & \text{if} \quad a_i = 1 \end{cases}$$

Definition 8 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of $V_3^n - V_2^n$. Then, $\boldsymbol{a}$ and a complementary minterm $\beta = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ correspond to each other if the following conditions hold.*

$$x_i^{a_i} = \begin{cases} \sim x_i & \text{if} \quad a_i = 0, \\ x_i \sim x_i & \text{if} \quad a_i = 1/2, \\ x_i & \text{if} \quad a_i = 1 \end{cases}$$

From the above definitions, the correspondence between elements of V_3^n and the simple terms is clearly one-to-one, and also between elements of $V_3^n - V_2^n$ and the complementary minterm.

Example 7 *The simple term corresponding to* (0, 1/2, 1) *is* $\sim x_1 x_3$ *and the complementary minterm is* $\sim x_1 x_2 \sim x_2 x_3$.

Lemma 2 *Let* $\boldsymbol{a} \in V_3^n$ *and* α *be the corresponding simple term to* $\boldsymbol{a}$.

(1) $\boldsymbol{a} \succ \boldsymbol{b}$ *if and only if* $\alpha(\boldsymbol{b}) = 1$,

(2) $\boldsymbol{a} \Delta \boldsymbol{b} = \emptyset$ *if and only if* $\alpha(\boldsymbol{b}) = 0$,

(3) $\boldsymbol{a} \not\succ \boldsymbol{b}$ *and* $\boldsymbol{a} \Delta \boldsymbol{b} \neq \emptyset$ *if and only if* $\alpha(\boldsymbol{b}) = 1/2$,

where $\boldsymbol{b} \in V_3^n$.

Proof: Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$. Then $\alpha(\boldsymbol{b}) = 1$ if and only if $a_i \succ b_i$ for any $i = 1, \ldots, n$, that is, $\boldsymbol{a} \succ \boldsymbol{b}$. Therefore, (1) stands true. Next, $\alpha(\boldsymbol{b}) = 0$ if and only if $a_i = 0$ and $b_i = 1$, or $a_i = 1$ and $b_i = 0$ for some $i = 1, \ldots, n$, that is, $a_i \Delta b_i = \emptyset$, and this implies $\boldsymbol{a} \Delta \boldsymbol{b} = \emptyset$. Accordingly, (2) stands true. We can derive (3) directly for (1) and (2). This completes the proof of the lemma. ∎

Lemma 3 *Let* $\boldsymbol{a} \in V_3^n - V_2^n$ *and* α *be the corresponding complementary minterm to* $\boldsymbol{a}$.

(1) $\boldsymbol{b} \succ \boldsymbol{a}$ *if and only if* $\alpha(\boldsymbol{b}) = 1/2$,

(2) $\boldsymbol{b} \not\succ \boldsymbol{a}$ *if and only if* $\alpha(\boldsymbol{b}) = 0$,

where $\boldsymbol{b} \in V_3^n$.

Proof: Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$. Then $\alpha(\boldsymbol{b}) = 1/2$ if and only if $b_i \succ a_i$ for any $i = 1, \ldots, n$, that is, $\boldsymbol{b} \succ \boldsymbol{a}$. Therefore, (1) stands true. We can derive (2) directly from (1). This completes the proof of the lemma. ∎

Theorem 4 *Let* F *be a Kleenean function. Then* F *can be represented as*

$$F(\boldsymbol{x}) = \bigvee_{\mathbf{c} \in V_3^n} \left\{ \bigwedge_{\mathbf{c} \succ \mathbf{c}'} F(\mathbf{c}')\alpha_c(\boldsymbol{x}) \vee F(\mathbf{c})\beta_c(\boldsymbol{x}) \right\}$$

where α_c *is the simple term corresponding to* $\mathbf{c}$ *and* β_c *is its complementary minterm* ($\beta_c = 0$ *when* $\mathbf{c} \in V_2^n$*). The notation* $\bigvee_{\mathbf{c} \in V_3^n}$ *signifies the disjunction (OR) of every element of* V_3^n, *while* $\bigwedge_{\mathbf{c} \succ \mathbf{c}'}$ *means the conjunction (AND) of every element* $\mathbf{c}'$ *such that* $\mathbf{c} \succ \mathbf{c}'$.

Proof: First, it is shown the following relation holds for all elements $\boldsymbol{a}$ of V_3^n such that $\boldsymbol{a} \neq \mathbf{c}$.

$$(1) \quad F(\boldsymbol{a}) \geq \bigwedge_{\mathbf{c} \succ \mathbf{c}'} F(\mathbf{c}')\alpha_c(\boldsymbol{a}) \vee F(\mathbf{c})\beta_c(\boldsymbol{a})$$

Let assume $F(\boldsymbol{a}) < \bigwedge_{\mathbf{c} \succ \mathbf{c}'} F(\mathbf{c}')\alpha_c(\boldsymbol{a}) \vee F(\mathbf{c})\beta_c(\boldsymbol{a})$, that is, one of

$$(a)\quad F(\boldsymbol{a}) < \bigwedge_{\boldsymbol{c}' \succ \boldsymbol{c}} F(\boldsymbol{a})\alpha_c(\boldsymbol{a}) \text{ or } \quad (b)\quad F(\boldsymbol{a}) < F(\boldsymbol{c})\beta_c(\boldsymbol{a}).$$

It is impossible (b) holds. Because, let suppose $\beta_c(\boldsymbol{a}) = 1/2$ (it is evident when $\beta_c(\boldsymbol{a}) = 0$), then $\boldsymbol{a} \succ \boldsymbol{c}$ by Lemma 3.1 and therefore $F(\boldsymbol{a}) \succ F(\boldsymbol{c})$. Accordingly, in order to satisfy (b), $F(\boldsymbol{c}) > F(\boldsymbol{a}) \geq 1/2$ should be held. This contradicts to (b) since $F(\boldsymbol{c})\beta_c(\boldsymbol{a}) \leq 1/2$. Therefore, (a) should be held. Suppose $\alpha_c(\boldsymbol{a}) = 1$ or $1/2$, since (a) never holds when $\alpha_c(\boldsymbol{a}) = 0$. If $\alpha_c(\boldsymbol{a}) = 1$ then $\boldsymbol{c} \succ \boldsymbol{a}$ by Lemma 2.1 and therefore $F(\boldsymbol{a}) \geq \bigwedge_{\boldsymbol{c} \succ \boldsymbol{c}'} F(\boldsymbol{c}')\alpha_c(\boldsymbol{a})$. If $\alpha_c(\boldsymbol{a}) = 1/2$ then $F(\boldsymbol{a}) \leq 1/2$ should be held. This, however, contradicts to (a) since $F(\boldsymbol{a}) \geq \bigwedge_{\boldsymbol{c} \succ \boldsymbol{c}'} F(\boldsymbol{c}')$ from $\boldsymbol{a}\Delta\boldsymbol{c} \neq \emptyset$. From the above we have the equation (1).

Next, we show the following relation holds.

$$(2)\quad F(\boldsymbol{a}) = \bigwedge_{\boldsymbol{a} \succ \boldsymbol{c}'} F(\boldsymbol{c}')\alpha_a(\boldsymbol{a}) \vee F(\boldsymbol{a})\beta_a(\boldsymbol{a})$$

In this case, $\alpha_a(\boldsymbol{a}) = 1$ and $\beta_a(\boldsymbol{a}) = 1/2$ ($\beta_a(\boldsymbol{a}) = 0$ when $\boldsymbol{a} \in V_2^n$) by Lemma 2.1 and 3.1. When $F(\boldsymbol{a}) > 1/2$, (2) stands true since $\bigwedge_{\boldsymbol{a} \succ \boldsymbol{c}} F(\boldsymbol{c}') = F(\boldsymbol{a})$. When $F(\boldsymbol{a}) \leq 1/2$, $\bigwedge_{\boldsymbol{a} \succ \boldsymbol{c}'} F(\boldsymbol{c}') \leq F(\boldsymbol{a})$. If $\boldsymbol{a} \in V_2^n$ then $\bigwedge_{\boldsymbol{a} \succ \boldsymbol{c}'} F(\boldsymbol{c}') = F(\boldsymbol{a})$ and $\beta_a(\boldsymbol{a}) = 0$, and therefore (2) stands true. If $\boldsymbol{a} \notin V_2^n$ then since $F(\boldsymbol{a})\beta_a(\boldsymbol{a}) = F(\boldsymbol{a}) \cdot 1/2 = F(\boldsymbol{a})$ and the equation (2) stands true. Form the above, we have the equation (2). Consequently, it holds that for all element $\boldsymbol{a} \in V_3^n$,

$$F(\boldsymbol{a}) = \bigvee_{\boldsymbol{c} \in V_3^n} \left\{ \bigwedge_{\boldsymbol{c} \succ \boldsymbol{c}'} F(\boldsymbol{c}')\alpha_c(\boldsymbol{a}) \vee F(\boldsymbol{c})\beta_c(\boldsymbol{a}) \right\}$$

■

Example 8 *Table 1 is a truth table of a 2-variable 5-valued Kleenean function F, and Table 4.1 shows the logic formulas $\bigwedge_{\boldsymbol{c} \succ \boldsymbol{c}'} F(\boldsymbol{c}')\alpha_c(x_1,\, x_2) \vee F(\boldsymbol{c})\beta_c(x_1,\, x_2)$ for all element* **c** *of V_3^n. Therefore, from Theorem 4 we have*

$$\begin{aligned} F(x_1,\, x_2) &= \bigvee_{\boldsymbol{c} \in V_3^2} \left\{ \bigwedge_{\boldsymbol{c} \succ \boldsymbol{c}'} F(\boldsymbol{c}')\alpha_c(x_1,\, x_2) \vee F(\boldsymbol{c})\beta_c(x_1,\, x_2) \right\} \\ &= 1/4 \sim x_1x_2 \sim x_2 \vee 1/4 \sim x_1x_2 \vee 1/2x_1 \sim x_1 \sim x_2 \vee 1/2x_1 \sim x_1x_2 \sim x_2 \vee \\ &\quad (1/4x_2 \vee 1/2x_1 \sim x_1x_2) \vee x_1 \sim x_2 \vee (3/4x_1 \vee 3/4x_1x_2 \sim x_2) \vee 3/4x_1x_2 \\ &= 1/4x_2 \vee 3/4x_1 \vee x_1 \sim x_2. \end{aligned}$$

A simplification of logic formula will appear in the next section.

4.3.2 Canonical Disjunctive Form for Kleenean Functions

This section presents an algorithm with respect to find a canonical disjunctive form of a given Kleenean function, and in addition shows that this form is uniquely determined.

As stated in the previous section, any Kleenean function F can be expanded into disjunctive form $F = \gamma_1 \vee \gamma_2 \vee \ldots \vee \gamma_s$. Each term γ_i can be classified as one of the following three types.

Table 4.1: Truth Table of Kleenean Function F in Example 8

x_1 \ x_2	0	1/4	1/2	3/4	1
0	0	1/4	1/4	1/4	1/4
1/4	1/4	1/4	1/4	1/4	1/4
1/2	1/2	1/2	1/2	1/2	1/2
3/4	3/4	3/4	3/4	3/4	3/4
1	1	3/4	3/4	3/4	3/4

Table 4.2: Each Logic Formula of Example 8

$\boldsymbol{c} = (x_1, x_2)$	$\alpha_c(\boldsymbol{x})$	$\beta_c(\boldsymbol{x})$	$\bigwedge_{\boldsymbol{c} \succ \boldsymbol{c}'} F(\boldsymbol{c}')\alpha_c(\boldsymbol{x}) \vee F(\boldsymbol{c})\beta_(\boldsymbol{x})$
(0, 0)	$\sim x_1 \sim x_2$	0	0
(0, 1/2)	$\sim x_1$	$\sim x_1x_2 \sim x_2$	$1/4 \sim x_1x_2 \sim x_2$
(0, 1)	$\sim x_1x_2$	0	$1/4 \sim x_1x_2$
(1/2, 0)	$\sim x_2$	$x_1 \sim x_1 \sim x_2$	$1/2x_1 \sim x_1 \sim x_2$
(1/2, 1/2)	1	$x_1 \sim x_1x_2 \sim x_2$	$1/2x_1 \sim x_1x_2 \sim x_2$
(1/2, 1)	x_2	$x_1 \sim x_1x_2$	$1/2x_2 \vee 1/2x_1 \sim x_1x_2$
(1, 0)	$x_1 \sim x_2$	0	$x_1 \sim x_2$
(1, 1/2)	x_1	$x_1x_2 \sim x_2$	$3/4x_1 \vee 3/4x_1x_2 \sim x_2$
(1, 1)	x_1x_2	0	$3/4x_1x_2$

type 1: $c \cdot \alpha$, where c is a constant such that $c > 1/2$ and α is a simple term.

type 2': $c \cdot \alpha$, where c is a constant such that $c \leq 1/2$ and α is a simple term.

type 3': $c \cdot \beta$, where c is a constant such that $c \leq 1/2$ and β is a complementary term.

Now, when the constant of a type 3' term is such that $c \geq 1/2$, it may be omitted, since β is a complementary term and therefore $x_i \sim x_i \leq 1/2$ holds for some variable x_i in β. Now suppose that there is not a variable x_i in a type 2' term $c\alpha$. Then it holds that $c\alpha = c\alpha(x_i \vee \sim x_i) = c\alpha x_i \vee c\alpha \sim x_i$ since $x_i \vee \sim x_i \geq 1/2 \geq c$. Similarly suppose there is not a variable x_i in a type 3' term $c\beta$. Then is holds that $c\beta = c\beta(x_i \vee \sim x_i) = c\beta x_i \vee c\beta \sim x_i$ since there is a product $x_j \sim x_j$ for some j in β and $x_j \sim x_j \leq 1/2 \leq x_i \sim x_i$. Accordingly, any α and β of type 2' and 3', respectively, can be expanded into a disjunction of minterms, or complementary minterms. By Definition 7, minterms in particular correspond one-to-one with the elements of V_2^n. Therefore, a Kleenean function can be expanded into a disjunction of the following three types of terms.

type 1,

type 2: $c \cdot \alpha$, where c is a constant such that $c \leq 1/2$ and α is a minterm, and

type 3: $c \cdot \beta$, where c is a constant such that $c \leq 1/2$ and β is a complementary minterm.

In type 1 term, the constant c may be omitted when $c = 1$, and also the constant c may be omitted when $c = 1/2$ if it is in type 2 or type 3. Next, let us examine the relation of each type of term.

Lemma 4 *Let $\gamma = c\alpha$ and $\gamma' = c'\alpha'$ be terms where c and c' are constants and α and α' are products of literals. $\gamma \sqsupseteq \gamma'$ (i.e. $\gamma \vee \gamma' = \gamma$) if and only if all of the literals in α exist in α' as well and $c \geq c'$.*
(Proof is omitted)

Let $\alpha = c\alpha_0$ be one of the three types of terms, where c is a constant and α_0 is the product of literals. Then, the element corresponding to α_0 is said to be the corresponding element to α. This being so, elements of V_3^n correspond to type 1 terms, elements of V_2^n to type 2 terms and elements of $V_3^n - V_2^n$ to type 3 terms. Moreover, if α, β and γ are type 1, 2 and 3 terms, respectively, then $\alpha \not\sqsupseteq \beta$, $\alpha \not\sqsupseteq \gamma$ and $\beta \not\sqsupseteq \gamma$ from the above and the definition of three types of terms.

Lemma 5 *Let $\alpha = a \cdot \alpha_0$ and $\beta = b \cdot \beta_0$ each be one of the three type of terms, and let $\boldsymbol{a}$ and $\boldsymbol{b}$ be elements of V_3^n that correspond to them, respectively. Then $\alpha \sqsupseteq \beta$ if and only if*

(1) $a \geq b$ and $\boldsymbol{a} \succ \boldsymbol{b}$, when α is a type 1 and β is a type 1 or 2,

(2) $a \geq b$ and $\boldsymbol{a}\Delta\boldsymbol{b} \neq \emptyset$, when α is a type 1 or 2 and β is a type 3,

(3) $a \geq b$ and $\boldsymbol{a} = \boldsymbol{b}$, when α and β are both type 2, and

(4) $a \geq b$ and $\boldsymbol{b} \succ \boldsymbol{a}$, when α and β are both type 3.

Proof: Let $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$, $\alpha_0 = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ and $\beta_0 = x_1^{b_1} \cdot \ldots \cdot x_n^{b_n}$. First, assume α is a type 1 term and β is a type 1 or 2 term. Suppose $a \geq b$ and $\boldsymbol{a} \succ \boldsymbol{b}$, then $\boldsymbol{a} \succ \boldsymbol{b}$ implies $a_i \succ b_i$ for all i $(i = 1, \ldots, n)$, that is, $x_i^{a_i} \geq x_i^{b_i}$ for all i. Therefore, $\alpha \sqsupseteq \beta$. Next, suppose $\alpha \sqsupseteq \beta$, that is, $\alpha \vee \beta = \alpha$. In this case, it is clear that $a \geq b$ and $\alpha(\boldsymbol{b}) \geq \beta(\boldsymbol{b})$. Therefore, $\alpha_0(\boldsymbol{b}) = 1$ since $\beta_0(\boldsymbol{b}) = 1$, and $\boldsymbol{a} \succ \boldsymbol{b}$ from Lemma 2.1. From the above, we have (1). (2), (3) and (4) are proved in the same manner. ∎

Definition 9 *A Kleenean function F represented as $F = \gamma_1 \vee \gamma_2 \vee \ldots \vee \gamma_s$ is said to be in canonical disjunctive form when each term γ_i $(i = 1, \ldots, s)$ is of type 1, 2, or 3 and $\gamma_i \not\sqsupseteq \gamma_j$ for all i and j $(i \neq j)$.*

Example 9 *Let $F = x_1(1/4 \cdot x_2 \vee 3/4 \cdot x_1 \vee x_3 \sim x_3)$.*

$$\begin{aligned} F &= 1/4 \cdot x_1x_2 \vee 3/4 \cdot x_1 \vee x_1x_3 \sim x_3 \\ &= 1/4 \cdot x_1x_2(x_3 \cdot x_3) \vee 3/4 \cdot x_1 \vee x_1(x_2 \vee \sim x_2)x_3 \sim x_3 \\ &= 1/4 \cdot x_1x_2x_3 \vee 1/4 \cdot x_1x_2 \sim x_3 \vee 3/4 \cdot x_1 \vee x_1x_2x_3 \sim x_3 \vee x_1 \sim x_2x_3 \sim x_3 \\ &= 3/4 \cdot x_1 \vee x_1x_2x_3 \sim x_3 \vee x_1 \sim x_2x_3 \sim x_3. \end{aligned}$$

Lemma 6 *Let the canonical disjunctive form for a Kleenean function F be $F_f = \alpha_1 \vee \ldots \vee \alpha_s$. If $\boldsymbol{a}$ is the element corresponding to the term α_i $(i = 1, \ldots, s)$ then $F_f(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$.*

Proof: First, let suppose α_i be a type 1 term and $F_f(\boldsymbol{a}) > \alpha_i(\boldsymbol{a})$. Then there is a type 1 term α_j such that $\alpha_j(\boldsymbol{a}) > \alpha_i(\boldsymbol{a})$ in F_f. Let $\alpha_i = c_i \cdot \alpha_i'$ and $\alpha_j = c_j \cdot \alpha_j'$, where c_i and c_j are constants, and α_i' and α_j' are products of literals in α_i and α_j, respectively. Then $c_i > c_j$ and $\alpha_j'(\boldsymbol{a}) = 1$. Therefore, $\boldsymbol{b} \succ \boldsymbol{a}$ from Lemma 2.1 where $\boldsymbol{b}$ is the corresponding element to α_j, that is, α_i is included in α_j from Lemma 5.1. This contradicts to that F_f is canonical disjunctive form. We can prove in the similar manner when α_i is a type 2 or 3 term. ∎

Theorem 5 *Any Kleenean function can be represented uniquely by canonical disjunctive form (ignoring order of terms).*

Proof: Let suppose there are two different canonical disjunctive forms F_1 and F_2 of a Kleenean function F (It is evident from the above discussion that there is at least one canonical disjunctive form of F). Now, we can suppose that a term α exists in F_1 but not F_2, and let $\boldsymbol{a}$ be the corresponding element to α. First, let α be a type 1 term. $F_1(\boldsymbol{a}) = \alpha(\boldsymbol{a})$ by Lemma 6, and hence $F_2(\boldsymbol{a}) = \alpha(\boldsymbol{a})$. Therefore, F_2 contains a type 1 term α' such that $\alpha'(\boldsymbol{a}) = \alpha(\boldsymbol{a})$, and $\boldsymbol{a}' \succ \boldsymbol{a}$ (Lemma 2.1) where $\boldsymbol{a}'$ is the corresponding element to α'. By assumption $\alpha \neq \alpha'$, and hence $\boldsymbol{a} \neq \boldsymbol{a}'$. Similarly, F_1 contains a type 1 term α'' such that $\alpha''(\boldsymbol{a}') = \alpha'(\boldsymbol{a}')$ and $\boldsymbol{a}'' \succ \boldsymbol{a}'$ (Lemma 2.1) where $\boldsymbol{a}''$ is the corresponding element to α''. Therefore, α is included in α'' (Lemma 5.1) since $\boldsymbol{a}'' \succ \boldsymbol{a}$ and $\boldsymbol{a} \neq \boldsymbol{a}''$. This contradicts that F_1 is canonical disjunctive form. Next, let α be a type 2 or type 3 term. $F_2(\boldsymbol{a}) = \alpha(\boldsymbol{a})$ since $F_1(\boldsymbol{a}) = \alpha(\boldsymbol{a})$ by Lemma 6. Therefore, F_2 contains at least one of type 1 $\sim$ type 3 term α' such that $\alpha'(\boldsymbol{a}) = \alpha(\boldsymbol{a})$. Let $\boldsymbol{a}'$ be the corresponding element to α'. When α' is a type 1 term, $\boldsymbol{a}\Delta\boldsymbol{a}' \neq \emptyset$ by Lemma 2.2, and $F_1(\boldsymbol{a}') = \alpha'(\boldsymbol{a}')$ since $F_2(\boldsymbol{a}') = \alpha'(\boldsymbol{a}')$ by Lemma 6. Therefore, F_1 contains a type 1 term α'' such that $\alpha''(\boldsymbol{a}') = \alpha'(\boldsymbol{a}')$, and $\boldsymbol{a}'' \succ \boldsymbol{a}'$ by Lemma 2.1 where $\boldsymbol{a}''$ is the corresponding element to α''. Since $\boldsymbol{a}'' \succ \boldsymbol{a}'$ and $\boldsymbol{a}\Delta\boldsymbol{a}' \neq \emptyset$, it holds $\boldsymbol{a}\Delta\boldsymbol{a}'' \neq \emptyset$, and accordingly α is included in α'' by Lemma 5.1 and 5.2. This contradicts to the assumption. Thus, α' can not be a type 1 term. When α' is a type 2 or type 3 term, $\boldsymbol{a} \succ \boldsymbol{a}'$ by Lemma 2.2 and 3.1 and we can show similarly that F_1 contains a type 2 or type 3 term α'' such that $\boldsymbol{a}' \succ \boldsymbol{a}''$ where $\boldsymbol{a}''$ is the corresponding element to α''. Since $\boldsymbol{a} \succ \boldsymbol{a}'$ and $\boldsymbol{a}' \succ \boldsymbol{a}''$, it holds $\boldsymbol{a} \succ \boldsymbol{a}''$, and accordingly α is included in α'' by Lemma 5.2, 5.3 and 5.4. This contradicts to the assumption. Therefore, all terms contained in F_1 exist also in F_2, and conversely, any term in F_2 is also in F_1. Accordingly, the canonical disjunctive from of a Kleenean function is uniquely determined. ∎

4.4 Minimization for Kleenean Functions

In this section, we describe minimization for Kleenean functions. A minimal form described in this section is defined as follows, and motivated by Boolean functions and fuzzy logic functions[9] $\sim$ [14].

Definition 10 *Let F be a disjunctive form of a Kleenean function. Then, F is said to be a minimal form of F if and only if no other equivalent disjunctive form involving a smaller total number of literals and constant.*

Note that a minimal form of a Kleenean function F does not determined uniquely as well as the cases of Boolean functions and fuzzy logic functions. Let α be a term, then α is said to be a implicant of F if and only if $F \sqsupseteq \alpha$. Moreover, a implicant α of F is said to be prime if and only if there is no term α' such that $F \sqsupseteq \alpha' \sqsupseteq \alpha$ and $\alpha \neq \alpha'$.

Theorem 6 *A minimal form of a Kleenean function F is represented by disjunction of prime implicants of F.*

Proof: Let $\alpha_1 \vee \ldots \vee \alpha_s$ be a minimal form of F, and suppose α_i is not a prime implicant of F. Then, there is a implicant α of F such that $\alpha \sqsupseteq \alpha_i$ and $\alpha \neq \alpha_i$. By Lemma 4 this contradicts to $\alpha_1 \vee \ldots \vee \alpha_s$ is a minimal form of F. ∎

From the above theorem, in order to find a minimal form of F, we have to get all prime implicants of F.

Lemma 7 *Let F be a disjunctive form of a Kleenean function, and $\boldsymbol{a}$ the element corresponding to a type 1 term α. Then $F(\boldsymbol{a}) \geq \alpha(\boldsymbol{a}) > 1/2$ if and only if there is a type 1 term α' in F such that $\alpha' \sqsupseteq \alpha$.*
(Proof is omitted)

Theorem 7 *Let F be canonical disjunctive form of a Kleenean function. Then, every type 1 term of F is a prime implicant of F. Conversely, if a type 1 term is a prime implicant of F, then it is sure to appear in F.*

Proof: Let suppose a type 1 term α in F is not a prime implicant of F. Then there is a type 1 term β such that $F \sqsupseteq \beta \sqsupseteq \alpha$ and $\beta \neq \alpha$. Therefore, $F(\boldsymbol{b}) \geq \beta(\boldsymbol{b}) > 1/2$ for the corresponding element $\boldsymbol{b}$ of β. Thus, by Lemma 7 there is a type 1 term β' in F such that $\beta' \sqsupseteq \beta \sqsupseteq \alpha$. Therefore α is included in β', and this contradicts to F is canonical disjunctive form. Conversely, let a type 1 term α be a prime implicant of F and let $\boldsymbol{a}$ the corresponding element to α. Then there is a type 1 term β in F such that $\beta \sqsupseteq \alpha$ from $F(\boldsymbol{a}) \geq \alpha(\boldsymbol{a}) > 1/2$ and Lemma 7. Since α is a prime implicant of F, $\alpha = \beta$. Therefore, α exists in F. ■

From the above theorem, we can easily find all prime implicants of F which are type 1 terms. Namely each type 1 terms existing in canonical disjunctive form of F is prime implicant of F. Therefore, it comes into question that how to find prime implicants of F which are type 2' or type 3' terms.

Definition 11 *Let α and β be terms on $x_1, \ldots, x_n$. The consensus of α and β is a type 2' or type 3' term defined as follows (The symbol $C_{\alpha\beta}$ means a set of all consensus of α and β). When there is a variable x_i such that $\alpha = x_i^* \alpha_0$ $(\sim x_i^* \not\sqsupseteq \alpha_0)$ and $\beta = \sim x_i^* \beta_0$ $(x_i^* \not\sqsupseteq \beta_0)$, where x_i^* means one of x_i or $\sim x_i$*

(1) If $\alpha_0 \cdot \beta_0$ is a type 1 term, then $1/2 \cdot \alpha_0 \cdot \beta_0 \in C_{\alpha\beta}$

(2) If $\alpha_0 \cdot \beta_0$ is a type 2' or type 3' term, then $\alpha_0 \cdot \beta_0 \in C_{\alpha\beta}$

(3) $C_{\alpha\beta}$ is given only by (1) and (2)

Any repeated literals are removed from the consensus of α and β, and its constant is minimum of that of α and β.

Example 10 *Let α and β be terms on x and y.*

(1) When $\alpha = x \sim y$ and $\beta = 3/4 \sim xy$, $C_{\alpha\beta} = \{x \sim x,\ y \sim y\}$,

(2) When $\alpha = xy$ and $\beta = x \sim y$, $C_{\alpha\beta} = \{1/2x\}$,

(3) When $\alpha = x$ and $\beta = \sim x$, $C_{\alpha\beta} = \{1/2\}$,

(4) When $\alpha = 1/4x \sim y$ and $\beta = xy$, $C_{\alpha\beta} = \{1/4x\}$.

Lemma 8 *Let α and β be terms. If $\gamma \in C_{\alpha\beta}$, then $\gamma \sqsubseteq \alpha \vee \beta$, that is, $\alpha \vee \beta \vee \gamma = \alpha \vee \beta$.*
(Proof is omitted)

Lemma 9 *Let $F = \alpha_1 \vee \ldots \vee \alpha_s$ be a disjunctive form of a Kleenean function F and let $\beta = \beta_1 \vee \ldots \vee \beta_t$ be canonical disjunctive form of a term β. If there is a term β_j $(j = 1, \ldots, t)$ for all terms α_i $(i = 1, \ldots, s)$ such that $\alpha_i \not\sqsupseteq \beta_j$, then $F \not\sqsupseteq \beta$.*
(Proof is omitted)

Theorem 8 *Let $F = \alpha_1 \vee \cdots \vee \alpha_s$ be a disjunctive form of a Kleenean function. Then, F is the disjunction of all prime implicants of F if and only if*

(1) no term includes any other term, that is, $\alpha_i \not\sqsupseteq \alpha_j$ for all i, j $(i,\ j = 1, \ldots, s$ and $i \neq j)$ and

(2) the consensus of any two terms dose not exist or is included in any other term α_i $(i = 1, \ldots, s)$.

Proof: It is evident that (1) and (2) hold when $F = \alpha_1 \vee \ldots \vee \alpha_s$ is disjunction of all prime implicants of F. Conversely, let assume $F = \alpha_1 \vee \ldots \vee \alpha_s$ is not a disjunction of all prime implicants of F when F holds (1) and (2). This means one of the following relations holds.

(*a*) a prime implicant α of F does not appear in $\alpha_1, \ldots, \alpha_s$, or

(*b*) a term α_i $(i = 1, \ldots, s)$ is not a prime implicant of F.

First, assume (*a*) holds, then it is impossible α is a type 1 term. Because, there is a type 1 term α_i $(i = 1, \ldots, s)$ in F such that $\alpha_i \sqsupseteq \alpha$ since $F(\boldsymbol{a}) \geq \alpha(\boldsymbol{a}) > 1/2$ and Lemma 7 where $\boldsymbol{a}$ is an element corresponding to α. The relation $\alpha_i \sqsupseteq \alpha$ is equivalent to $\alpha_i = \alpha$ since α is a prime implicant of F. Therefore, let suppose α is a type 2' or type 3', and let $\alpha = c \cdot \alpha_0$ where c $(\leq 1/2)$ is a constant and α_0 is product of literals. It may be possible to add some literals to α_0 if their negations do not exist in α_0, forming a term β which still has the property that $\alpha_i \not\sqsupseteq c\beta$ for all i $(i = 1, \ldots, s)$ (i.e. a literal x_i^* can be added to α_0 if variable x_i does not exist in α_0). Finally, we can get a term α_0' from α_0 satisfying the following conditions.

Condition 1: $\alpha_i \not\sqsupseteq c\alpha_0'$ for all i $(i = 1, \ldots, s)$.

Condition 2: For any possible x_i there are α_j and α_k such that $\alpha_j \sqsupseteq c\alpha_0' x_i$ and $\alpha_k \sqsupseteq c\alpha_0' \sim x_i$ $(j,\ k = 1, \ldots, s$ and $j \neq k)$.

It is impossible α_0' becomes a minterm or a complementary minterm from Lemma 9. Let $\alpha_j = c_j \alpha_j' x_i$ and $\alpha_k = c_k \alpha_k' \sim x_i$ where c_j and c_k are constants, and α_j' and α_k' are product of literals which satisfy $\sim x_i \not\sqsupseteq \alpha_j'$ and $x_i \not\sqsupseteq \alpha_k'$ from *Condition* 1 and 2. Then we have $\alpha_j \alpha_k \sqsupseteq c\alpha_0' x_i \sim x_i$ from *Condition* 2, that is, $c' \alpha_j' \alpha_k' \sqsupseteq c\alpha_0'$ where $c' = \min(c_j,\ c_k)$. If $c'\alpha_j'\alpha_k'$ is a type 2' or type 3' term then it is a consensus of α_j and α_k. In accordance with the assumption (2), there is a term α_t $(t = 1, \ldots, s)$ such that $\alpha_t \sqsupseteq c'\alpha_j'\alpha_k'$. Therefore, $\alpha_t \sqsupseteq c\alpha_0'$ holds, however, this is contradictory to *Condition* 1. If $c'\alpha_j'\alpha_k'$ is a type 1 term then $1/2\alpha_j'\alpha_k'$ is a consensus of α_j and α_k. Since $c \leq 1/2$ and the assumption (2), $\alpha_t \sqsupseteq 1/2\alpha_j'\alpha_k' \sqsupseteq c\alpha_0'$ holds for some t $(t = 1, \ldots, s)$, and this is contradictory to *Condition* 1. Therefore, no term α_t $(t = 1, \ldots, s)$ includes any consensus of α_j and α_k, however, this is contradictory to the assumption (2). Thus, all prime implicants of F appear in $\alpha_1, \ldots, \alpha_s$. Next, assume (*b*) holds, then there is a prime implicant α_j in $\alpha_1, \ldots, \alpha_s$ such that $\alpha_j \sqsupseteq \alpha_i$ since all prime implicants of F appear in $\alpha_1, \ldots, \alpha_s$ from the above discussions. Therefore, α_j includes α_i, and this is contradictory to the assumption (1). Thus, $F = \alpha_1 \vee \cdots \vee \alpha_s$ is the disjunction of all prime implicants of F. ■

From Theorem 8, we have the following algorithm to find all prime implicants of a Kleenean function F which is a disjunctive form.

Algorithm A

Step 1: Remove any term that are included in any other term, and let $\alpha_1, \ldots, \alpha_s$ be remaining terms.

Step 2: Find all of the consensus of any two terms α_i and α_j $(i, j = 1, \ldots, s)$ If no consensus exists, then in Step 4, otherwise in Step 3.

Step 3: Construct the disjunctive form from $\alpha_1, \ldots, \alpha_s$ and all of the consensus getting in Step 2. If any consensus is included in one of $\alpha_1, \ldots, \alpha_s$, then in Step 4, otherwise in Step 1.

Step 4: The remaining terms are all of the prime implicants of F.

Definition 12 *Let α be a prime implicant of a Kleenean function F then it is said that α is an essential if it exists all minimal form of F. Conversely, α said to be an unessential if α does not exist any minimal form of F.*

Theorem 9 *If a type 1 term α is a prime implicant of a Kleenean function F then α is an essential one.*

Proof: Let $F = \alpha_1 \vee \cdots \vee \alpha_s$ be a minimal form of F and let a type 1 term α be a prime implicant of F. Then $F(\boldsymbol{a}) \geq \alpha(\boldsymbol{a}) > 1/2$ where $\boldsymbol{a}$ is the corresponding element to α, and by Lemma 7 there is a type 1 term α_i $(i = 1, \ldots, s)$ such that $\alpha_i \sqsupseteq \alpha$. This implies $\alpha_i = \alpha$ since α is a prime implicant of F. Therefore, α is an essential one. ∎

Theorem 10 *Let type 1 term α and β, and type 2' or type 3' term γ be prime implicants of a Kleenean function F. If γ is a consensus of α and β then γ is an unessential one.*

Proof: By Lemma 8 $\alpha \vee \beta \sqsupseteq \gamma$, and by Theorem 9 α and β are essential. Therefore, γ is an unessential one. ∎

Lemma 10 *Let α be a type 2 or 3 term and $\boldsymbol{a} = (a_1, \ldots, a_n)$ be the corresponding element to $\alpha = a\alpha'$ where a is the constant appearing in α and α' is a minterm when α is type 2 and a complementary minterm when α is type 3. If $\beta = b\beta'$ where b and β' are, respectively, the constant and a term appearing in β, is an arbitrary term, then*

(1) when α is a type 2 term, $\alpha \sqsubseteq \beta$ if and only if $a \leq b$ and $\beta'(\boldsymbol{a}) = 1$,

(2) when α is a type 3 term, $\alpha \sqsubseteq \beta$ if and only if $a \leq b$ and $\beta'(\boldsymbol{a}) \geq 1/2$.

Proof: First suppose α is a type 2 term, and $a \leq b$ and $\beta'(\boldsymbol{a}) = 1$. Let $\beta' = x_1^{\beta} \cdot \ldots \cdot x_n^{\beta}$ where each x_i^{β} denotes one of x_i, $\sim x_i$, $x_i \sim x_i$ or 1, and let $\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. Then, in order to keep the condition $\beta'(\boldsymbol{a}) = 1$, one of the following relations has to hold for every $i = 1, \ldots, n$.

$$\begin{array}{ll} x_i^{\beta} = x_i \text{ or } 1 & \text{when } a_i = 0, \\ x_i^{\beta} = 1 & \text{when } a_i = 1/2 \text{ or} \\ x_i^{\beta} = x_i \text{ or } 1 & \text{when } a_i = 1. \end{array}$$

Therefore, $x_i^{\alpha_i} \sqsubseteq x_i^{\beta}$ for any i, and this implies $\alpha' \sqsubseteq \beta'$. Accordingly we have $\alpha \sqsubseteq \beta$ form $a \leq b$ and $\alpha' \sqsubseteq \beta'$. Conversely suppose $\alpha \sqsubseteq \beta$. Then it is evident $a \leq b$, and moreover we have $\beta'(\boldsymbol{a}) = 1$ by $\alpha'(\boldsymbol{a}) = 1$ and $\alpha \sqsubseteq \beta$. It has been shown the proof of the lemma (1).

Next suppose α is a type 3 term, and $a \leq b$ and $\beta'(\boldsymbol{a}) \geq 1/2$. Let $\beta' = x_1^{\beta} \cdot \ldots \cdot x_n^{\beta}$ where each x_i^{β} denotes one of x_i, $\sim x_i$, $x_i \sim x_i$ or 1, and let $\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. Then, in order to keep the condition $\beta'(\boldsymbol{a}) \geq 1/2$, one of the following relations has to hold for every $i = 1, \ldots, n$.

$$\begin{array}{ll} x_i^{\beta} =\sim x_i \text{ or } 1 & \text{when } a_i = 0, \\ x_i^{\beta} = x_i, \sim x_i, x_i \sim x_i \text{ or } 1 & \text{when } a_i = 1/2 \text{ or} \\ x_i^{\beta} = x_i \text{ or } 1 & \text{when } a_i = 1. \end{array}$$

Therefore, $x_i^{\alpha_i} \sqsubseteq x_i^{\beta}$ for any i, and this implies $\alpha' \sqsubseteq \beta'$. Accordingly we have $\alpha \sqsubseteq \beta$ from $a \leq b$ and $\alpha' \sqsubseteq \beta'$. Conversely suppose $\alpha \sqsubseteq \beta$. Then it is evident $a \leq b$, and moreover we obtain $\beta'(\boldsymbol{a}) \geq 1/2$ by $\alpha'(\boldsymbol{a}) = 1/2$ and $\alpha \sqsubseteq \beta$. It has been shown the proof of the lemma (2).

This completes the proof of the lemma. ■

Theorem 11 *Let $\alpha_1 \vee \ldots \vee \alpha_s$ be the canonical disjunctive form of a Kleenean function F, and let $\beta_1 \vee \ldots \vee \beta_t$ be a minimal form of F. Then each minterm α_i $(i = 1, \ldots, s)$ is included in some product term β_j $(j = 1, \ldots, t)$.*

Proof: First suppose α_i is a type 1 term, and let $\boldsymbol{a}$ be the corresponding element to α_i. Then $F(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$ by Lemma 6, and this implies $(\beta_1 \vee \ldots \vee \beta_t)(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$, that is, there is a type 1 term β_j such that $\beta_j(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$. This means that $a = b$ and $\beta_j'(\boldsymbol{a}) = 1$ where a and b are constants appearing in α_i and β_j, respectively, and β_j' is a simple term appearing in β_j. Accordingly by Lemma 2.1 we have $\boldsymbol{a} \succ \boldsymbol{b}$, and therefore by Lemma 5.1 $\alpha_i \sqsubseteq \beta_j$.

Next suppose α_i is a type 2 term, and let $\boldsymbol{a}$ be the corresponding element to α_i. Then by Lemma 6 $F(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$, and therefore there is a type 2' or type 3' term β_j such that $\beta_j(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$. It is impossible β_j is a type 3' term. Because, let $\beta_{1_j} \vee \ldots \vee \beta_{q_j}$ be the canonical disjunctive form of β_j, that is, each β_{l_j} $(l = 1, \ldots, q)$ is a type 3 term. Then by $\beta_j(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$ there is a term β_{l_j} such that $\beta_{l_j}(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$. This implies $\beta_{l_j}'(\boldsymbol{a}) = 1/2$ where β_{l_j}' is a complementary minterm appearing in β_{l_j}, and therefore by Lemma 3.1 we obtain $\boldsymbol{a} \succ \boldsymbol{b}$ for the corresponding element $\boldsymbol{b}$ to β_{l_j}'. Since $\boldsymbol{a} \in V_2^n$, $\boldsymbol{a} \succ \boldsymbol{b}$ implies $\boldsymbol{a} = \boldsymbol{b}$, and this contradicts to $\boldsymbol{b} \in V_3^n - V_2^n$. Therefore, β_j is never type 3' term. When β_j is a type 2' term, then obviously $a = b$ where a and b are constants appearing in α_i and β_j, respectively. Moreover, let β_j' be a simple term appearing in β_j, then $\beta_j'(\boldsymbol{a}) = 1$ since $\boldsymbol{a} \in V_2^n$. Therefore by Lemma 10.1, we obtain $\alpha_i \sqsubseteq \beta_j$.

Finally suppose α_i is a type 3 term, and let $\boldsymbol{a}$ be the corresponding element to α_i. Then $F(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$ be Lemma 6, and therefore there is a type 3' term β_j such that $\beta_j(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$. This implies that $a = b$ and $\beta_j'(\boldsymbol{a}) = 1/2$ where a and b are constants appearing in α_i and β_j, respectively, and β_j' is a complementary term appearing in β_j. Therefore by Lemma 10.2 we have $\alpha_i \sqsubseteq b_j$.

From the above we have been shown the proof of the theorem. ■

Any type 1 term of canonical disjunctive form of a Kleenean function F is an essential prime implicant of F from Theorem 8 and 9. Moreover, in order to find a minimal form of a Kleenean function F, it is not necessary to make a consensus of a pair of type 1 terms from Theorem 10. Therefore, we have the following algorithm to find a minimal form of any Kleenean function F which is a disjunctive form.

Algorithm B

Step 1: By generating consensus of any two terms which are not both type 1, find all prime implicants $\alpha_1, \dots, \alpha_s$, which are type 2' or type 3', of F.

Step 2: Expand to canonical disjunctive form, and let $\beta_1, \dots, \beta_j$ be type 1 terms and $\gamma_1, \dots, \gamma_k$ type 2 or type 3 terms.

Step 3: Find a minimal group $\alpha'_1, \dots, \alpha'_t$ from $\alpha_1, \dots, \alpha_s$ such that $\alpha'_1 \vee \cdots \vee \alpha'_t \sqsupseteq \gamma_1 \vee \cdots \vee \gamma_k$.

Step 4: $F = \beta_1 \vee \cdots \vee \beta_j \vee \alpha'_1 \vee \cdots \vee \alpha'_t$ is a minimal form of the given Kleenean function F.

By Theorem 11 Step 3 of Algorithm B corresponds to the minimum covering problem for Boolean functions. Therefore, Step 3 is solved by using tables like Boolean functions.

Example 11 *Let a 4-variable 5-valued Kleenean function F be*

$$\begin{aligned} F &= \sim x_2 \sim x_4 \vee 3/4 x_1 \sim x_2 \sim x_3 \vee \sim x_1 x_2 x_4 \vee 1/2 x_1 \sim x_2 x_3 x_4 \vee \\ &\quad \sim x_1 x_2 x_3 \sim x_3 \sim x_4 \vee 1/4 x_1 \sim x_1 \sim x_2 \sim x_3 x_4 \vee x_1 x_2 x_3 x_4 \sim x_4 \end{aligned}$$

: canonical disjunctive form.

A minimal form of F is found as follows. By Step 1 of Algorithm B, we have prime implicants of type 2' or type 3' denoted below.

$$\begin{array}{llll} 1/2x_1 \sim x_2x_3, & 1/4x_1 \sim x_1 \sim x_3, & 1/4x_1 \sim x_1x_4, & 1/4x_1 \sim x_1 \sim x_2, \\ x_3 \sim x_3 \sim x_4, & x_2x_3 \sim x_3, & x_4 \sim x_4, & x_2 \sim x_2, \\ x_1 \sim x_1x_3x_4, & x_1 \sim x_1 \sim x_2x_3, & x_1 \sim x_1x_2x_4, & x_1x_3 \sim x_3. \end{array}$$

We get the following 6 minimal forms of F by finding a minimal group from Table 4.3, where in Table 4.3 α_1, α_2, α_3 and α_4 denote $1/2x_1 \sim x_2x_3x_4$, $\sim x_1x_2x_3 \sim x_3 \sim x_4$, $1/4x_1 \sim x_1 \sim x_2 \sim x_3x_4$ and $x_1x_2x_3x_4 \sim x_4$, respectively, which are type 2 or 3 terms appearing in the canonical disjunctive form.

$$\begin{aligned} F &= \sim x_2 \sim x_4 \vee 3/4x_1 \sim x_2 \sim x_3 \vee \sim x_1x_2x_4 \vee 1/2x_1 \sim x_2x_3x_4 \sim x_4 \vee \\ &\quad \left\{ \begin{array}{c} x_3 \sim x_3 \sim x_4 \\ x_2x_3 \sim x_3 \end{array} \right\} \vee \left\{ \begin{array}{c} 1/4x_1 \sim x_1 \sim x_3 \\ 1/4x_1 \sim x_1 \sim x_4 \\ 1/4x_1 \sim x_1 \sim x_2 \end{array} \right\} \end{aligned}$$

The terms $x_1 \sim x_1x_3x_4$, $x_1 \sim x_1 \sim x_2x_3$, $x_1 \sim x_1x_2x_4$, $x_1x_3 \sim x_3$, and $x_2 \sim x_2$ are unessential prime implicants.

4.5 P-Type Logic Functions in Kleenean Functions

Definition 13 *Let F_1 and F_2 be Kleenean functions. It is said that F_1 and F_2 are V_2-equivalent if $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_2^n, and we denote it as $F_1 \approx F_2$. $V_{eq}(F)$ is defined as a set of all Kleenean functions which are V_2-equivalent to a Kleenean function F.*

Definition 14 *Let F be a Kleenean function, and consider the following Boolean function f_s.*

$$f_s(\boldsymbol{a}) = \begin{cases} 1 & F(\boldsymbol{a}) \geq s/(m-1), \\ 0 & \text{otherwise} \end{cases}$$

where $\boldsymbol{a} \in V_2^n$ and $s = 1, \dots, m-1$. Let f'_s be the disjunctive form of all prime implicants in the sense of Boolean functions of f_s. A function ξ_s is defined a Kleenean function represented by the logic formula f'_s.

Table 4.3: Prime Implicants Table of Example 11

	α_1	α_2	α_3	α_4
$1/2x_1 \sim x_2x_3$	$\surd$			
$1/4x_1 \sim x_1 \sim x_3$			$\surd$	
$1/4x_1 \sim x_1x_4$			$\surd$	
$1/4x_1 \sim x_1 \sim x_2$			$\surd$	
$x_3 \sim x_3 \sim x_4$		$\surd$		
$x_2x_3 \sim x_3$		$\surd$		
$x_4 \sim x_4$				$\surd$
$x_1 \sim x_1x_3x_4$				
$x_1 \sim x_1 \sim x_2x_3$				
$x_1 \sim x_1x_2x_4$				
$x_1x_3 \sim x_3$				
$x_2 \sim x_2$				

Note that the above Kleenean functions ξ_s whose domain restricts to V_3^n is P-type logic functions in B-ternary logic functions (P-type B-ternary logic functions, for short)[3]. Here, an n-variable B-ternary logic function is a mapping from V_3^n to V_3, which is represented by a logic formula consisting of constants 0, 1, and variables $x_1, \ldots, x_n$, and logic operations AND($\cdot$), OR($\vee$) and NOT($\sim$). Therefore, B-ternary logic functions are special case of Kleenean functions.

Example 12 *Let F_1 and F_2 be Kleenean functions such that $F_1 = 1/4 \sim x_1x_2 \vee 3/4x_1 \vee x_1 \sim x2$, $F_2 = 1/4x_2 \vee 1/2x_1 \vee 3/4x_1x_2 \vee x_1 \sim x_2 \vee \sim x_1x_2 \sim x_2$, where m=5. Table 4.4 and Table 4.5 show truth tables of F_1 and F_2 respectively. Then, it is evident that $F_1 \approx F_2$. Moreover, Kleenean functions ξ_s (s=1, 2, 3, 4), which are given from F_1 or F_2 by Definition 14, are $\xi_1 = x_1 \vee x_2$, $\xi_2 = \xi_3 = x_1$, $\xi_4 = x_1 \sim x_2$, respectively.*

Table 4.4: Truth Table of F_1 of Example 12

x_1 \ x_2	0	1/4	1/2	3/4	1
0	0	1/4	1/4	1/4	1/4
1/4	1/4	1/4	1/4	1/4	1/4
1/2	1/2	1/2	1/2	1/2	1/2
3/4	3/4	3/4	3/4	3/4	3/4
1	1	3/4	3/4	3/4	3/4

The set V_3^n makes a finite partial ordering set concerning with the partial order relation $\succ$. The maximal element of V_3^n is $(1/2, \ldots, 1/2)$, which is also maximum. The number of minimal elements of V_3^n is 2^n, and they are also elements of V_2^n.

Definition 15 *Let $\boldsymbol{a}$ be an element of V_3^n, and F Kleenean function. Then, we define $\boldsymbol{a}^*$ as a set of minimal elements $\boldsymbol{a}'$ such as $\boldsymbol{a} \succ \boldsymbol{a}'$, that is, $\boldsymbol{a}^* = \{\boldsymbol{a}' \in V_2^n | \boldsymbol{a} \succ \boldsymbol{a}'\}$. $F(\boldsymbol{a}^*)$ is defined as a set of $F(\boldsymbol{a}')$, that is, $F(\boldsymbol{a}^*) = \{F(\boldsymbol{a}') | \boldsymbol{a}' \in \boldsymbol{a}^*\}$.*

Table 4.5: Truth Table of F_2 of Example 12

x_1 \ x_2	0	1/4	1/2	3/4	1
0	0	1/4	1/2	1/4	1/4
1/4	1/4	1/4	1/2	1/4	1/4
1/2	1/2	1/2	1/2	1/2	1/2
3/4	3/4	3/4	1/2	3/4	3/4
1	1	3/4	1/2	3/4	3/4

ξ_s is the same representation as a P-type B-ternary logic functions. Therefore, we have the following lemma given in [3].

Lemma 11 *P-type B-ternary logic functions ξ_s satisfy the following conditions.*

(1) $\xi_s(\boldsymbol{a}) = 1$ *if and only if* $\xi_s(\boldsymbol{a}^*) = \{1\}$

(2) $\xi_s(\boldsymbol{a}) = 0$ *if and only if* $\xi_s(\boldsymbol{a}^*) = \{0\}$

(3) $\xi_s(\boldsymbol{a}) = 1/2$ *if and only if* $\xi_s(\boldsymbol{a}*) = \{0,\ 1\}$

where $\boldsymbol{a} \in V_3^n$.

Definition 16 *A Kleenean function F is said to be a P-type logic function if and only if $F' \succ F$ holds for all elements F' of $V_{eq}(F)$, where $F' \succ F$ means that $F'(\boldsymbol{a}) \succ F(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_m^n.*

Lemma 12 *Let F be a Kleenean function and ξ_i $(i = 1, \ldots, m-1)$ Kleenean functions given from F by Definition 14. Then, $\xi_1 \sqsupseteq \xi_2 \sqsupseteq \ldots \sqsupseteq \xi_{m-1}$.*
(Proof is omitted)

Theorem 12 *Let F_0 be a Kleenean function. Then, $F_P = 1/(m-1) \cdot \xi_1 \vee \ldots \vee i/(m-1) \cdot \xi_i \vee \ldots \vee \xi_{m-1}$ is P-type logic function such that $F_P \approx F_0$, where ξ_i $(i = 1, \ldots, m-1)$ are Kleenean functions given from F_0 by Definition 14.*

Proof: First, we prove $F_P \approx F_0$. Suppose $F_0(\boldsymbol{a}) = i/(m-1)$ for an element $\boldsymbol{a} \in V_2^n$. Then, $\xi_1(\boldsymbol{a}) = \ldots = \xi_i(\boldsymbol{a}) = 1$ and $\xi_{i+1}(\boldsymbol{a}) = \ldots = \xi_{m-1}(\boldsymbol{a}) = 0$ from Definition 14. Therefore, $F_P(\boldsymbol{a}) = i/(m-1)$, that is, $F_P \approx F_0$. Next, we prove that F_P is P-type logic function. Let assume F_P is not P-type logic function, that is, $F \succ F_P$ for an element F of $V_{eq}(F_P)$. This means that there is at least an element $\boldsymbol{a}$ satisfying one of

(1) $F_P(\boldsymbol{a}) \succ F(\boldsymbol{a})$ and $F_P(\boldsymbol{a}) \neq F(\boldsymbol{a})$, or

(2) $F(\boldsymbol{a})$ and $F_P(\boldsymbol{a})$ are not comparable.

First, assume (1) hold and let $F_P(\boldsymbol{a}) = i/(m-1)$. In this case we may assume $\boldsymbol{a}$ is an element of V_3^n without loss of generality by Corollary 2. (1) implies one of

$$(a)\ F(\boldsymbol{a}) > F_P(\boldsymbol{a}) > 1/2 \text{ or } \quad (b)\ F(\boldsymbol{a}) < F_P(\boldsymbol{a}) \leq 1/2$$

Table 4.6: Outputs of F for Elements of V_2^2 of Example 13

x_2 / x_1	0	1
0	0	1/4
1	1/2	1

holds. Suppose (a) holds, then $\xi_i(\boldsymbol{a}) = 1$ and $\xi_j(\boldsymbol{a}) =1/2$ or 0 for all j $(j > i)$. By Lemma 10 $\xi_i(\boldsymbol{a}^*) = \{1\}$ and $\xi_j(\boldsymbol{a}^*) = \{0,\ 1\}$ or $\{0\}$. Therefore, $\xi_i(\boldsymbol{b}) = 1$ and $\xi_{i+1}(\boldsymbol{b}) = 0$ for an element $\boldsymbol{b}$ of $\boldsymbol{a}^*$, and by Lemma 11 $\xi_{i+1}(\boldsymbol{b}) = \ldots = \xi_{m-1}(\boldsymbol{b}) = 0$. Thus, $F_P(\boldsymbol{b}) = i/(m-1)$. On the other hand, $F(\boldsymbol{b}) > i/(m-1)$ from $\boldsymbol{a} \succ \boldsymbol{b}$. This contradicts to $F_P \approx F$. Next suppose (b) holds, then $\xi_i(\boldsymbol{a}) = 1/2$ or 1. By Lemma 10 $\xi_i(\boldsymbol{a}^*) = \{0,\ 1\}$ or $\{1\}$. Therefore $F_P(\boldsymbol{b}) > i/(m-1)$ since $\xi_i(\boldsymbol{b}) = 1$ for an element $\boldsymbol{b}$ of $\boldsymbol{a}^*$. On the other hand, $F(\boldsymbol{b}) < i/(m-1)$ from $\boldsymbol{a} \succ \boldsymbol{b}$. This contradicts to $F_P \approx F$. From the above (1) does not hold. Second assume (2), that is, one of the following relations holds. $F_P(\boldsymbol{a}) < 1/2$ and $1/2 < F(\boldsymbol{a})$, or $1/2 < F_P(\boldsymbol{a})$ and $1/2 > F(\boldsymbol{a})$. When first half holds, $F_P(\boldsymbol{b}) < 1/2$ and $1/2 < F(\boldsymbol{b})$ for any element $\boldsymbol{b}$ such that $\boldsymbol{b} \in \boldsymbol{a}^*$. This is contradictory to $F_P \approx F$. The later half is also contradictory to $F_P \approx F$. Therefore, (2) does not hold. Accordingly F_P is P-type logic function such that $F_0 \approx F_P$. ■

Example 13 *Table 4.6 shows the outputs of a 2-variable Kleenean function F for the input of V_2^2, where m=5. A P-type logic function F_P such that $F_P \approx F$ is $F_P = 1/4(x_1 \vee x_2) \vee 1/2 x_1 \vee x_1 x_2$. Table 4.7 shows truth table of F_P. Here, the above logic formula is simplified like $F_P = 1/4 x_2 \vee 1/2 x_1 \vee x_1 x_2$.*

Table 4.7: Truth Table of P-Type Logic Function F_P of Example 13

x_2 / x_1	0	1/4	1/2	3/4	1
0	0	1/4	1/4	1/4	1/4
1/4	1/4	1/4	1/4	1/4	1/4
1/2	1/2	1/2	1/2	1/2	1/2
3/4	1/2	1/2	1/2	3/4	3/4
1	1/2	1/2	1/2	3/4	1

Let the truth vale 1/2 of $V_3 = \{0, 1/2, 1\}$ represent a indeterminate state besides 0 or 1, that is, a normal input is an element whose components take 0 or 1. Now, let consider a Kleenean function F, and assume $F(\boldsymbol{a}_1) = F(\boldsymbol{a}_2) = 1$ for the normal inputs $\boldsymbol{a}_1 = (1,\ 0,\ 0)$ and $\boldsymbol{a}_2 = (1,\ 1,\ 0)$. Then we consider an input $\boldsymbol{a}_3 = (1,\ 1/2,\ 0)$ whose second component is indeterminate state. What value should be assigned to $F(\boldsymbol{a}_3)$? Note that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are all of the inputs obtained when all the ambiguities involved in $\boldsymbol{a}_3$ are cleared, that is, obtained by replacing any 1/2 in $\boldsymbol{a}_3$ with 0 or 1. If we assign 1/2 to $F(\boldsymbol{a}_3)$ even though $F(\boldsymbol{a}_1) = F(\boldsymbol{a}_2) = 1$, then we can say that some information is lost in a sense. P-type logic functions require that $F(\boldsymbol{a}_1) = F(\boldsymbol{a}_2) = F(\boldsymbol{a}_3)$ in such a case, that is, we can say that they are Kleenean functions whose information loss is minimum.

Theorem 13 *Let F_1 and F_2 be P-type logic functions in Kleenean functions. Then, $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n if and only if $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_2^n.*

Proof: Let suppose that $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_2^n. This means that $F_1 \approx F_2$, that is, $F_2 \in V_{eq}(F_1)$ (or $F_1 \in V_{eq}(F_2)$). Then, we have $F_2 \succ F_1$ because F_1 is P-type logic function, and also $F_2 \succ F_1$ because F_2 is P-type logic function. Therefore, we have $F_1 = F_2$. The converse is trivial. ■

From Theorem 3, any Kleenean function is determined uniquely for the inputs of V_3^n. Moreover, any P-type logic function is determined uniquely for the inputs of V_2^n from the above theorem. Therefore, by Theorem 12 P-type logic function which is V_2-equivalent to a Kleenean function is sure to exist, and by Theorem 13 such a function is only one.

4.6 Conclusions

In the chapter, we described Kleenean functions which are an effective means to treat ambiguities, and almost of their basically properties have been cleared.

Any Kleenean function can be determined uniquely for all assignments of only three kinds truth values 0 1/2 and 1 to the variables (Theorem 3). This result implies that the number of n-variable Kleenean functions is at most 3^{3^n}. Actually, the number of n-variable m-valued Kleenean functions is enumerating until $n \leq 3$ and $m \leq 8$ by the reference [9] and [10], and the following table shows the number of n-variable m-valued Kleenean functions.

Table 4.8: Number of Kleenean Functions

m \ n	1	2	3
2	4	16	256
3	11	197	129,615
4	18	436	413,454
5	35	2,807	73,846,955
6	46	4,102	103,628,502
7	77	18,897	12,184,733,823
8	92	22,832	13,537,461,438

Chapter 5

Stone Logic Functions

5.1 Introduction

In the previous chapter, we showed some properties of B-ternary logic functions, regular ternary logic functions, fuzzy logic functions and multiple-valued Kleenean functions. However, among their functions, the operations *min*, *max* and (1−) are employed their logical operations AND, OR and NOT, respectively. These three kinds of operations are originally based on the set-theoretical operations of fuzzy set theory [52] or logical operations of Kleene's ternary logic [16]. As the result by employing the above three kinds operations, they each forms some algebraic system, that is, a Kleene algebra. Kleene algebras are obtained by replacing the complementary laws ($x \vee \sim x = 1$ and $x \cdot \sim x = 0$) with Kleene's law ($x \cdot \sim x \leq y \vee \sim y$) only, and therefore, Kleene algebras are weaker than Boolean algebras, but Kleene algebras are very close to Boolean algebras. Indeed, we do not know the existence of an algebraic system between a Kleene algebra and a Boolean algebra.

In the chapter, we describe multiple-valued logic functions employing the operations *min*, *max* and

$$\neg x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

as their logical operations AND, OR and NOT. The unary operation $\neg$ corresponds to that of Gödel's ternary logic (see Table 1.3 of Chapter 1). However, note that functions described in the chapter do not correspond to Gödel's ternary logic as looking at the whole system, since implication of Gödel's system can not represented in the terms of *min*, *max* and $\neg$. Moreover, the unary operation $\neg$ is also related with the intutionistic negation [43], [44].

Functions discussed in the chapter are called Stone logic functions, since the set of them forms an algebraic system called a Stone algebra [2]. A Kleene algebra is an algebraic system satisfying the axioms of Boolean algebras excluding the complementary laws, but Kleene's law. On the other hand, Stone algebras satisfy the different properties from Kleene algebras, but are motivated by the properties of Boolean algebras. Therefore, Stone algebras are very close to Boolean algebras. Likewise the case of Kleene algebras, we do not know the existence of an algebraic system between a Stone algebra and a Kleene algebra.

The chapter aims to clarify some basic properties of Stone logic functions. First, in Section 5.2, we give a definition of Stone logic functions in the term of a logic formula, then some properties of Stone logic functions appear in the section. Especially, we will show that they are monotone for a partial order relation denoted by the symbol $\leq_S$ in the chapter, and moreover any Stone logic function is uniquely determined only 3-valued inputs, in spite of it is originally defined as an infinite-valued functions. In Section 5.3, we show a canonical disjunctive form which allows for the unique representation of any Stone logic function. A necessary and sufficient condition for Stone logic functions will be cleared in Section 5.4 by using the term of the partial order relation $\leq_S$. Minimization for Stone logic functions is discussed in Section 5.5. Finally, the number of n-variable Stone logic functions is cleared in Section 5.6, and it closely connects with monotone Boolean functions. Because, the number of n-variable Stone logic functions is represented in the term of monotone Boolean functions.

5.2 Stone Logic Functions and Their Properties

Let V be a set such that $[0, 1]$, that is, V is a closed interval 0 to 1. Then an n-variable function is a mapping from V^n into V. Here, the logic operations AND($\cdot$), OR($\vee$) and NOT($\neg$) of Boolean functions are expanded among the set V as follows.

$$a \cdot b = \min(a, b), \quad a \vee b = \max(a, b), \quad \neg a = \begin{cases} 1 & \text{if} \quad a = 0 \\ 0 & \text{if} \quad a > 0 \end{cases}$$

where $a, b \in V$.

An n-variable Stone logic function is defined to be a mapping from V^n into V, which is represented by a logic formula defined below.

Definition 1 *Logic formulas are defined inductively as follows.*

(1) Constants 0, 1, and variables $x_1, \ldots, x_n$ are logic formulas.

(2) If G and H are logic formulas, then $(G \cdot H)$, $(G \vee H)$ and $(\neg G)$ are also logic formulas.

(3) The only logic formulas are given by (1) and (2)

Definition 2 *A mapping $F : V^n \to V$ represented by a logic formula is called an n-variable Stone logic function.*

The notion $\cdot$ may be often omitted. Hereafter, for short, we call an n-variable Stone logic function a Stone logic function, and identify a Stone logic function with the logic formula which represents it, if no confusion arises. In writing logic formulas, we assume that symbol $\neg$ is stronger than $\cdot$, and $\cdot$ is stronger than $\vee$.

Example 1 *The 2-variable logic formula $\neg\neg x_2 \cdot (\neg x_1 \vee x_2)$ represents a 2-variable Stone logic function $F(x_1, x_2)$. For $(0.9, 0.4) \in V^2$ $F(0.9, 0.4) = \neg\neg 0.4 \cdot (\neg 0.9 \vee 0.4) = 1 \cdot (0 \vee 0.4) = 0.4$.*

Let $\mathcal{S}$ be a set of all n-variable Stone logic functions, then an algebraic system $< \mathcal{S}; \cdot, \vee, \neg, 0, 1 >$ satisfies almost of the equations holding Boolean algebras and Kleene algebras. The following equations are properties holding $\mathcal{S}$, where a, b, $cin\mathcal{S}$ below.

1. $a \cdot b = b \cdot a$, $a \vee b = b \vee a$ (the commutative laws)
2. $a \cdot a = a$, $a \vee a = a$ (the idempotent laws)
3. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, $a \vee (b \vee c) = (a \vee b) \vee c$ (the associative laws)
4. $a \vee a \cdot b = a$, $a \cdot (a \vee b) = a$ (the absorption laws)

5. $a \cdot (b \vee c) = a \cdot b \vee a \cdot c$, $a \vee b \cdot c = (a \vee b) \cdot (a \vee c)$ (the distributive laws)

6. $\neg(a \cdot b) = \neg a \vee \neg b$, $\neg(a \vee b) = \neg a \cdot \neg b$ (De Morgan's laws)

7. $0 \cdot a = 0$, $0 \vee a = a$ (the least element)

8. $1 \cdot a = a$, $1 \vee a = 1$ (the greatest element)

9. $a \cdot \neg a = 0$

10. $\neg\neg a \vee \neg a = 1$

11. $a \cdot b = 0$ implies $a \leq \neg b$

12. $\neg\neg\neg a = \neg a$

The equations $1 \sim 12$ are identical with the axioms of Stone algebras, given in [2]. Therefore a set of Stone logic functions is one of the models of Stone algebras, and this is the reason why a function represented by a logic formula is called a Stone logic function. Note that in Boolean algebras, the equations $\neg\neg a = a$, $a \cdot \neg a = 0$ and $a \vee \neg a = 1$ (the later two equations are called the complementary laws) are valid instead of $9 \sim 12$. Also, in Kleene algebras, the equations $\neg\neg a = a$, $a \cdot \neg a \cdot (b \vee \neg b) = a \cdot \neg a$ and $a \cdot \neg a \vee (b \vee \neg b) = b \vee \neg b$ (the later two equations are called Kleene's laws, and they are weaker conditions for complementary laws) are valid instead of $9 \sim 12$.

Definition 3 *Let a and b be elements of V. Then $a \leq_S b$ if and only if one of the following conditions holds (see Figure 5.1).*

$$(1)\ a = 0 \text{ and } b = 0, \text{ or } \quad (2)\ 0 < a \leq b \leq 1$$

It is expanded among the elements of V^n as follows. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V^n, then $\boldsymbol{a} \leq_S \boldsymbol{b}$ if and only if $a_i \leq_S b_i$ for any i $(i = 1, \ldots, n)$.

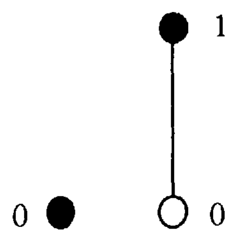

Figure 5.1: Partial Order Relation $\leq_S$

Here, $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be comparable to each other under the relation $\leq_S$ if $\boldsymbol{a} \leq_S \boldsymbol{b}$ or $\boldsymbol{b} \leq_S \boldsymbol{a}$, otherwise not comparable.

Example 2 *Let $\boldsymbol{a}_1 = (0, 0.3, 0.8)$, $\boldsymbol{a}_2 = (0, 0.4, 0.9)$ and $\boldsymbol{a}_3 = (0, 0.5, 0.8)$ be elements of V^3. Then $\boldsymbol{a}_1 \leq_S \boldsymbol{a}_2$, and $\boldsymbol{a}_1 \leq_S \boldsymbol{a}_3$, whereas $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ are not comparable to each other under $\leq_S$.*

Theorem 1 *Let F be a Stone logic function, and $\boldsymbol{a}$ and $\boldsymbol{b}$ elements of V^n. If $\boldsymbol{a} \leq_S \boldsymbol{b}$ then $F(\boldsymbol{a}) \leq_S F(\boldsymbol{b})$.*

Proof: It will be shown by induction concerning the number of logic operations. It is clear that the constants 0 and 1, and each variable $x_1, \ldots, x_n$ satisfy the theorem. Suppose G and H satisfy the theorem, then $(\neg G)$, $(G \vee H)$ and $(G \cdot H)$ also satisfy the theorem. Because, $G(\boldsymbol{b}) > 0$ when $G(\boldsymbol{a}) > 0$ from the assumption, and therefore $\neg G(\boldsymbol{a}) = \neg G(\boldsymbol{b}) = 0$. Similarly $G(\boldsymbol{b}) = 0$ when $G(\boldsymbol{a}) = 0$, and therefore $\neg G(\boldsymbol{a}) = \neg G(\boldsymbol{b}) = 1$. Thus, $(\neg G)$ satisfies the theorem. Next, we show $(G \vee H)$ satisfies the theorem. First suppose $G(\boldsymbol{a}) = 0$ then $G(\boldsymbol{b}) = 0$, and therefore $(G \vee H)(\boldsymbol{a}) = G(\boldsymbol{a}) \vee H(\boldsymbol{a}) = H(\boldsymbol{a}) \leq_S H(\boldsymbol{b}) = G(\boldsymbol{b}) \vee H(\boldsymbol{b}) = (G \vee H)(\boldsymbol{b})$. Next, suppose $G(\boldsymbol{a}) > 0$ then $G(\boldsymbol{b}) \geq G(\boldsymbol{a}) > 0$. If $H(\boldsymbol{a}) = 0$ then $H(\boldsymbol{b}) = 0$, and we have $(G \vee H)(\boldsymbol{a}) = G(\boldsymbol{a}) \vee H(\boldsymbol{a}) = G(\boldsymbol{a}) \leq_S G(\boldsymbol{b}) = G(\boldsymbol{b}) \vee H(\boldsymbol{b}) = (G \vee H)(\boldsymbol{b})$. If $H(\boldsymbol{a}) > 0$ then $H(\boldsymbol{b}) \geq H(\boldsymbol{a}) > 0$, and we have $0 < G(\boldsymbol{a}) \vee H(\boldsymbol{a}) \leq G(\boldsymbol{b}) \vee H(\boldsymbol{b})$. Thus, $(G \vee H)$ satisfy the theorem. $(G \cdot H)$ is proved similarly. ∎

Example 3 *Let* $F = x_1 \vee \neg\neg x_1 \cdot x_2$, *and let* $(0.3, 0.6)$ *and* $(0.4, 0.8)$ *be elements of* V^2. *Then* $(0.3, 0.6) \leq_S (0.4, 0.8)$ *and* $F(0.3, 0.6) = 0.6 \leq_S 0.8 = F(0.4, 0.8)$.

Let V_3 be the set $\{0, 1/2, 1\}$, and we define the following mapping from V onto V_3.

Definition 4 *Let* a *be an element of* V. *Then* $\overline{a}^\varepsilon$ *is defined as follows, where* $0 < \varepsilon < 1$ *(see Figure 5.2).*

$$\overline{a}^\varepsilon = \begin{cases} 0 & \text{if } a = 0, \\ 1/2 & \text{if } 0 < a \leq \varepsilon, \\ 1 & \text{if } \varepsilon < a \leq 1 \end{cases}$$

It can be expanded into the set V^n *as follows. Let* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *be an element of* V^n *then* $\overline{\boldsymbol{a}}^\varepsilon$ *is defined by* $(\overline{a_1}^\varepsilon, \ldots, \overline{a_n}^\varepsilon)$ *of* V_3^n.

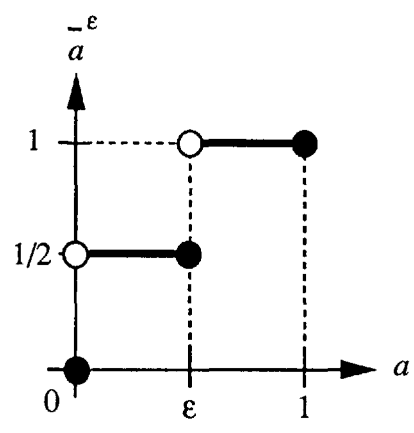

Figure 5.2: Mapping $\overline{a}^\varepsilon$ $(0 < \varepsilon < 1)$

Example 4 *Let* $\boldsymbol{a} = (0.4, 0, 0.6)$. $\overline{\boldsymbol{a}}^\varepsilon = (1/2, 0, 1)$ *when* $\varepsilon = 0.5$ *and* $\overline{\boldsymbol{a}}^\varepsilon = (1, 0, 1)$ *when* $\varepsilon = 0.3$.

Theorem 2 *Let* F *be a Stone logic function and* $\boldsymbol{a}$ *an element of* V^n. *Then* $F(\overline{\boldsymbol{a}}^\varepsilon) = \overline{F(\boldsymbol{a})}^\varepsilon$ *for any value* ε $(0 < \varepsilon < 1)$.

Proof: It will be shown by induction concerning the number of logic operations. It is clear that the constants 0, 1, and each variable $x_1, \dots, x_n$ satisfy the theorem. Suppose G and H satisfy the theorem, then $(\neg G)$, $(G \vee H)$ and $(G \cdot H)$ also satisfy the theorem. Because if $G(\boldsymbol{a}) > 0$ then $\overline{\neg G(\boldsymbol{a})}^{\varepsilon} = 0$ since $\neg G(\boldsymbol{a}) = 0$, and $\neg G(\overline{\boldsymbol{a}}^{\varepsilon}) = 0$ since $\overline{G(\boldsymbol{a})}^{\varepsilon} = G(\overline{\boldsymbol{a}}^{\varepsilon}) > 0$ for all $\varepsilon (0 < \varepsilon < 1)$. If $G(\boldsymbol{a}) = 0$ then $\overline{\neg G(\boldsymbol{a})}^{\varepsilon} = 1$ since $\neg G(\boldsymbol{a}) = 1$, and $\neg G(\overline{\boldsymbol{a}}^{\varepsilon}) = 1$ since $\overline{G(\boldsymbol{a})}^{\varepsilon} = G(\overline{\boldsymbol{a}}^{\varepsilon}) = 0$ for all $\varepsilon (0 < \varepsilon < 1)$. Thus, $(\neg G)$ satisfy the theorem. Next, we show $(G \vee H)$ satisfy the theorem. First, suppose $G(\boldsymbol{a}) \geq H(\boldsymbol{a})$ then $\overline{G(\boldsymbol{a})}^{\varepsilon} \geq \overline{H(\boldsymbol{a})}^{\varepsilon}$ and from the assumption $G(\overline{\boldsymbol{a}}^{\varepsilon}) \geq H(\overline{\boldsymbol{a}}^{\varepsilon})$. Therefore, $\overline{(G \vee H)(\boldsymbol{a})}^{\varepsilon} = \overline{G(\boldsymbol{a}) \vee H(\boldsymbol{a})}^{\varepsilon} = \overline{G(\boldsymbol{a})}^{\varepsilon} = G(\overline{\boldsymbol{a}}^{\varepsilon}) = G(\overline{\boldsymbol{a}}^{\varepsilon}) \vee H(\overline{\boldsymbol{a}}^{\varepsilon}) = (G \vee H)(\overline{\boldsymbol{a}}^{\varepsilon})$. We have same result when $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$. Thus, $(G \vee H)$ satisfy the theorem. $(G \cdot H)$ is proved similarly. ■

Example 5 *Let a Stone logic function F be represented by such a logic formula $x_1 \vee \neg\neg x_1 \cdot x_2$, and let $(0.4, 0.9)$ be element of V^2 and $\varepsilon = 0.5$. Then $\overline{F(0.4, 0.9)}^{\varepsilon} = \overline{0.9}^{\varepsilon} = 1$ and $F(\overline{0.4}^{\varepsilon}, \overline{0.9}^{\varepsilon}) = F(1/2, 1) = 1$.*

Theorem 3 *Let G and H be Stone logic functions. $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V^n if and only if $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n.*

Proof: Let Suppose that $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n and the theorem dose not hold. Then we can assume that there is at least one element $\boldsymbol{a}$ of V^n such that $G(\boldsymbol{a}) \neq H(\boldsymbol{a})$. This means either one of $G(\boldsymbol{a}) > H(\boldsymbol{a})$ or $G(\boldsymbol{a}) < H(\boldsymbol{a})$. Suppose the former case holds. By Theorem 2 $G(\overline{\boldsymbol{a}}^{\varepsilon}) = \overline{G(\boldsymbol{a})}^{\varepsilon} > \overline{H(\boldsymbol{a})}^{\varepsilon} = H(\overline{\boldsymbol{a}}^{\varepsilon})$ for $\varepsilon = (G(\boldsymbol{a}) + H(\boldsymbol{a}))/2$, and this contradicts to the assumption since $\overline{\boldsymbol{a}}^{\varepsilon}$ is an elements of V_3^n. The later case is proved similarly. Converse is trivial. This completes the proof of the theorem. ■

Corollary 1 *Let G and H be Stone logic functions. $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V_3^n if and only if $G(\boldsymbol{a}) \leq H(\boldsymbol{a})f$ for every element $\boldsymbol{a}$ of V^n.*
(The proof is omitted.)

Definition 5 *Let G and H be Stone logic functions. Then H includes G (or G is included in H) if and only if $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V^n, and we denote it as $G \sqsubseteq H$ (or $H \sqsupseteq G$).*

In accordance with Corollary 1, $G \sqsubseteq H$ if and only if $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V_3^n.

5.3 Canonical Disjunctive Forms

As discussed in Section 5.2, any logic formula representing a Stone logic function can be expanded into a disjunctive form (sum-of-product form) since the idempotent laws, the absorption laws, the distributive laws, and so on are valid in the logic formulas. A disjunctive form of any Stone logic function, however, dose not determine uniquely. In this section we describe a canonical disjunctive form which enables us for uniquely representation of any Stone logic function, and this form corresponds to minterm expressions of Boolean functions.

A variable x, a negation $\neg x$ or a double negation $\neg\neg x$ is called a *literal.* Note that the double negation law $\neg\neg x = x$ is not valid, but the equation $\neg\neg\neg x = \neg x$. A conjunction of one or more literals is called a product term. In any product term α, it is assumed that α dose not contain two or more deferent literals simultaneously for each variable appearing in it. Because, any repeated literals are removed from the idempotent laws and the equations $x\neg\neg x = x$, $x\neg x = 0$ and $\neg x\neg\neg x = 0$ stand always true. If a variable x does not exist in a product term α then the relation $\alpha = \alpha(\neg x \vee \neg\neg x) = \alpha\neg x \vee \alpha\neg\neg x$ holds since $\neg x \vee \neg\neg x = 1$ stands always true. Therefore, any product term α can expand into a disjunction of product terms appearing all variables. Hereafter, a product term appearing all variables is called a *minterm*. Thus, any logic formula representing a Stone logic function F can always be expanded into the disjunctive form $F = \alpha_1 \vee \ldots \vee \alpha_s$ where α_i is a minterm.

In the following it is assumed that any disjunctive form $F = \alpha_1 \vee \ldots \vee \alpha_s$ where α_i is a product term satisfies the property $\alpha_i \not\sqsubseteq \alpha_j$ for all $i, j = 1, \ldots, s$ such that $i \neq j$.

Definition 6 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V_3^n. Then the element $\boldsymbol{a}$ corresponds to a minterm $\alpha = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ if the following condition hold for every i.*

$$x_i^{a_i} = \begin{cases} \neg x_i & \text{if } a_i = 0, \\ \neg\neg x_i & \text{if } a_i = 1/2, \\ x_i & \text{if } a_i = 1 \end{cases}$$

Obviously $\alpha(\boldsymbol{a}) = 1$ and there is one-to-one correspondence between minterms and the elements of V_3^n.

Example 6 *$\neg x_1\neg\neg x_2 x_3$ and $\neg x_1\neg\neg x_2\neg\neg x_3$ respectively correspond to* (0, 1/2,1) *and* (0, 1/2, 1/2) *if they are minterms on variables x_1, x_2 and x_3.*

In the relation $\leq_S$, if elements a and b of V_3 are comparable to each other, then there is the greatest lower bound of a and b, otherwise there is not. We will write the greatest lower bound of a and b as $a\Delta_S b$, and if it dose not exist then write $a\Delta_S b = \emptyset$. This can be expanded among V_3^n as follows. For two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of V_3^n, we define $\boldsymbol{a}\Delta_S\boldsymbol{b}$ as $(a_1\Delta_S b_1, \ldots, a_n\Delta_S b_n)$, and if $a_i\Delta_S b_i = \emptyset$ for some i $(i = 1, \ldots, n)$, then define it as $\boldsymbol{a}\Delta_S\boldsymbol{b} = \emptyset$.

Example 7 *Let $\boldsymbol{a}_1 = (0, 1, 1/2)$, $\boldsymbol{a}_2 = (0, 1, 1)$ and $\boldsymbol{a}_3 = (0, 1/2, 1)$. Then $\boldsymbol{a}_1 \leq_S \boldsymbol{a}_2$, $\boldsymbol{a}_3 \leq_S \boldsymbol{a}_2$, whereas $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$ are not comparable to each other. However, $\boldsymbol{a}_1\Delta_S\boldsymbol{a}_3 = (0, 1, 1)$, and especially $\boldsymbol{a}_1\Delta_S\boldsymbol{a}_2 = \boldsymbol{a}_2$ and $\boldsymbol{a}_2\Delta_S\boldsymbol{a}_3 = \boldsymbol{a}_2$.*

Lemma 1 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be any element of V_3^n and α the corresponding minterm to $\boldsymbol{a}$. For any element $\boldsymbol{b} = (b_1, \ldots, b_n)$ of V_3^n*

(1) $\alpha(\boldsymbol{b}) = 1$ if and only if $\boldsymbol{a} \leq_S \boldsymbol{b}$,

(2) $\alpha(\boldsymbol{b}) = 1/2$ if and only if $\boldsymbol{a} \not\leq_S \boldsymbol{b}$ and $\boldsymbol{a}\Delta_S\boldsymbol{b} \neq \emptyset$,

(3) $\alpha(\boldsymbol{b}) = 0$ if and only if $\boldsymbol{a}\Delta_S\boldsymbol{b} = \emptyset$.

Proof: $\alpha(\boldsymbol{b}) = 1$ if and only if $a_i = b_i$ for all i such that $a_i = 0$ or 1, and $a_i \leq_S b_i$ for all i such that $a_i = 1/2$. Therefore, (1) is true. Next, $\alpha(\boldsymbol{b}) = 0$ if and only if there is at least one i such that $b_i \neq 0$ when $a_i = 0$, or $b_i = 0$ when $a_i \neq 0$. Therefore, (3) is true. (2) is derived directly from (1) and (3). ∎

Lemma 2 *Let α and β be minterms, and $\boldsymbol{a}$ and $\boldsymbol{b}$ the corresponding elements of V_3^n, respectively. Then $\boldsymbol{a} \leq_S \boldsymbol{b}$ if and only if $\alpha \vee \beta = \alpha$, that is, $\alpha \sqsupseteq \beta$.*

Proof: Let $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$, $\alpha = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ and $\beta = x_1^{b_1} \cdot \ldots \cdot x_n^{b_n}$. First suppose $\boldsymbol{a} \leq_S \boldsymbol{b}$, that is $a_i \leq_S b_i$ for all i ($i = 1, \ldots, n$). This implies $x_i^{a_i} \sqsupseteq x_i^{b_i}$ for all i ($i = 1, \ldots, n$), and therefore $\alpha \sqsupseteq \beta$. Next, suppose $\alpha \vee \beta = \alpha$, and this implies $\alpha(\boldsymbol{a}') \geq \beta(\boldsymbol{a}')$ for all elements $\boldsymbol{a}'$ of V_3^n. Thus, $\alpha(\boldsymbol{b}) \geq \beta(\boldsymbol{b}) = 1$ and we have $\alpha(\boldsymbol{b}) = 1$. Therefore, by Lemma 1.1 we have $\boldsymbol{a} \leq_S \boldsymbol{b}$. ■

Definition 7 *If a Stone logic function F is represented by a logic formula $F = \alpha_1 \cdot \ldots \cdot \alpha_s$, then it is said that F is in the canonical disjunctive form where every α_i $(i = 1, \ldots, s)$ is a minterm and $\alpha_i \not\sqsubseteq \alpha_j$ for all i, j such that $i \neq j$.*

Example 8 *The canonical disjunctive form of a 3-variable Stone logic function $F = (\neg\neg x_1 \vee x_2)x_3$ is obtained as follows.*

$$\begin{aligned}
F &= (\neg\neg x_1 \vee x_2)x_3 \\
&= \neg\neg x_1 x_3 \vee x_2 x_3 \\
&= \neg\neg x_1(\neg x_2 \vee \neg\neg x_2)x_3 \vee (\neg x_1 \vee \neg\neg x_1)x_2 x_3 \\
&= \neg\neg x_1 \neg x_2 x_3 \vee \neg\neg x_1 \neg\neg x_2 x_3 \vee \neg x_1 x_2 x_3 \vee \neg\neg x_1 x_2 x_3 \\
&= \neg\neg x_1 \neg x_2 x_3 \vee \neg\neg x_1 \neg\neg x_2 x_3 \vee \neg x_1 x_2 x_3
\end{aligned}$$

Here, $(1/2, 0, 1)$, $(1/2, 1/2, 1)$ and $(0, 1, 1)$ correspond to $\neg\neg x_1 \neg x_2 x_3$, $\neg\neg x_1 \neg\neg x_2 x_3$ and $\neg x_1 x_2 x_3$, respectively, and therefore, by Lemma 2 each three minterms is not included in any other minterm. Thus, the canonical disjunctive form of F is $\neg\neg x_1 \neg x_2 x_3 \vee \neg\neg x_1 \neg\neg x_2 x_3 \vee \neg x_1 x_2 x_3$.

Theorem 4 *Any Stone logic function F can be represented uniquely (ignoring the order of the minterms) by the canonical disjunctive form.*

Proof: It is evident from the above discussion that there is at least one canonical disjunctive form of F. Let suppose G and H are two deferent canonical disjunctive form of F, and suppose a minterm α exists in G and not H. Let $\boldsymbol{a}$ be the corresponding element to α, and then $\alpha(\boldsymbol{a}) = 1$. Thus, there is a minterm $\beta(\neq \alpha)$ in H such that $\beta(\boldsymbol{a}) = 1$ since $G(\boldsymbol{a}) = H(\boldsymbol{a}) = 1$. Let $\boldsymbol{b}$ be the corresponding element to β, then by Lemma 1.1 $\boldsymbol{b} \leq_S \boldsymbol{a}$ since $\beta(\boldsymbol{a}) = 1$, and by the assumption of $\alpha \neq \beta$, $\boldsymbol{a} \neq \boldsymbol{b}$. In the similar manner, $G(\boldsymbol{b}) = H(\boldsymbol{b}) = 1$ from $\beta(\boldsymbol{b}) = 1$, and therefore there is a minterm γ in G such that $\gamma(\boldsymbol{b}) = 1$. Let $\boldsymbol{c}$ be the corresponding element to γ, then by Lemma 1.1 $\boldsymbol{c} \leq_S \boldsymbol{b}$. From the above, we have $\boldsymbol{c} \leq_S \boldsymbol{b} \leq_S \boldsymbol{a}$ and $\boldsymbol{a} \neq \boldsymbol{b}$, and accordingly by Lemma 2 α is included in γ. This contradicts to the assumption that G is a canonical disjunctive form. Therefore, all minterms existing in G also exist in H. In the similar manner, we can prove that all minterms existing in H also exist in G. This completes the proof of the theorem. ■

5.4 A Characterization of Stone Logic Functions

5.4.1 Necessary and Sufficient Condition for Stone-3 Logic Functions

By Theorem 3, any Stone logic function is determined uniquely only for every input in V_3^n, and it is clear that V_3 is the range of Stone logic functions whose domain is restricted to the set V_3^n. Hereafter, we call a Stone logic function whose domain is restricted to V_3^n a Stone-3 logic function, i.e., a Stone-3 logic function is a function $F : V_3^n \to V_3$ (called a ternary function below) represented by a logic formula. Obviously, it is not true that every ternary function can obtain by means of a logic formula, i.e., Stone-3 logic functions are not functionally complete. Therefore, first, we discuss a necessary and sufficient condition for a ternary function to be a Stone-3 logic function.

The following set of two conditions is a necessary and sufficient condition for a ternary function to be a Stone-3 logic function, where V_2 denotes the set $\{0, 1\}$ below.

(Sa) $\boldsymbol{a} \in V_2^n$ implies $F(\boldsymbol{a}) \in V_2$

(Sb) $\boldsymbol{a} \leq_S \boldsymbol{b}$ implies $F(\boldsymbol{a}) \leq_S F(\boldsymbol{b})$

First, we discuss ternary functions satisfying Conditions (Sa) and (Sb). Let F be a ternary function satisfying Conditions (Sa) and (Sb). Then we consider specific three subsets $F^{-1}(0)$, $F^{-1}(1/2)$ and $F^{-1}(1)$ of V_3^n as follows.

$$F^{-1}(i) = \{\boldsymbol{a} \in V_3^n \mid F(\boldsymbol{a}) = i\},$$

where $i \in V_3$. It is clear that $F^{-1}(i) \cap F^{-1}(j) = \emptyset$ and $\bigcup_{i \in V_3} F^{-1}(i) = V_3^n$. Suppose $\boldsymbol{a}$ be an element of $F^{-1}(1)$, then by Condition (Sb) all elements $\boldsymbol{b}$ such that $\boldsymbol{a} \leq_S \boldsymbol{b}$ are also elements of $F^{-1}(1)$. Similarly if $\boldsymbol{a}$ is an element of $F^{-1}(1/2)$, then all elements $\boldsymbol{b}$ such that $\boldsymbol{b} \leq_S \boldsymbol{a}$ are also elements of $F^{-1}(1/2)$, and moreover, if $\boldsymbol{a}$ is an element of $F^{-1}(0)$, then all elements $\boldsymbol{b}$ such that $\boldsymbol{a} \leq_S \boldsymbol{b}$ or $\boldsymbol{b} \leq_S \boldsymbol{a}$ are also elements of $F^{-1}(0)$. Therefore, $F^{-1}(0)$, $F^{-1}(1/2)$ and $F^{-1}(1)$ each form a partial order finite set concerning with the relation $\leq_S$. Thus, for any given Stone-3 logic function F the set of maximal elements of $F^{-1}(0)$ and $F^{-1}(1/2)$, and the set of minimal elements $F^{-1}(1)$ are uniquely determined, and they are denoted by $\partial F^{-1}(0)$, $\partial F^{-1}(1/2)$ and $\partial F^{-1}(1)$, respectively.

Lemma 3 *Let F be a ternary function satisfying Conditions (Sa) and (Sb). Then,*

(1) $F^{-1}(1) = \emptyset$ implies $F(\boldsymbol{a}) = 0$ for every element $\boldsymbol{a}$ of V_3^n,

(2) Let $\boldsymbol{a}$ be an element of V_3^n, then $F(\boldsymbol{a}) = 0$ implies $\boldsymbol{a} \Delta_S \boldsymbol{b} = \emptyset$ for every element $\boldsymbol{b}$ of $F^{-1}(1)$,

(3) Let $\boldsymbol{a}$ be an element of V_3^n, then $F(\boldsymbol{a}) = 1/2$ implies that there is at least one $\boldsymbol{b} \in F^{-1}(1)$ such that $\boldsymbol{a} \leq_S \boldsymbol{b}$.

Proof: (1): By $F^{-1}(1) = \emptyset$ $F(\boldsymbol{a}) \neq 1$ for all elements $\boldsymbol{a}$ of V_2^n, that is, $F(\boldsymbol{a}) = 0$ from Condition (Sa). Therefore by Condition (Sb) $F(\boldsymbol{a}) = 0$ for all elements $\boldsymbol{a}$ of V_3^n. (2): Assume that there is an element $\boldsymbol{b}$ of $F^{-1}(1)$ such that $\boldsymbol{a} \Delta_S \boldsymbol{b} \neq \emptyset$. Then $F(\boldsymbol{a} \Delta_S \boldsymbol{b}) \neq 0$ since $\boldsymbol{a} \Delta_S \boldsymbol{b} \leq_S \boldsymbol{b}$, (Sb) and $\boldsymbol{b} \in F^{-1}(1)$, and therefore $F(\boldsymbol{a}) \neq 0$ since $\boldsymbol{a} \Delta_S \boldsymbol{b} \leq_S \boldsymbol{a}$, (Sb) and $F(\boldsymbol{a} \Delta_S \boldsymbol{b}) \neq 0$. This contradicts to the assumption. (3): By $F(\boldsymbol{a}) = 1/2$ there is an element $\boldsymbol{b}$ of V_2^n such that $\boldsymbol{a} \leq_S \boldsymbol{b}$, and therefore by Conditions (Sa) and (Sb) $F(\boldsymbol{b}) = 1$. ■

Theorem 5 *If F is a Stone-3 logic function, then F also satisfies Condition (Sa) and (Sb).*

Proof: It is evident from Theorem 1 and the definitions of each operation AND($\cdot$), OR($\vee$) and NOT($\neg$). ■

Theorem 6 *If F is a ternary function satisfying Conditions (Sa) and (Sb) then there is a Stone-3 logic function F_s such that $F(\boldsymbol{a}) = F_s(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V_3^n.*

Proof: For any given ternary function F satisfying Conditions (Sa) and (Sb) we construct a Stone-3 logic function F_s such that $F(\boldsymbol{a}) = F_s(\boldsymbol{a})$ for every element $\boldsymbol{a}$ of V_3^n. We can assume without loss of generality that $\partial F^{-1}(1) \neq \emptyset$, because if $\partial F^{-1}(1) = \emptyset$ then by Lemma 3.1 $F(\boldsymbol{a}) = 0$ for all elements $\boldsymbol{a}$ of V_3^n, and this is a Stone-3 logic function. Let F_f be a logic formula constructed by the disjunction of all minterms corresponding to elements of $\partial F^{-1}(1)$. First, we shall show $F(\boldsymbol{a}) = 1$ if and only if $F_f(\boldsymbol{a}) = 1$. Suppose $F(\boldsymbol{a}) = 1$. Then there is an element $\boldsymbol{b}$ of $\partial F^{-1}(1)$ such that $\boldsymbol{b} \leq_S \boldsymbol{a}$, and therefore by Lemma 1.1 $\beta(\boldsymbol{a}) = 1$ where β is a corresponding minterm to $\boldsymbol{b}$. This implies $F_f(\boldsymbol{a}) = 1$. Conversely suppose $F_f(\boldsymbol{a}) = 1$. Then there is a minterm β in F_f such that $\beta(\boldsymbol{a}) = 1$, and let $\boldsymbol{b}$ be the corresponding element to β then by Lemma 1.1 $\boldsymbol{b} \leq_S \boldsymbol{a}$. On the other hand, $F(\boldsymbol{b}) = 1$ since $\boldsymbol{b} \in \partial F^{-1}(1)$. Therefore, we have $F(\boldsymbol{a}) = 1$ by $\boldsymbol{b} \leq_S \boldsymbol{a}$, $F(\boldsymbol{b}) = 1$ and (Sb). Thus, $F(\boldsymbol{a}) = 1$ if and only if $F_f(\boldsymbol{a}) = 1$. Next, we shall show $F(\boldsymbol{a}) = 1/2$ if and only if $F_f(\boldsymbol{a}) = 1/2$. Suppose $F(\boldsymbol{a}) = 1/2$. By Lemma 3.3 there is an element $\boldsymbol{b}'$ of $F^{-1}(1)$ such that $\boldsymbol{a} \leq_S \boldsymbol{b}'$, and is an element $\boldsymbol{b}$ of $\partial F^{-1}(1)$ such that $\boldsymbol{b} \leq_S \boldsymbol{b}'$. Let β be the corresponding minterm to $\boldsymbol{b}$ then $\beta(\boldsymbol{b}') = 1$, and therefore $\beta(\boldsymbol{a}) = 1$ or $1/2$ from $\boldsymbol{a} \leq_S \boldsymbol{b}'$ and Theorem 1. $\beta(\boldsymbol{a}) = 1$, however, dose not hold since this implies $F_f(\boldsymbol{a}) = 1$, and this contradicts to the assumption since $F_f(\boldsymbol{a}) = 1$ leads $F(\boldsymbol{a}) = 1$. Therefore, $\beta(\boldsymbol{a}) = 1/2$, that is, $F_f(\boldsymbol{a}) = 1/2$. Conversely suppose $F_f(\boldsymbol{a}) = 1/2$. It is clear that $F(\boldsymbol{a}) \neq 1$ from the above discussion. Let assume $F(\boldsymbol{a}) = 0$, then by Lemma 3.2 $\boldsymbol{a}\Delta_S\boldsymbol{b} = \emptyset$ for all elements $\boldsymbol{b}$ of $F^{-1}(1)$, and therefore by Lemma 1.3 $\beta(\boldsymbol{a}) = 0$ for all minterms β existing in F_f, i.e., $F_f(\boldsymbol{a}) = 0$. This contradicts to the assumption. Therefore, $F(\boldsymbol{a}) = 0$ dose not hold, and thus we have $F(\boldsymbol{a}) = 1/2$. Finally, it is derived directly from the above discussion that $F(\boldsymbol{a}) = 0$ if and only if $F_f(\boldsymbol{a}) = 0$. This completes the proof of the theorem. ∎

Conditions (Sa) and (Sb) are independent of each other. Because, the ternary function F_a appearing in Table 1 is satisfies Condition (Sa) but not (Sb), and moreover, F_b of Table 5.1 only satisfies Condition (Sb). Therefore, Condition (Sa) and (Sb) are independent of each other.

Table 5.1: Truth Table of Ternary Functions F_a and F_b

x	0	1/2	1
$F_a(x)$	0	0	1
$F_b(x)$	0	1/2	1/2

5.4.2 Necessary and Sufficient Condition for Stone Logic Functions

In this section, we describe a necessary and sufficient condition for Stone logic functions.

The following set of three conditions is a necessary and sufficient condition for a function $F : V^n \to V$, called an infinite-valued function below, to be a Stone logic function.

(SA) $\boldsymbol{a} \in V_2^n$ implies $F(\boldsymbol{a}) \in V_2$

(SB) $\boldsymbol{a} \leq_S \boldsymbol{b}$ implies $F(\boldsymbol{a}) \leq_S F(\boldsymbol{b})$

(SC) $F(\overline{\boldsymbol{a}}^{\varepsilon}) = \overline{F(\boldsymbol{a})}^{\varepsilon}$ for any ε such that $0 < \varepsilon < 1$

Lemma 4 *Let F be an infinite-valued function and satisfy Condition (SC). If $\boldsymbol{a} \in V_3^n$ then $F(\boldsymbol{a}) \in V_3$.*

Proof: Let assume $\boldsymbol{a} \in V_3^n$ and $F(\boldsymbol{a}) \notin V_3$. Then for any ε such that $1/2 \le \varepsilon < 1$ we have $\overline{\boldsymbol{a}}^{\varepsilon} = \boldsymbol{a}$, and therefore by Condition (SC) $F(\boldsymbol{a}) = F(\overline{\boldsymbol{a}}^{\varepsilon}) = \overline{F(\boldsymbol{a})}^{\varepsilon} \in V_3$. This contradicts to the assumption. ∎

Theorem 7 *If F is a Stone logic function then F satisfies Conditions (SA), (SB) and (SC).*

Proof: It is evident from Theorem 1, 2 and the definitions of each logic operations. ∎

Theorem 8 *If F is an infinite-valued function satisfying Conditions (SA), (SB) and (SC) then F is a Stone logic function.*

Proof: By Lemma 4 and Condition (SC) if $\boldsymbol{a} \in V_3^n$ then $F(\boldsymbol{a}) \in V_3$, and therefore the restriction $F|_{V_3^n}$ of F to the set V_3^n satisfies Condition (SA) and

(SB') $\boldsymbol{a}, \boldsymbol{b} \in V^n \cap V_3^n = V_3^n$ and $\boldsymbol{a} \le_S \boldsymbol{b}$ imply $F|_{V_3^n}(\boldsymbol{a}) \le_S F|_{V_3^n}(\boldsymbol{b})$.

Thus, by Theorem 6 there is a logic formula F_s representing a Stone-3 logic function such that $F|_{V_3^n}(\boldsymbol{a}) = F_s(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n. We can prove that the logic formula F_s also represents a Stone logic function F. Because, let assume $F(\boldsymbol{a}) \ne F_s(\boldsymbol{a})$ for an element $\boldsymbol{a}$ of $V^n - V_3^n$, then this implies one of $F(\boldsymbol{a}) > F_s(\boldsymbol{a})$ or $F(\boldsymbol{a}) < F_s(\boldsymbol{a})$. If former case is hold then $\overline{F(\boldsymbol{a})}^{\varepsilon} = F(\overline{\boldsymbol{a}}^{\varepsilon}) > F_s(\overline{\boldsymbol{a}}^{\varepsilon}) = \overline{F_s(\boldsymbol{a})}^{\varepsilon}$, and this contradicts to $F|_{V_3^n}(\boldsymbol{a}) = F_s(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_3^n since $\overline{\boldsymbol{a}}^{\varepsilon} \in V_3^n$. For the later case we can leads contradiction in the similar manner. Therefore, F and F_s are same Stone logic function, and this completes the proof. ∎

Conditions (SA), (SB) and (SC) are not independent of each other since Condition (SC) can lead Conditions (SA) and (SB). Consequently, we have an interesting result, that is, the set of functions represented by logic formulas and the set of functions satisfying Condition (SC) are equivalent to each other.

Theorem 9 *Let F be an infinite-valued function and satisfy Condition (SC). Then F also satisfies Condition (SA).*

Proof: By Lemma 4 and Condition (SC) if $\boldsymbol{a} \in V_3^n$ then $F(\boldsymbol{a}) \in V_3$. Let assume there is an element $\boldsymbol{b} \in V_2^n$ such that $F(\boldsymbol{b}) \notin V_2$, that is, $F(\boldsymbol{b}) = 1/2$. Then $\boldsymbol{b} = \overline{\boldsymbol{b}}^{\varepsilon}$ for any ε such that $0 < \varepsilon < 1$, and therefore $F(\boldsymbol{b}) = F(\overline{\boldsymbol{b}}^{\varepsilon}) = \overline{F(\boldsymbol{b})}^{\varepsilon}$. On the other hand $\overline{F(\boldsymbol{b})}^{\varepsilon} = 1$ for any ε such that $0 < \varepsilon < 1/2$ since $F(\boldsymbol{b}) = 1/2$, and this is a contradiction. Thus, this completes the proof of the theorem. ∎

Lemma 5 *Let $\boldsymbol{a}, \boldsymbol{b} \in V_3^n$ and $\boldsymbol{a} \le_S \boldsymbol{b}$. Then there are an element $\boldsymbol{t}$ of V^n and constants ε_1 and ε_2 such that $\boldsymbol{a} = \overline{\boldsymbol{t}}^{\varepsilon_1}$, $\boldsymbol{b} = \overline{\boldsymbol{t}}^{\varepsilon_2}$ and $0 < \varepsilon_2 < \varepsilon_1 < 1$.*

Proof: Let $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$ and $\boldsymbol{t} = (t_1, \ldots, t_n)$. By $\boldsymbol{a} \le_S \boldsymbol{b}$, $a_i \le_S b_i$ holds for each i $(i = 1, \ldots, n)$. Therefore, one of

$$(1)\ a_i = 0 \text{ and } b_i = 0, \quad (2)\ a_i = 1 \text{ and } b_i = 1 \text{ or } \quad (3)\ a_i = 1/2 \text{ and } b_i = 1/2 \text{ or } 1$$

holds for each i. Suppose (1), then set $t_i = 0$, and we obtain $a_i = \overline{t_i}^{\varepsilon_1}$ and $b_i = \overline{t_i}^{\varepsilon_2}$ for any ε_1 and ε_2 such that $0 < \varepsilon_1 < 1$ and $0 < \varepsilon_2 < 1$. Next, suppose (2), then set $\varepsilon_1 < t_i \le 1$ and $\varepsilon_2 < t_i \le 1$, and we obtain $a_i = \overline{t_i}^{\varepsilon_1}$ and $b_i = \overline{t_i}^{\varepsilon_2}$ for any ε_1 and ε_2 such that $0 < \varepsilon_1 \le 1/2$ and $0 < \varepsilon_2 \le 1/2$. Finally, suppose (3). Then $a_i = \overline{t_i}^{\varepsilon_1}$ for any t_i such that $0 < t_i \le \varepsilon_1$ from $a_i = 1/2$. Therefore, if $b_i = 1/2$ then $b_i = \overline{t_i}^{\varepsilon_2}$ for any t_i such that $0 < t_i \le \varepsilon_2$. If $b_i = 1$ then $b_i = \overline{t_i}^{\varepsilon_2}$ for any t_i such that $\varepsilon_2 < t_i \le 1$, and therefore in order to exist t_i the relation $\varepsilon_2 < t_i \le \varepsilon_1$ must be hold. Thus, we have $\varepsilon_2 < \varepsilon_1$. From the above there are constants ε_1 and ε_2 and an element $\boldsymbol{t}$ of V^n such that $\boldsymbol{a} = \overline{\boldsymbol{t}}^{\varepsilon_1}$, $\boldsymbol{b} = \overline{\boldsymbol{t}}^{\varepsilon_2}$ and $0 < \varepsilon_2 < \varepsilon_1 < 1$. ∎

Lemma 6 *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be elements of V^n. $\boldsymbol{a} \leq_S \boldsymbol{b}$ if and only if $\overline{\boldsymbol{a}}^{\varepsilon} \leq_S \overline{\boldsymbol{b}}^{\varepsilon}$ for any ε such that $0 < \varepsilon < 1$.*

Proof: Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$. Assume that $\boldsymbol{a} \leq_S \boldsymbol{b}$, that is, $a_i \leq_S b_i$ for all i. Then one of

$$(1)\ \overline{a_i}^{\varepsilon} = 0, \quad (2)\ \overline{a_i}^{\varepsilon} = 1 \text{ or } \quad (3)\ \overline{a_i}^{\varepsilon} = 1/2$$

holds. Suppose (1) then $a_i = 0$. Thus, by $a_i = 0$ and $a_i \leq_S b_i$ we have $b_i = 0$, and therefore $\overline{b_i}^{\varepsilon} = 0$, that is, $\overline{a_i}^{\varepsilon} \leq_S \overline{b_i}^{\varepsilon}$. Suppose (2) then $\varepsilon < a_i \leq 1$. Thus, by $\varepsilon < a_i \leq 1$ and $a_i \leq_S b_i$ we have $a_i \leq b_i \leq 1$, and therefore $\overline{b_i}^{\varepsilon} = 1$, that is, $\overline{a_i}^{\varepsilon} \leq_S \overline{b_i}^{\varepsilon}$. Suppose (3) then $0 < a_i \leq \varepsilon$. Thus, by $0 < a_i \leq \varepsilon$ and $a_i \leq_S b_i$ we have $0 < b_i \leq 1$, and therefore $\overline{b_i}^{\varepsilon} = 1/2$ or 1, that is, $\overline{a_i}^{\varepsilon} \leq_S \overline{b_i}^{\varepsilon}$. From the above $\boldsymbol{a} \leq_S \boldsymbol{b}$ for any ε such that $0 < \varepsilon < 1$. Conversely, assume $\boldsymbol{a} \not\leq_S \boldsymbol{b}$, that is, for some i $a_i = b_i = 0$ dose not hold and also $0 < a_i \leq b_i \leq 1$ does not. Therefore, one of each $a_i \neq 0$ or $b_i \neq 0$ holds, and moreover $b_i < a_i$. Thus for such i we have $\overline{b_i}^{\varepsilon} < \overline{a_i}^{\varepsilon}$ where $\varepsilon = (a_i + b_i)/2$ (which is $0 < \varepsilon < 1$), and accordingly $\overline{\boldsymbol{a}}^{\varepsilon} \leq_S \overline{\boldsymbol{b}}^{\varepsilon}$ never hold for such ε. Thus $\boldsymbol{a} \not\leq_S \boldsymbol{b}$ implies $\overline{\boldsymbol{a}}^{\varepsilon} \not\leq_S \overline{\boldsymbol{b}}^{\varepsilon}$ for some ε, that is, $\overline{\boldsymbol{a}}^{\varepsilon} \leq_S \overline{\boldsymbol{b}}^{\varepsilon}$ for all ε implies $\boldsymbol{a} \leq_S \boldsymbol{b}$. ∎

Lemma 7 *Let $\boldsymbol{a}$ be an element of V^n and ε_1, ε_2 any constants such that $0 < \varepsilon_1 < 1$ and $0 < \varepsilon_2 < 1$. Then $\varepsilon_1 \leq_S \varepsilon_2$ implies $\overline{\boldsymbol{a}}^{\varepsilon_2} \leq_S \overline{\boldsymbol{a}}^{\varepsilon_1}$.*

Proof: It is clear when $\varepsilon_1 = \varepsilon_2$, and therefore let suppose $\varepsilon_1 \leq_S \varepsilon_2$ and $\varepsilon_1 \neq \varepsilon_2$. Since $0 < \varepsilon_1$ and $0 < \varepsilon_2$, $\varepsilon_1 \leq_S \varepsilon_2$ and $\varepsilon_1 \neq \varepsilon_2$ mean $\varepsilon_1 < \varepsilon_2$. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$, then

(1) if $a_i = 0$ then $\overline{a_i}^{\varepsilon_1} = \overline{a_i}^{\varepsilon_2} = 0$,

(2) if $0 < a_i \leq \varepsilon_1$ then $\overline{a_i}^{\varepsilon_1} = \overline{a_i}^{\varepsilon_2} = 1/2$,

(3) if $\varepsilon_1 < a_i \leq \varepsilon_2$ then $\overline{a_i}^{\varepsilon_1} = 1$ and $\overline{a_i}^{\varepsilon_2} = 1/2$ and

(4) if $\varepsilon_2 < a_i \leq 1$ then $\overline{a_i}^{\varepsilon_1} = \overline{a_i}^{\varepsilon_2} = 1$.

Therefore, $\overline{a_i}^{\varepsilon_1} \leq_S \overline{a_i}^{\varepsilon_2}$ is hold in any case. Thus $\overline{\boldsymbol{a}}^{\varepsilon_2} \leq_S \overline{\boldsymbol{a}}^{\varepsilon_1}$. ∎

Theorem 10 *Let F be an infinite-valued function and satisfy Condition (SC). Then F also satisfies Condition (SB).*

Proof: Let $\boldsymbol{a}$, $\boldsymbol{b} \in V^n$ and $\boldsymbol{a} \leq_S \boldsymbol{b}$, then by Lemma 6 $\overline{\boldsymbol{a}}^{\varepsilon} \leq_S \overline{\boldsymbol{b}}^{\varepsilon}$ for any ε such that $0 < \varepsilon < 1$. Therefore, by Lemma 5 there are constants ε_1 and ε_2 and $\boldsymbol{t} \in V^n$ such that $0 < \varepsilon_2 < \varepsilon_1 < 1$ and $\overline{\boldsymbol{a}}^{\varepsilon} = \overline{\boldsymbol{t}}^{\varepsilon_1}$, $\overline{\boldsymbol{b}}^{\varepsilon} = \overline{\boldsymbol{t}}^{\varepsilon_2}$. Thus, by the assumption of (SC) the following relations hold.

$$\begin{aligned} \overline{F(\boldsymbol{a})}^{\varepsilon} = F(\overline{\boldsymbol{a}}^{\varepsilon}) &= F(\overline{\boldsymbol{t}}^{\varepsilon_1}) = \overline{F(\boldsymbol{t})}^{\varepsilon_2}, \\ \overline{F(\boldsymbol{b})}^{\varepsilon} = F(\overline{\boldsymbol{b}}^{\varepsilon}) &= F(\overline{\boldsymbol{t}}^{\varepsilon_2}) = \overline{F(\boldsymbol{t})}^{\varepsilon_2}. \end{aligned}$$

On the other hand, $\varepsilon_2 \leq_S \varepsilon_1$ since $0 < \varepsilon_2 < \varepsilon_1 < 1$, and therefore by Lemma 7 $\overline{F(\boldsymbol{t})}^{\varepsilon_1} \leq_S \overline{F(\boldsymbol{t})}^{\varepsilon_2}$. Therefore, $\overline{F(\boldsymbol{a})}^{\varepsilon} \leq_S \overline{F(\boldsymbol{b})}^{\varepsilon}$ for any ε such that $0 < \varepsilon < 1$, and by Lemma 6 we have $F(\boldsymbol{a}) \leq_S F(\boldsymbol{b})$. This completes the proof of the theorem. ∎

5.5 Minimization for Stone Logic Functions

In this section we describe minimization for Stone logic functions. A minimal form described in this section is motivated by Boolean functions and fuzzy logic functions.

5.5.1 Prime Implicants and Minimal Forms

A minimal form of a Stone logic function F is defined as follows.

Definition 8 *Let F be a disjunctive form of a Stone logic function. Then F is called a minimal form if and only if no other equivalent disjunctive form involving a smaller total number of literals.*

Note that a minimal form of any Stone logic function does not determined uniquely as well as Boolean functions and fuzzy logic functions. Let α be a product term then α is said to be an implicant of F if and only if $F \sqsupseteq \alpha$. Moreover, an implicant α of F is said to be prime if and only if there is no product term β such that $F \sqsupseteq \beta \sqsupseteq \alpha$ and $\alpha \not\sqsupseteq \beta$.

Let define a new partial order relation $\leq_{\#}$ on the set $\{0, 1/2, 1, \#\}$ denoted $V_{\#}$ below.

Definition 9 *Let a and b be elements of $V_{\#}$. Then a partial order relation $\leq_{\#}$ is defined as follows (see Figure 5.3).*

$$\# \leq_{\#} 0, \quad \# \leq_{\#} 1/2, \quad \# \leq_{\#} 1, \quad 1/2 \leq_{\#} 1 \text{ and } \quad i \leq_{\#} i,$$

where $i \in V_{\#}$.

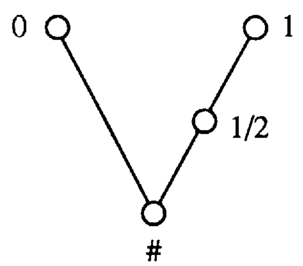

Figure 5.3: Partial Order Relation *sorderd*

The relation $\leq_{\#}$ is expanded among the elements of $V_{\#}^n$. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of $V_{\#}^n$ then $\boldsymbol{a} \leq_{\#} \boldsymbol{b}$ if and only if $a_i \leq_{\#} b_i$ for all i $(i = 1, \ldots, n)$. The set $V_{\#}$ forms the partial order set concerning the relation $\leq_{\#}$, and therefore we find the least upper bound of any elements a and b of $V_{\#}$ if it exists. We write the least upper bound of a and b as $a \nabla_{\#} b$, and if it dose not exist then write $a \nabla_{\#} b = \emptyset$. This can be expanded among the elements of $V_{\#}^n$ as follows. For two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ we define $\boldsymbol{a} \nabla_{\#} \boldsymbol{b}$ as $(a_1 \nabla_{\#} b_1, \ldots, a_n \nabla_{\#} b_n)$, and if $a_i \nabla_{\#} b_i = \emptyset$ for some i $(i = 1, \ldots, n)$ then define it as $\boldsymbol{a} \nabla_{\#} \boldsymbol{b} = \emptyset$.

Example 9 *Let $\boldsymbol{a}_1 = (0, \#, 1/2)$, $\boldsymbol{a}_2 = (\#, 1/2, \#)$ and $\boldsymbol{a}_3 = (1, 1/2, \#)$. Then $\boldsymbol{a}_2 \leq_{\#} \boldsymbol{a}_3$ and neither $\boldsymbol{a}_1 \leq_{\#} \boldsymbol{a}_2$ nor $\boldsymbol{a}_2 \leq_{\#} \boldsymbol{a}_1$, and also both $\boldsymbol{a}_1 \leq_{\#} \boldsymbol{a}_3$ and $\boldsymbol{a}_3 \leq_{\#} \boldsymbol{a}_1$ dose not hold. Moreover, $\boldsymbol{a}_1 \leq_{\#} \boldsymbol{a}_2 = (0, 1/2, 1/2)$, $\boldsymbol{a}_1 \leq_{\#} \boldsymbol{a}_3 = \emptyset$ and $\boldsymbol{a}_2 \leq_{\#} \boldsymbol{a}_3 = \boldsymbol{a}_3$.*

Definition 10 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of $V_{\#}^n$. Then $\boldsymbol{a}$ corresponds to a product term $\alpha = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ if the following condition holds.*

$$x_i^{a_i} = \begin{cases} \neg x_i & \text{if} \quad a_i = 0, \\ \neg\neg x_i & \text{if} \quad a_i = 1/2, \\ x_i & \text{if} \quad a_i = 1, \\ 1 & \text{if} \quad a_i = \# \end{cases}$$

Obviously there is one-to-one correspondence between product terms and the elements of $V_{\#}^n$.

Example 10 *The element* $(0, 1/2, \#, 1)$ *correspond to the product term* $\neg x_1 \neg\neg x_2 x_4$.

Definition 11 *Let* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *be an element of* $V_{\#}^n$. *Then* $\boldsymbol{a}^*$ *is defined as a set of all elements* $\boldsymbol{b} = (b_1, \ldots, b_n) \in V_3^n$ *such that* $b_i = 0$, *1/2 or 1 when* $a_i = \#$, *and otherwise* $b_i = a_i$, *i.e.,*

$$\boldsymbol{a}^* = \left\{ (b_1, \ldots, b_n) \in V_3^n \;\middle|\; b_i = \begin{cases} 0, 1/2 \quad or \quad 1 & if \;\; a_i = \#, \\ a_i & otherwise \end{cases} \right\}$$

Example 11 *Let* $\boldsymbol{a} = (0, 1, \#)$ *then* $\boldsymbol{a}^* = \{(0,1,0), (0,1,1/2), (0,1,1)\}$.

Lemma 8 *Let* α *be a product term and* $\boldsymbol{a}$ *the corresponding element to* α. *For an element* $\boldsymbol{b}$ *of* V_3^n,

(1) $\alpha(\boldsymbol{b}) = 1$ *if and only if* $\boldsymbol{a} \leq_{\#} \boldsymbol{b}$,

(2) $\alpha(\boldsymbol{b}) = 1/2$ *if and only if* $\boldsymbol{a} \not\leq_{\#} \boldsymbol{b}$ *and* $\boldsymbol{a} \nabla_{\#} \boldsymbol{b} \neq \emptyset$,

(3) $\alpha(\boldsymbol{b}) = 0$ *if and only if* $\boldsymbol{a} \nabla_{\#} \boldsymbol{b} = \emptyset$.

(The proof is omitted)

Lemma 9 *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be elements of* $V_{\#}^n$ *and* α *and* β *product terms corresponding to* $\boldsymbol{a}$ *and* $\boldsymbol{b}$, *respectively. Then* $\alpha \sqsupseteq \beta$ *if and only if* $\boldsymbol{a} \leq_{\#} \boldsymbol{b}$ *if and only if* $\alpha(\boldsymbol{b}^*) = \{1\}$ *where* $\alpha(\boldsymbol{b}^*)$ *is the image of* $\boldsymbol{b}^*$.

(The proof is omitted)

Lemma 10 *Let* $\alpha_1 \vee \ldots \vee \alpha_s$ *be a disjunctive form of a Stone logic function* F, *and let* β *be a product term and* $\boldsymbol{b}$ *the corresponding element to* β. *Then* $F(\boldsymbol{b}^*) = \{1\}$, *which is the image of* $\boldsymbol{b}^*$, *implies that there is a sequence* $i_1, \ldots, i_k$ *such that* $\alpha_{i_1} \vee \ldots \vee \alpha_{i_k} \sqsupseteq \beta$, $i_j \in \{1, \ldots, s\}$ *and* $1 \leq j \leq k \leq s$.

Proof: $F(\boldsymbol{b}^*) = \{1\}$ implies that there is a sequence $i_1, \ldots, i_k$ such that $(\alpha_{i_1} \vee \ldots \vee \alpha_{i_k})(\boldsymbol{b}^*) = \{1\}$, $i_j \in \{1, \ldots, s\}$ and $1 \leq j \leq k \leq s$. Let $\boldsymbol{a}$ be an element of V_3^n, and suppose $\beta(\boldsymbol{a}) = 1$. Then by Lemma 3.1 $\boldsymbol{b} \leq_{\#} \boldsymbol{a}$, that is, $\boldsymbol{a} \in \boldsymbol{b}^*$, and we have $(\alpha_{i_1} \vee \ldots \vee \alpha_{i_k})(\boldsymbol{a}) = 1$. Next suppose $\beta(\boldsymbol{a}) = 1/2$. Then $(\alpha_{i_1} \vee \ldots \vee \alpha_{i_k})(\boldsymbol{a}) \neq 0$, because if $(\alpha_{i_1} \vee \ldots \vee \alpha_{i_k})(\boldsymbol{a}) = 0$ holds then $\alpha_{i_j}(\boldsymbol{a}) = 0$ for all j ($j = 1, \ldots, k$), and therefore by Lemma 83 $\boldsymbol{a} \nabla_{\#} \boldsymbol{a}_j = \emptyset$ where $\boldsymbol{a}_j$ is the corresponding element to α_{i_j}. By $\beta(\boldsymbol{a}) = 1/2$ and Lemma 8.2, $\boldsymbol{a} \nabla_{\#} \boldsymbol{b} \neq \emptyset$, and accordingly $\boldsymbol{b} \nabla_{\#} \boldsymbol{a}_j = \emptyset$ for all j. Therefore, for all elements $\boldsymbol{b}'$ of $\boldsymbol{b}^*$ we have $\alpha_{i_j}(\boldsymbol{b}') = 0$, and this contradicts to $(\alpha_{i_1} \vee \ldots \vee \alpha_{i_k})(\boldsymbol{b}^*) = \{1\}$. Thus, $(\alpha_{i_1} \vee \ldots \vee \alpha_{i_k})(\boldsymbol{a}) \neq 0$ when $\beta(\boldsymbol{a}) = 1/2$. Form the above $\alpha_{i_1} \vee \ldots \vee \alpha_{i_k} \sqsupseteq \beta$. ■

Theorem 11 *Let* $\alpha_1 \vee \ldots \vee \alpha_s$ *be a minimal form of a Stone logic function* F. *Then each* α_i ($i = 1, \ldots, s$) *is a prime implicant of* F.

Proof: Let assume that there is a product term α_j ($j = 1, \ldots, s$) which is not a prime implicant of F, that is, there is a prime implicant β of F such that $\beta \sqsupseteq \alpha_j$ and $\alpha_j \neq \beta$. Let $\boldsymbol{b} = (b_1, \ldots, b_n)$ be corresponding elements to β then by Lemma 9 $\beta(\boldsymbol{b}^*) = \{1\}$, and therefore $F(\boldsymbol{b}^*) = \{1\}$ since β is an implicant of F. Thus by Lemma 10 there is a sequence $i_1, \ldots, i_k$ such that $\alpha_{i_1} \vee \ldots \vee \alpha_{i_k} \sqsupseteq \beta$,

$i_t \in \{1, \ldots, s\}$ and $1 \leq t \leq k \leq s$. Here, j never appears in the sequence $i_1, \ldots, i_k$. Because, let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be the corresponding element to α_j then $\boldsymbol{b} \leq_\# \boldsymbol{a}$ since $\beta \sqsupseteq \alpha_j$ and Lemma 9. Therefore, $a_i = \#$ implies $b_i = \#$. Also $b_i = \#$ implies $a_i = \#$, because if it dose not hold then the number of literals appearing in β is smaller then that of α_j, and this contradicts that F is a minimal form. Therefore $a_i = \#$ if and only if $b_i = \#$. Thus,

$$\begin{aligned} a_i = 0 &\quad \text{if and only if} \quad b_i = 0, \\ a_i = 1/2 &\quad \text{if and only if} \quad b_i = 1/2, \text{ and} \\ a_i = 1 &\quad \text{if and only if} \quad b_i = 1/2 \text{ or } 1. \end{aligned}$$

If $b_i = 1$ for all i such that $a_i = 1$, then $\alpha_j = \beta$, and this contradicts that $\alpha_j \neq \beta$. Therefore, for at least one i we have $a_i = 1$ and $b_i = 1/2$. Thus, $\alpha_j(\boldsymbol{b}^*) = \{1/2\}$, and this implies j never appears in the sequence $i_1, \ldots, i_k$. Accordingly α_j is not included in $\alpha_{i_1} \vee \ldots \vee \alpha_{i_k}$ since $\alpha_{i_1} \vee \ldots \vee \alpha_{i_k} \sqsupseteq \beta \sqsupseteq \alpha_j$ and this contradicts that F is a minimal form. This completes the proof of the theorem. ∎

From the above theorem, in order to find any minimal form of F we need to get all of the prime implicants of F.

5.5.2 Consensus and Algorithms

Definition 12 *Let α and β be product terms and $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ the corresponding elements to α and β, respectively. Moreover, let α and β satisfy the following two conditions.*

(1) For only one i $(i = 1, \ldots, n)$, $a_i = 0$ and $b_i = 1/2$, or $a_i = 1/2$ and $b_i = 0$, and

(2) For all j such that $j \neq i$, $a_j \nabla_\# b_j \neq \emptyset$.

Let $\boldsymbol{a}'$ and $\boldsymbol{b}'$ be the elements of $V_\#^n$ such that the i-th components of $\boldsymbol{a}$ and $\boldsymbol{b}$ are replaced by $\#$, respectively, i.e., $\boldsymbol{a}' = (a_1, \ldots, a_{i-1}, \#, a_{i+1}, \ldots, a_n)$ and $\boldsymbol{b}' = (b_1, \ldots, b_{i-1}, \#, b_{i+1}, \ldots, b_n)$. Then the consensus of α and β is defined such a product term $\alpha'\beta'$ which are corresponding to $\boldsymbol{a}'$ and $\boldsymbol{b}'$, respectively.

In the above definition, it is assumed that any consensus dose not contain two or more different literals simultaneously for each variable appearing in it.

Example 12 *Let $\alpha = \neg x_1 \neg\neg x_2 x_4$, $\beta = \neg x_2 x_3 \neg\neg x_4$ and $\gamma = \neg\neg x_1 \neg x_3 x_4$, then $\neg x_1 x_3 x_4$ is the consensus of α and β, and $\neg\neg x_2 \neg x_3 x_4$ is that of α and γ. However, there is no consensus of β and γ.*

Lemma 11 *Let α and β be product terms and γ the consensus of α and β. Then $\alpha \vee \beta \sqsupseteq \gamma$, that is, $\alpha \vee \beta \vee \gamma = \alpha \vee \beta$.*

Proof: We can assume without loss of generality that $\alpha = \alpha' x_i^*$ and $\beta = \beta' \neg x_i^*$ for some i where x_i^* is one of $\neg x_i$ or $\neg\neg x_i$. Since $\neg x_i^* \vee \neg\neg x_i^* = 1$, $(\alpha \vee \beta)(\boldsymbol{a}) = (\alpha' \vee \beta')(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_3^n. Therefore, $\gamma(\boldsymbol{a}) = (\alpha'\beta')(\boldsymbol{a}) \leq (\alpha' \vee \beta')(\boldsymbol{a}) = (\alpha \vee \beta)(\boldsymbol{a})$, i.e., we have $\alpha \vee \beta \sqsupseteq \gamma$. ∎

Theorem 12 *Let $\alpha_1 \vee \ldots \vee \alpha_s$ be a disjunctive form of a Stone logic function F. Then $\alpha_1 \vee \ldots \vee \alpha_s$ is the disjunction of all prime implicants of F if and only if*

(1) no product term includes any other product term, i.e., $\alpha_i \not\sqsupseteq \alpha_j$ *for all* i, j $(i,\ j = 1,\ldots,s,\ \ and\ \ i \neq j)$, *and*

(2) the consensus of any two product terms dose not exist or is included in other product term α_i $(i = 1,\ldots,s)$.

Proof: It is evident that (1) and (2) hold when $F = \alpha_1 \vee \ldots \vee \alpha_s$ is the disjunction of all prime implicants of F. Conversely, let assume $F = \alpha_1 \vee \ldots \vee \alpha_s$ is not the disjunction of all prime implicants of F when F holds (1) and (2). This means one of the following relations holds.

(a) a prime implicant α of F dose not appear in $\alpha_1, \vee \ldots \vee, \alpha_s$, or

(b) a product term α_i $(i = 1,\ldots,s)$ is not a prime implicant of F.

First, assume (a) holds. Then it may be possible to add some literals $\neg x_i$ or $\neg\neg x_i$ to α if the variable x_i dose not appear in α, forming a product term α' which sill has the property that $\alpha_i \not\sqsupseteq \alpha'$ for all i $(i = 1,\ldots,s)$. Let β be a product term constructed by the above way and satisfying the following two conditions.

Condition 1: $\alpha_i \not\sqsupseteq \beta$ for all i $(i = 1,\ldots,s)$, and

Condition 2: For any possible x_i there are α_j and α_k such that $\alpha_j \sqsupseteq \beta\neg x_i$ and $\alpha_k \sqsupseteq \beta\neg\neg x_i$ $(j,\ k = 1,\ldots,s\ \ and\ \ j \neq k)$.

It is impossible β become a minterm, because if β is a minterm then $F(\boldsymbol{b}) = 1$ for the corresponding element $\boldsymbol{b} \in V_3^n$ to β since $\beta(\boldsymbol{b}) = 1$ and β is an implicant of F. Therefore there is a product term α_i $(i = 1,\ldots,s)$ such that $\alpha_i(\boldsymbol{b}) = 1$, and thus $\boldsymbol{a}_i \leq_\# \boldsymbol{b}$ where $\boldsymbol{a}_i$ is the corresponding element to α_i. Accordingly by Lemma 9 β is included in α_i, and this contradicts to *Condition* 1. Therefore β is not a minterm. Since β is not a minterm there is at least one variable x_i which dose not appear in β, and by *Condition* 2 there are α_j and α_k such that $\alpha_j \sqsupseteq \beta\neg x_i$ and $\alpha_k \sqsupseteq \beta\neg\neg x_i$ $(j,\ k = 1,\ldots,s\ \ and\ \ j \neq k)$. Let $\boldsymbol{b} = (b_1,\ldots,b_n)$, $\boldsymbol{a}_j = (a_{j_1},\ldots,a_{j_n})$ and $\boldsymbol{a}_k = (a_{k_1},\ldots,a_{k_n})$ be respectively the corresponding elements to β, α_j and α_k. Then $\boldsymbol{b}_j = (b_1,\ldots,b_{i-1},0,b_{i+1},\ldots,b_n)$ and $\boldsymbol{b}_k = (b_1,\ldots,b_{i-1},1/2,b_{i+1},\ldots,b_n)$ are the corresponding elements to $\beta\neg x_i$ and $\beta\neg\neg x_i$, respectively, and we have $\boldsymbol{a}_j \leq_\# \boldsymbol{b}_j$ and $\boldsymbol{a}_k \leq_\# \boldsymbol{b}_k$. Therefore it holds that $a_{j_i} = 0$ and $a_{k_i} = 1/2$ (by *Condition* 1 and Lemma 9 neither $a_{j_i} \neq \#$ nor $a_{k_i} \neq \#$), and for all m $(m = 1,\ldots,n)$ such that $m \neq i$

$$b_m = \begin{cases} 0, \\ 1/2, \\ 1, \\ \# \end{cases} \quad \text{implies} \quad a_{j_m}, a_{k_m} = \begin{cases} 0\ \ or\ \ \#, \\ 1/2\ \ or\ \ \#, \\ 1/2,\ \ 1\ \ or\ \ \#, \\ \# \end{cases}$$

respectively. Thus $a_{j_m} \nabla_\# a_{k_m}$ for all m such that $m \neq i$, and accordingly there exists the consensus of α_j and α_k. Set $\alpha_j = \neg x_i \alpha'_j$ and $\alpha_k = \neg x_i \alpha'_k$, and then the consensus of α_j and α_k is $\alpha'_j \cdot \alpha'_k$ and obviously $\alpha'_j \alpha'_k \sqsupseteq \beta$. By the assumption (2) there is a product term α_t $(t = 1,\ldots,s)$ such that $\alpha_t \sqsupseteq \alpha'_j \alpha'_k$ and this contradicts to *Condition* 1. Therefore no product term α_t $(t = 1,\ldots,s)$ includes the consensus $\alpha'_j \alpha'_k$ of α_j and α_k, and this contradicts to (2). Thus all prime implicants of F appear in $\alpha_1,\ldots,\alpha_s$. Next, assume (b) holds then there is a prime implicant α_j in $\alpha_1,\ldots,\alpha_s$ such that $\alpha_j \sqsupseteq \alpha_i$ since all prime implicants of F appear in $\alpha_1,\ldots,\alpha_s$ from the above discussion, and this contradicts to (1). Thus $F = \alpha_1 \vee \ldots \vee \alpha_s$ is the disjunction of all prime implicants of F. ■

From the above theorem, we have the following algorithm to find all prime implicants of a given Stone logic function F.

Algorithm A

Step 1: Expand F into a disjunctive form.

Step 2: Remove any product term that is included in another product term, and let $\alpha_1, \ldots, \alpha_s$ be remaining product terms.

Step 3: Find all of the consensus of any two product terms α_i and α_j $(i, j = 1, \ldots, s)$. If no consensus exists, then in Step 5, otherwise in Step 4.

Step 4: Construct the disjunctive form from $\alpha_1, \ldots, \alpha_s$ and all of the consensus getting in Step 3. If any consensus is included in one of $\alpha_1, \ldots, \alpha_s$, then in Step 5, otherwise in Step 2.

Step 5: The remaining product terms are all of the prime implicants of F.

Theorem 13 *Let $\alpha_1 \vee \ldots \vee \alpha_s$ be the canonical disjunctive form of a Stone logic function F, and let $\beta_1 \vee \ldots \vee \beta_t$ be a disjunctive form of F. Then each minterm α_i $(i = 1, \ldots, s)$ is included in a product term β_j $(j = 1, \ldots, t)$.*

Proof: Let $\boldsymbol{a}$ be the corresponding element to α_i, then $\alpha_i(\boldsymbol{a}) = 1$ and this implies $F(\boldsymbol{a}) = 1$. Suppose no product term β_j includes α_i, i.e., $\beta_j \not\supseteq \alpha_i$ for all j $(j = 1, \ldots, t)$. Then by Lemma 9 $\boldsymbol{b} \not\leq_{\#} \boldsymbol{a}$ where $\boldsymbol{b}$ is the corresponding element to β_j, and therefore by Lemma 8.1 $\beta_j(\boldsymbol{a}) \neq 1$ for all j. Thus we have $F(\boldsymbol{a}) \neq 1$, and this is contradiction. This completes the proof of the theorem. ■

From Theorem 13, we have the algorithm to find a minimal form of a given Stone logic function F.

Algorithm B

Step 1: Expand F into the canonical disjunctive form, and let $\alpha_1, \ldots, \alpha_s$ be minterms appearing in the canonical disjunctive form of F.

Step 2: Find all prime implicants $\beta_1, \ldots, \beta_t$ of F by applying Algorithm A.

Step 3: Find a minimal group $\beta_{i_1}, \ldots, \beta_{i_k}$ $(i_j = 1, \ldots, t,\ 1 \leq j \leq k \leq t)$ of prime implicants such that each α_i $(i = 1, \ldots, s)$ is included in a prime implicant β_{i_j} $(j = 1, \ldots, k)$, then $\beta_{i_1} \vee \ldots \vee \beta_{i_k}$ is a minimal form of F.

Step 3 of Algorithm B corresponds to the minimum covering problem for Boolean functions. Therefore, Step 3 is solved by using a prime implicants table like Boolean functions.

Example 13 *Let a 3-variable Stone logic function*

$$F = \neg\neg x_1 \neg\neg x_2 \neg x_3 \vee \neg\neg x_1 \neg x_2 \neg x_3 \vee \neg x_1 \neg\neg x_2 \neg\neg x_3 \vee \neg x_1 \neg x_2 \neg\neg x_3 \vee \neg\neg x_1 x_2 \neg\neg x_3,$$

which is the canonical disjunctive form. By the above algorithm we have all of the prime implicants of F denoted below.

$$\neg\neg x_1 \neg x_3, \quad \neg x_1 \neg\neg x_3, \quad \neg\neg x_1 x_2, \quad x_2 \neg\neg x_3.$$

Then we get the following 2 minimal forms of F by finding a minimal group form Table 5.2.

$$F = \neg\neg x_1 \neg x_3 \vee \neg x_1 \neg\neg x_3 \vee \left\{ \begin{array}{c} \neg\neg x_1 x_2 \\ x_2 \neg\neg x_3 \end{array} \right\}$$

Table 5.2: Prime Implicants Table of Example 13

	$\neg\neg x_1 \neg x_3$	$\neg x_1 \neg\neg x_3$	$\neg\neg x_1 x_2$	$x_2 \neg\neg x_3$
$\neg\neg x_1 \neg\neg x_2 \neg x_3$	√			
$\neg\neg x_1 \neg x_2 \neg x_3$	√			
$\neg x_1 \neg\neg x_2 \neg\neg x_3$		√		
$\neg x_1 \neg x_2 \neg\neg x_3$		√		
$\neg\neg x_1 x_2 \neg\neg x_3$			√	√

5.6 Number of n-Variable Stone Logic Functions

The number of n-variable Stone logic functions is equivalent to that of Stone-3 logic functions from Theorem 3. Therefore, in this section, we discuss the number of n-variable Stone-3 logic functions. The formula finally obtained in the section is represented in the term of monotone Boolean functions, that is, the number of Stone logic functions closely connects with monotone Boolean functions.

Definition 13 *A relation $\equiv_S$ on the set V_3 is defined as follows.*

$$1 \equiv_S 1/2,\ 1/2 \equiv_S 1 \text{ and } i \equiv_S i,$$

where $i \in V_3$. The relation $\equiv_S$ is expanded into V_3^n as follows. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V_3^n. Then $\boldsymbol{a} \equiv_S \boldsymbol{b}$ holds if and only if $a_i \equiv_S b_i$ for every $i = 1, \ldots, n$.

It is clear that the relation $\equiv_S$ is an equivalence relation by easy verification. Therefore, we define the equivalence class $[\boldsymbol{c}]_{\equiv_S}$ of $\boldsymbol{c} \in V_3^n$ by $[\boldsymbol{c}]_{\equiv_S} = \{\boldsymbol{c}' \in V_3^n | \boldsymbol{c} \equiv_S \boldsymbol{c}'\}$, and then obviously $a_i = 0$ if and only if $b_i = 0$ for any $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of $[\boldsymbol{c}]_{\equiv_S}$. Moreover, each $[\mathrm{c}]_{\equiv_S}$ forms a finite partial order set under $\leq_S$, and it is sure there exists a maximum element in each $[\mathrm{c}]_{\equiv_S}$, which is also an element of V_2^n. Therefore, the relation $\bigcup_{\mathrm{c} \in V_2^n} [\mathrm{c}]_{\equiv_S} = V_3^n$ holds. Thus, we can choose the element of V_2^n as the representative of each equivalence class $[\mathrm{c}]_{\equiv_S}$. Obviously, if $\mathrm{c}_1, \mathrm{c}_2 \in V_2^n$ such that $\mathrm{c}_1 \neq \mathrm{c}_2$, then $[\mathrm{c}_1]_{\equiv_S} \cap [\mathrm{c}_2]_{\equiv_S} = \emptyset$.

Example 14 *The set V_3^2 is classified by four different kinds of equivalence classes such that* $[(0,0)] = \{(0,0)\}$, $[(0,1)] = \{(0,1), (0,1/2)\}$, $[(1,0)] = \{(1,0), (1/2,0)\}$ *and* $[(1,1)] = \{(1,1), (1,1/2), (1/2,1), (1,1)\}$. *Figure 5.4 and 5.5 show Hasse diagrams of* $[(0,1)]$ *and* $[(1,1)]$, *respectively, under the relation $\leq_S$.*

Recall that the set of Conditions (Sa) and (Sb) in Section 5.4 is a necessary and sufficient condition for a ternary function to be a Stone-3 logic function. Among Conditions (Sa) and (Sb), only Condition (Sb) describes the relation between the outputs of a Stone-3 logic function

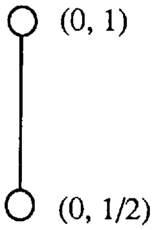

Figure 5.4: Hasse Diagram of $[(0,1)]$ in Example 14

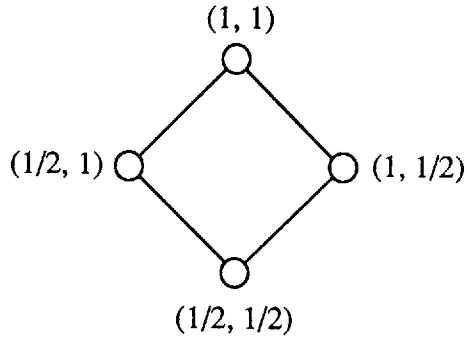

Figure 5.5: Hasse Diagram of $[(0,1)]$ in Example 14

for different kinds of inputs. Now, let $[\boldsymbol{c}_1]_{\equiv_S}$ and $[\boldsymbol{c}_2]_{\equiv_S}$ be the equivalence classes such that $[\boldsymbol{c}_1]_{\equiv_S} \neq [\boldsymbol{c}_2]_{\equiv_S}$. Then any $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, which are respectively elements of $[\boldsymbol{c}_1]_{\equiv_S}$ and $[\boldsymbol{c}_2]_{\equiv_S}$, are not comparable to each other under the relation $\leq_{\mathrm{S}}$. Then the antecedent of (Sb) stands always false for such two elements $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, and thus Condition (Sb) stands always true whether the consequence of (Sb) is true or false. Therefore for any Stone-3 logic function f_s there is not any relation between $f_s(\boldsymbol{a}_1)$ and $f_s(\boldsymbol{a}_2)$ if $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are respectively elements of $[\boldsymbol{c}_1]_{\equiv_S}$ and $[\boldsymbol{c}_2]_{\equiv_S}$ such that $[\boldsymbol{c}_1]_{\equiv_S} \neq [\boldsymbol{c}_2]_{\equiv_S}$. From the above we have the following formula concerning with the number of n-variable Stone-3 logic functions.

$$|F_S(n)| = \prod_{\boldsymbol{c} \in V_2^n} |F_S([\boldsymbol{c}]_{\equiv_S})| \tag{5.1}$$

In the equation (5.1) $F_S(n)$ and $F_S([\boldsymbol{c}]_{\equiv_S})$ are sets defined below.

$$\begin{aligned} F_S(n) &= \{f_s \mid f_s \text{ is an } n\text{-variable Stone-3 logic function}\} \\ F_S([\boldsymbol{c}]_{\equiv_S}) &= \{f_s|([\boldsymbol{c}]_{\equiv_S}) \mid f_s \in F_S(n)\}, \end{aligned}$$

where $f_s|[\boldsymbol{c}]_{\equiv_S}$ is the restriction of f_s to the equivalence class $[\boldsymbol{c}]_{\equiv_S}$.

The set $F_S([\boldsymbol{c}]_{\equiv_S})$ has a connection with monotone Boolean functions. An n-variable monotone Boolean function is a Boolean function $f_b : V_2^n \to V_2$ satisfying

(M) $\boldsymbol{a}, \boldsymbol{b} \in V_2^n$ and $\boldsymbol{a} \leq_{\mathrm{B}} \boldsymbol{b}$ imply $f_b(\boldsymbol{a}) \leq_{\mathrm{B}} f_b(\boldsymbol{b})$.

In the above condition (M), $\leq_{\mathrm{B}}$ means a partial order relation on V_2 defined as $0 \leq_{\mathrm{B}} 1$ and $i \leq_{\mathrm{B}} i$ where $i \in V_2$. It can be expanded among V_2^n as follows. For any two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of V_2^n $\boldsymbol{a} \leq_{\mathrm{B}} \boldsymbol{b}$ if and only if $a_i \leq_{\mathrm{B}} b_i$ for all $i = 1, \ldots, n$.

In the following we discuss the relationship between Stone-3 logic functions and monotone Boolean functions. Below, $F_M(n)$ denotes the set of all n-variable monotone Boolean functions, that is, $F_M(n) = \{f_m \mid f_m \text{ is an } n\text{-variable monotone Boolean function}\}$.

Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V_2^n. Then a mapping $\lambda_i : V_2^n \to V_2$ is defined by $\lambda_i(\boldsymbol{a}) = a_i$ $(i = 1, \ldots, n)$. Obviously, $|[\mathbf{c}]_{\equiv_S}| = 2^k$ for any element $\boldsymbol{c} \in V_2^n$ where $k = \sum_{i=1}^{n} \lambda_i(\boldsymbol{a})$. Therefore, the number of all elements of the set $[\mathbf{c}]_{\equiv_S}$ and that of V_2^k is equivalent, that is, $|[\mathbf{c}]_{\equiv_S}| = |V_2^k|$. Then, next, we define a mapping φ_c as follows.

Definition 14 *Let* $\mathbf{c}$ *be an element of* $V_2^n - \{(\underbrace{0, \ldots, 0}_{n \text{ times}})\}$ *and* $k = \sum_{i=1}^{n} \lambda_i(\boldsymbol{c})$. *Then a mapping* $\varphi_c : [\mathbf{c}]_{\equiv_S} \to V_2^k$ *is defined as* $\varphi_c(\boldsymbol{a}) = (a'_1, \ldots, a'_k) \in V_2^k$ *for every* $\boldsymbol{a} \in [\mathbf{c}]_{\equiv_S}$, *which satisfies the following condition for any* $j = 1, \ldots, k$,

$$a'_j = \begin{cases} 0 & \text{if and only if } a_{i_j} = 1/2 \\ 1 & \text{if and only if } a_{i_j} = 1 \end{cases}$$

where the set $\{i_1, \ldots, i_k\}$ $(1 \leq i_1 < i_2 < \ldots < i_k \leq n)$ *is a subset of* $\{1, \ldots, n\}$ *such that* $a_{i_j} \neq 0$.

It is easy to show that φ_c is one-to-one and onto mapping, and $\boldsymbol{a} \leq_S \boldsymbol{b}$ if and only if $\varphi_c(\boldsymbol{a}) \leq_B \varphi_c(\boldsymbol{b})$. Therefore, the finite order set $[\mathbf{c}]_{\equiv_S}$ with $\leq_S$ and the finite order set V_2^k with $\leq_B$ are isomorphic, i.e., φ_c is an order isomorphism between $[\mathbf{c}]_{\equiv_S}$ and V_2^k.

Example 15 *Let* $\mathbf{c} = (0, 1, 1)$. *Then* $[\mathbf{c}]_{\equiv_S} = \{(0,1,1), (0,1,1/2), (0,1/2,1), (0,1/2,1/2)\}$. *Table 5.3 shows a mapping* φ_c, *and Figure 5.6 and 5.7 show Hasse diagrams of* $[\mathbf{c}]_{\equiv_S}$ *and* $\varphi_c([\mathbf{c}]_{\equiv_S}) = \{(0,0), (0,1), (1,0), (1,1)\}$.

Table 5.3: Mapping φ_c of Example 15

$\boldsymbol{a} \in [\mathbf{c}]_{\equiv_S}$	$\varphi_c(\boldsymbol{a})$
$(0,1,1)$	$(1,1)$
$(0,1,1/2)$	$(1,0)$
$(0,1/2,1)$	$(0,1)$
$(0,1/2,1/2)$	$(0,0)$

Definition 15 *Let* $\mathbf{c}$ *be an element of* $V_3^n - \{(\underbrace{0, \ldots, 0}_{n \text{ times}})\}$ *and* $k = \sum_{i=1}^{n} \lambda_i(\boldsymbol{c})$. *Then* $\xi_c : F_S([\mathbf{c}]_{\equiv_S}) \to F_B(k)$, *where* $F_B(k) = \{f_b | f_b : V_2^k \to V_2\}$, *is defined such a mapping that it satisfies one of the following conditions (1) and (2), where* $f_s \in F_S([\mathbf{c}]_{\equiv_S})$ *and* $f_b = \xi_c(f_s)$ *below.*

(1) $f_s(\boldsymbol{a}) = 0$ *for some* $\boldsymbol{a} \in [\mathbf{c}]_{\equiv_S}$ *if and only if* $f_b(\boldsymbol{a}') = 0$ *for every element* $\boldsymbol{a}' \in V_2^k$, *or*

(2) If $f_s(\boldsymbol{a}) \neq 0$ *for some* $\boldsymbol{a} \in [\mathbf{c}]_{\equiv_S}$ *then for every* $\boldsymbol{a} \in [\mathbf{c}]_{\equiv_S}$

(a) $f_b(\boldsymbol{a}') = 0$ *if and only if* $f_s(\boldsymbol{a}) = 1/2$, *and*

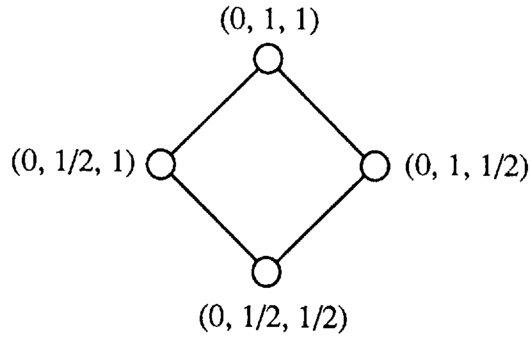

Figure 5.6: Hasse Diagram of $[c]_{\equiv_S}$ of Example 15

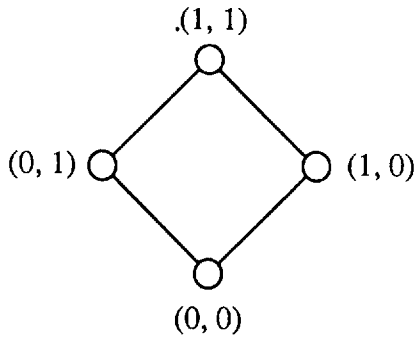

Figure 5.7: Hasse Diagram of $\varphi_c([c]_{\equiv_S})$ of Example 15

(b) $f_b(\boldsymbol{a}') = 1$ *if and only if* $f_s(\boldsymbol{a}) = 1$

where $\boldsymbol{a}' = \varphi_c(\boldsymbol{a})$.

Note that let f_s be Stone-3 logic function, then by Condition (Sb) $f_s(\boldsymbol{a}) = 0$ for some $\boldsymbol{a} \in [\boldsymbol{c}]_{\equiv_S}$ if and only if $f_s(\boldsymbol{a}) = 0$ for all $\boldsymbol{a} \in [\boldsymbol{c}]_{\equiv_S}$ for any $\boldsymbol{c}$ of V_3^n, and therefore, $f_s(\boldsymbol{a}) \neq 0$ for some $\boldsymbol{a} \in [\boldsymbol{c}]_{\equiv_S}$ if and only if $f_s(\boldsymbol{a}) \neq 0$ for all $\boldsymbol{a} \in [\boldsymbol{c}]_{\equiv_S}$.

The above definition is well defined and it is easy verification that ξ_c is a one-to-one mapping. Let $f_s \in F_S([\boldsymbol{c}]_{\equiv_S})$ and $\boldsymbol{a}, \boldsymbol{b} \in [\boldsymbol{c}]_{\equiv_S}$, then it is also evident that $f_s(\boldsymbol{a}) \leq_{\mathrm{S}} f_s(\boldsymbol{b})$ if and only if $\xi_c(f_s)(\boldsymbol{a}') \leq_{\mathrm{B}} \xi_c(f_s)(\boldsymbol{b}')$, where $\boldsymbol{a}' = \varphi_c(\boldsymbol{a})$ and $\boldsymbol{b}' = \varphi_c(\boldsymbol{b})$. Therefore, ξ_c preserves the monotonicity relation. Each associated function $\xi_c(f_s)$ is a monotone Boolean function since φ_c is an order isomorphism between $[\boldsymbol{c}]_{\equiv_S}$ and V_2^k, and

(Sa') $\boldsymbol{a} \in V_2^n \cap [\boldsymbol{c}]_{\equiv_S}$ implies $f_s(\boldsymbol{a}) \in V_2$, and

(Sb') $\boldsymbol{a}, \boldsymbol{b} \in [\boldsymbol{c}]_{\equiv_S}$ and $\boldsymbol{a} \leq_{\mathrm{S}} \boldsymbol{b}$ imply $f_s(\boldsymbol{a}) \leq_{\mathrm{S}} f_s(\boldsymbol{b})$

are satisfied for every $f_s \in F_S([\boldsymbol{c}]_{\equiv_S})$. Moreover, obviously there is a function $f_s \in F_S([\boldsymbol{c}]_{\equiv_S})$ for every monotone Boolean function f_m such that $\xi_c(f_s) = f_m$. From the above, the image of ξ_c, denoted by $\xi_c(F_S([\boldsymbol{c}]_{\equiv_S}))$, is equivalent to $F_M(k)$, i.e., $\xi_c(F_S([\boldsymbol{c}]_{\equiv_S})) = F_M(k)$. Since ξ_c is one-to-one, the relation $|F_S([\mathbf{c}]_{\equiv_S})| = |\xi_c(F_S([\mathbf{c}]_{\equiv_S}))|$ stands always true, and therefore we can conclude

$$\left|F_S([\mathbf{c}]_{\equiv_S})\right| = \left|F_M(k)\right| \tag{5.2}$$

(refer Figure 5.8).

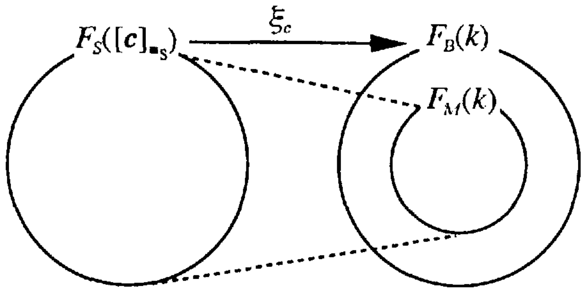

Figure 5.8: Relationship between $F_S([c]_{\equiv_S})$ and $F_M(k)$

Example 16 *Table 5.4 shows a truth table of a Stone-3 logic function F_s represented by such a logic formula $F_s = \neg x \vee \neg\neg xy$. Let $\mathbf{c} = (1,1)$, then $[\mathbf{c}]_{\equiv_S} = \{(1,1),(1,1/2),(1/2,1),(1/2,1/2)\}$ and the image of φ_c, denoted by $\varphi_c([\mathbf{c}]_{\equiv_S})$, is $\varphi_c([\mathbf{c}]_{\equiv_S}) = V_2^2 = \{(1,1),(1,0),(0,1),(0,0)\}$. Then Table 5.5 shows a truth table of a monotone Boolean function $\xi_c(F_S|[\mathbf{c}]_{\equiv_S})$.*

Table 5.4: Truth Table of Stone-3 Logic Function F_s of Example 16

y \ x	0	1/2	1
0	1	0	0
1/2	1	1/2	1/2
1	1	1	1

Table 5.5: Truth Table of Monotone Boolean Function of Example 16

y \ x	0	1
0	0	0
1	1	1

Finally, from the equations (5.1), (5.2) and

$$\left|\left\{\mathbf{c} \in V_2^n \mid \sum_{i=1}^{n} \lambda_i(\mathbf{c}) = k\right\}\right| =_n C_k$$

we have the following equation concerning with the number of n-variable Stone-3 logic functions.

$$\begin{aligned} \left|F_S(n)\right| &= \left|F_S([\mathbf{c}_0]_{\equiv_S})\right| \times \prod_{\mathbf{c}\in V_2^n, \mathbf{c}_0 \neq \mathbf{c}} \left|F_S([\mathbf{c}]_{\equiv_S})\right| \\ &= \left|F_S([\mathbf{c}_0]_{\equiv_S})\right| \times \prod_{k=1}^{n} \left|F_M(k)\right|^{{}_nC_k}, \end{aligned}$$

where $\mathbf{c}_0 = (0, \ldots, 0)$.

In the above equation, $F_S([\mathbf{c}_0]_{\equiv_S}) = \{0, 1\}$ because the constants 0 and 1 are only possible functions for the restriction $F_s|[\mathbf{c}_0]_{\equiv_S}$ by Conditions (Sa') and (Sb'). Also 0-variable monotone Boolean functions are only constants 0 and 1, i.e., $F_M(0) = \{0, 1\}$. Therefore, we have the number of n-variable Stone-3 logic functions as the following formula described by monotone Boolean functions below.

$$|F_S(n)| = \prod_{k=0}^{n} |F_M(k)|^{{}_nC_k}$$

The above formula also indicates the number of n-variable Stone logic functions by Theorem 3. Thus we have the accurate number of n-variable Stone logic functions until $n \leq 7$, because the number of n-variable monotone Boolean functions was calculated until $n \leq 7$ (see [50]). Table 5.6 shows the number of n-variable monotone Boolean functions until $n \leq 7$.

Table 5.6: The Number of Monotone Boolean Functions

n	The number of monotone Boolean Functions
0	2
1	3
2	6
3	20
4	168
5	7,581
6	7,828,354
7	2,414,682,040,998

5.7 Conclusions

In the chapter, we defined the new class of infinite-valued functions, called Stone logic functions, and the set of Stone logic functions is one of the models of Stone algebras. This is a reason why an infinite-valued function represented by a logic formula is called a Stone logic function.

Stone logic functions are essentially different from a series of multiple-valued logic functions based on the set-theoretical operations of fuzzy set theory, that is, different from the definition of negation operation.

Almost of the basic properties of Stone logic functions have been cleared as the results of the chapter.

Chapter 6

Kleene-Stone Logic Functions

6.1 Introduction

In Chapter 3 and Chapter 4, we discussed some of models of Kleene algebras, that is, B-ternary logic functions, regular ternary logic functions, fuzzy logic functions and multiple-valued Kleenean functions. Among them, *min*, *max*, and (1−) are employed as logical operations AND, OR and NOT, respectively. On the other hand, in Chapter 5, we discussed some properties of Stone logic functions, and among them the unary operation $\neg$ defined below is employed as logical operation for NOT.

$$\neg x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Moreover, the set of Stone logic functions forms an algebraic system called a Stone logic algebra, and it is different from Kleene algebras.

In the chapter, we introduce a new class of multiple-valued logic functions called Kleene-Stone logic functions which are fuzzy logic functions with the unary operation $\neg$. The set of them forms an algebraic system called a Kleene-Stone algebra [4], [5], and this is why fuzzy logic functions with $\neg$ called Kleene-Stone logic functions. In contrast to the existence of only one unary operation $\sim$ and $\neg$ in each model of Kleene algebras and Stone logic functions, respectively, Kleene-Stone logic functions employ both unary operations $\sim$ and $\neg$.

A Kleene-Stone algebra has been proposed as an algebraic system which satisfies the properties both a Kleene algebra and a Stone algebra.

It is interesting that Kleene-Stone logic functions have connection with modal logics. Because, they can represent the following two functions.

$$\neg\neg x = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \qquad \neg \sim x = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

The above two functions $\neg\neg$ and $\neg \sim$ respectively correspond to a possibility operation and a necessity operation [37], and they are usually denoted by the symbols $\Diamond$ and $\Box$, respectively. Therefore, Kleene-Stone logic functions relate to a logical system having a possibility operation and a necessity operation, that is, a modal logic. Of course, the following equations, which are the properties of a modal logic, hold in Kleene-Stone logic functions: $\Diamond x = *\Box * x$ and $\Box x = *\Diamond * x$, where $*$ denotes one of $\sim$ or $\neg$.

The chapter describes some of the following properties of Kleene-Stone logic functions rather than the relationship between modal logics and them.

First, in Section 6.2, we give a definition of Kleene-Stone logic functions in the term of logic formulas, then some properties of them appear in the section. Especially, we will show that they are monotone for a partial order relation denoted by the symbol $\leq_{KS}$ in the chapter, and moreover any Kleene-Stone logic function is uniquely determined only 5-valued inputs. In Section 6.3, we show a canonical disjunctive form which allows for the unique representation of any Kleene-Stone logic function. A necessary and sufficient condition for Kleene-Stone logic functions will be cleared in Section 6.4 by using the term of partial order relation $\leq_{KS}$. Minimization for Kleene-Stone logic functions is discussed in Section 6.5. Finally, the number of n-variable Kleene-Stone logic functions is appeared in Section 6.6, and it closely connects with B-ternary logic functions (or fuzzy logic functions). Because, the number of them is represented in the term of B-ternary logic functions (of fuzzy logic functions).

Recently, some results of the chapter are obtained in the paper [6].

6.2 Kleene-Stone logic functions and Their Properties

First, we give the definition of Kleene-Stone logic functions. Let V be the closed interval [0, 1]. An n-variable Kleene-Stone logic function is defined to be a mapping from V^n into V; $F : V \rightarrow V^n$, which is represented by a logic formula consisting of variables $x_1, \ldots, x_n$, constants 0 and 1, and logic operations AND($\cdot$), OR($\vee$) and NOT($\sim$), which are identical with those of fuzzy logic functions, and one more unary operation ($\neg$). Then, operations are defined as follows.

$$x \cdot y = \min(x, y), \qquad x \vee y = \max(x, y),$$
$$\sim x = 1 - x, \qquad \neg x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

where $x,\ y \in V$.

Namely, an n-variable Kleene-Stone logic function is defined strictly as follows,

Definition 1 *Logic formulas are defined inductively as follows.*

(1) Constants 0 and 1, and variables $x_1, \ldots, x_n$ are logic formulas.

(2) If G and H are logic formulas, then $(G \cdot H)$, $(G \vee H)$, $(\sim G)$ and $(\neg G)$ are also logic formulas.

(3) The only logic formulas are given by (1) and (2).

Definition 2 *A mapping $F : V \rightarrow V^n$ represented by a logic formula is called a Kleene-Stone logic function.*

The notation $\cdot$ may be often omitted. Hereafter, for simplicity, we will identify a Kleene-Stone logic function with the logic formula which represents it, and moreover, we will call an n-variable Kleene-Stone logic function a Kleene-Stone logic function. In writing logic formulas, we assume that symbols $\sim$ and $\neg$ are stronger than $\cdot$, and $\cdot$ is stronger than $\vee$, for omitting the parentheses.

Example 1 *Suppose 2-variable Kleene-Stone logic function $F = \neg \sim x_1 \vee \neg x_1 \cdot \sim x_2$ and $\boldsymbol{a} = (0.3, 0.9)$, then $F(\boldsymbol{a}) = \neg \sim 0.3 \vee \neg 0.3 \cdot \sim 0.9 = \neg 0.7 \vee 0 \cdot 0.1 = 0 \vee 0 = 0$.*

Looking at a set of Kleene-Stone logic functions as an algebraic system, the following equations, motivated by properties in Boolean algebra, Kleene algebra, and Stone algebra, are hold.

(1)	$a \cdot a = a, \quad a \vee a = a$	(the idempotent laws)
(2)	$a \cdot b = b \cdot a, \quad a \vee b = b \vee a$	(the commutative laws)
(3)	$a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad a \vee (b \vee c) = (a \vee b) \vee c$	(the associative laws)
(4)	$a \cdot (a \vee b) = a, \quad a \vee a \cdot b = a$	(the absorption laws)
(5)	$a \cdot (b \vee c) = a \cdot b \vee a \cdot c,$ $a \vee b \cdot c = (a \vee b) \cdot (a \vee c)$	(the distributive laws)
(6)	$\sim (a \cdot b) = \sim a \vee \sim b, \quad \sim (a \vee b) = \sim a \cdot \sim b,$ $\neg(a \cdot b) = \neg a \vee \neg b, \quad \neg(a \vee b) = \neg a \cdot \neg b$	(De Morgan's laws)
(7)	$\sim\sim a = a$	(the double negation law)
(8)	$0 \cdot a = 0, \quad 0 \vee a = a$	(the least element)
(9)	$1 \cdot a = a, \quad 1 \vee a = 1$	(the greatest element)
(10)	$a \cdot \sim a \cdot (b \vee \sim b) = a \cdot \sim a,$ $a \cdot \sim a \vee (b \vee \sim b) = b \vee \sim b$	(Kleene's laws)
(11)	$\neg\neg\neg a = \neg a$	
(12)	$a \cdot \neg a = 0, \quad \neg a \vee \neg\neg a = 1$	

The equations (1) $\sim$ (12) are identical to the axioms of Kleene-Stone algebra, given in [5]. Therefore a set of Kleene-Stone logic functions is one of the models of Kleene-Stone algebras. It is characteristic feature of Kleene-Stone logic functions that they allow two types of unary operations $\sim$ and $\neg$. These two operations are weaker than 2-valued one. In particular, $0 \leq a \cdot \sim a \leq b \vee \sim b \leq 1$, whereas $0 = a \cdot \neg a$. Also $\sim\sim a = a$, $\neg\neg\neg a = \neg a$, and $\sim \neg a = \neg\neg a$.

6.2.1 Fundamental Properties

First, a partial order relation is defined on the set V, and then any Kleene-Stone logic function satisfies the monotonicity for this partial order relation. Next, Theorem 2, which is an important theorem for Kleene-Stone logic functions, is described. Then, we show that a Kleene-Stone logic function is essentially a 5-valued logic function by using this theorem.

We define a partial order relation $\leq_{\mathrm{KS}}$ on V as follows.

Definition 3 *Let a and b be element of V. Then, $a \leq_{\mathrm{KS}} b$ if and only if one of the following relations holds (refer to Figure 6.2.1).*

$$(1)\ 0 < a \leq b \leq 1/2, \quad (2)\ 1/2 \leq b \leq a < 1, \quad (3)\ a = b.$$

It can be extended among V^n as follows. For two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of V^n, $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ if and only if $a_i \leq_{\mathrm{KS}} b_i$ for any $i = 1, \ldots, n$.

In the partial order relation $\leq_{\mathrm{KS}}$, 0 and 1 are only comparable with itself. Moreover, any two element of the open set (0, 1/2) and (1/2, 1), respectively, are always not comparable to each other, and 1/2 is not comparable only 0 and 1.

In the following, we show that any Kleene-Stone logic function satisfies the monotonicity for the partial order relation $\leq_{\mathrm{KS}}$.

Theorem 1 *Let F be any Kleene-Stone logic function and $\boldsymbol{a}$ and $\boldsymbol{b}$ be any element of V_5^n. If $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ then $F(\boldsymbol{a}) \leq_{\mathrm{KS}} F(\boldsymbol{b})$.*

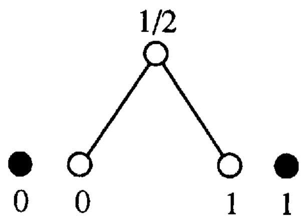

Figure 6.1: Partial Order Relation $\leq_{\mathrm{KS}}$

Proof: It will be show by induction concerning the number of operations. It is evident that the constants 0 and 1, and each variable $x_1, \ldots, x_n$ satisfy this theorem. Suppose that all Kleene-Stone logic functions, in which the number of operations is less than or equal to m, satisfy this theorem. Next, let us suppose that F is a Kleene-Stone logic function in which the number of operations is $m+1$. F is one of $(\sim G)$, $(\neg G)$, $(G \vee H)$ and $(G \cdot H)$. First, we show that $(\sim G)$ and $(\neg G)$ satisfy the theorem. It is evident that $G(\boldsymbol{a}) = G(\boldsymbol{b})$. When $G(\boldsymbol{a}) \neq G(\boldsymbol{b})$, we have the following two cases. (1) $0 < G(\boldsymbol{a}) < G(\boldsymbol{b}) \leq 1/2$ or (2) $1/2 \leq G(\boldsymbol{b}) < G(\boldsymbol{a}) < 1$. If (1) holds, then $1/2 \leq 1 - G(\boldsymbol{b}) = \sim G(\boldsymbol{b}) < \sim G(\boldsymbol{a}) = 1 - G(\boldsymbol{a}) < 1$ and $\neg G(\boldsymbol{a}) = \neg G(\boldsymbol{b}) = 0$. Therefore, we have $\sim G(\boldsymbol{a}) \leq_{\mathrm{KS}} \sim G(\boldsymbol{b})$ and $\neg G(\boldsymbol{a}) \leq_{\mathrm{KS}} \neg G(\boldsymbol{b})$. We can obtain $\sim G(\boldsymbol{a}) \leq_{\mathrm{KS}} \sim G(\boldsymbol{b})$ and $\neg G(\boldsymbol{a}) \leq_{\mathrm{KS}} \neg G(\boldsymbol{b})$ in the similar manner when (2) holds. Next, we prove that $(G \vee H)$ and $(G \cdot H)$ also satisfy this theorem. We can classify the following four cases.

(1) $0 \leq G(\boldsymbol{a}) \leq G(\boldsymbol{b}) \leq 1/2$ or (2) $1/2 \leq G(\boldsymbol{b}) \leq G(\boldsymbol{a}) \leq 1$, and
(3) $0 \leq H(\boldsymbol{a}) \leq H(\boldsymbol{b}) \leq 1/2$ or (4) $1/2 \leq H(\boldsymbol{b}) \leq H(\boldsymbol{a}) \leq 1$.

When (1) and (3) hold, then $0 \leq G(\boldsymbol{a}) \vee H(\boldsymbol{a}) \leq G(\boldsymbol{b}) \vee H(\boldsymbol{b}) \leq 1/2$, that is, we have $(G \vee H)(\boldsymbol{a}) \leq_{\mathrm{KS}} (G \vee H)(\boldsymbol{b})$. When (1) and (4) hold. then $(G \vee H)(\boldsymbol{a}) = G(\boldsymbol{a}) \vee H(\boldsymbol{a}) = H(\boldsymbol{a})$ and $(G \vee H)(\boldsymbol{b}) = G(\boldsymbol{b}) \vee H(\boldsymbol{b}) = H(\boldsymbol{b})$, that is, $(G \vee H)(\boldsymbol{a}) \leq_{\mathrm{KS}} (G \vee H)(\boldsymbol{b})$. In the remaining cases, we can also obtain $(G \vee H)(\boldsymbol{a}) \leq_{\mathrm{KS}} (G \vee H)(\boldsymbol{b})$ in the similar manner. Therefore, we have $(G \vee H)(\boldsymbol{a}) \leq_{\mathrm{KS}} (G \vee H)(\boldsymbol{b})$. $(G \cdot H)$ is evident since $(G \cdot H) = \sim(\sim G \cdot \sim H)$ stands always true. Thus, it has been shown that any Kleene-Stone logic function satisfies this theorem. ∎

Example 2 *Suppose 2-variable Kleene-Stone logic function* $F = \sim(x_1 \vee \neg x_1 \cdot \sim x_2)$, $\boldsymbol{a} = (0, 0.3)$, *and* $\boldsymbol{b} = (0, 0.4)$, *then* $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ *and* $F(\boldsymbol{a}) = 0.3 \leq_{\mathrm{KS}} 0.4 = F(\boldsymbol{b})$.

Let V_5 be the set $\{0, 1/4, 1/2, 3/4, 1\}$, then we give the following definition.

Definition 4 *Let a be an element of V. Then, $\overline{a}^{\varepsilon}$ is defined as follows where $0 < \varepsilon \leq 1/2$ (see Figure 6.2).*

$$\overline{a}^{\varepsilon} = \begin{cases} 0 & \text{if} \quad a = 0 \\ 1/4 & \text{if} \quad 0 < a < \varepsilon \\ 1/2 & \text{if} \quad \varepsilon \leq a \leq 1 - \varepsilon \\ 3/4 & \text{if} \quad 1 - \varepsilon < a < 1 \\ 1 & \text{if} \quad a = 1 \end{cases}$$

Moreover, it can be expanded among V_5^n as follows. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V^n, then $\overline{\boldsymbol{a}}^{\varepsilon} \in V_5$ is defined as $(\overline{a_1}^{\varepsilon}, \ldots, \overline{a_n}^{\varepsilon})$.

Example 3 *Suppose* $\boldsymbol{a} = (0, 0.1, 0.4)$ *and* $\varepsilon = 0.3$, *then* $\overline{\boldsymbol{a}}^{\varepsilon} = (0, 1/4, 1/2)$.

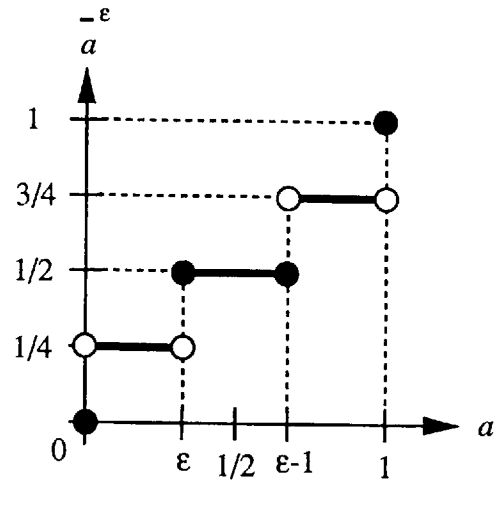

Figure 6.2: Mapping $\overline{a}^{\varepsilon}$ $(0 < \varepsilon \leq 1/2)$

Theorem 2 *Let F be a Kleene-Stone logic function and $\boldsymbol{a}$ be an element of V^n. Then $F(\overline{\boldsymbol{a}}^{\varepsilon}) = \overline{F(\boldsymbol{a})}^{\varepsilon}$, for any ε $(0 < \varepsilon \leq 1/2)$.*

Proof: It will be shown by induction concerning the number of operations. It is evident that the constants 0 and 1, and each variable $x_1, \ldots, x_n$ satisfy this theorem. Suppose that all Kleene-Stone logic functions, in which the number of operations is smaller than or equal to m, satisfy this theorem. Next let us suppose that F is a Kleene-Stone logic function in which the number of operations is $m+1$. F is one of $(\sim G)$, $(\neg G)$, $(G \cdot H)$ or $(G \vee H)$. $(\sim G)$ and $(\neg G)$ satisfy this theorem because $\overline{(\sim G)(\boldsymbol{a})}^{\varepsilon} = \overline{\sim G(\boldsymbol{a})}^{\varepsilon} = \overline{1 - G(\boldsymbol{a})}^{\varepsilon} = 1 - \overline{G(\boldsymbol{a})}^{\varepsilon} = 1 - G(\overline{\boldsymbol{a}}^{\varepsilon}) = (\sim G)(\overline{\boldsymbol{a}}^{\varepsilon})$. Therefore, $(\sim G)$ satisfies this theorem. If $G(\boldsymbol{a}) = 0$ then $\overline{\neg G(\boldsymbol{a})}^{\varepsilon} = 1$ because of $\neg G(\boldsymbol{a}) = 1$, and moreover we can obtain $\neg G(\overline{\boldsymbol{a}}^{\varepsilon}) = 1$ since $\overline{G(\boldsymbol{a})}^{\varepsilon} = G(\overline{\boldsymbol{a}}^{\varepsilon}) = 0$ by the assumption. If $G(\boldsymbol{a}) > 0$ then $\overline{G(\boldsymbol{a})}^{\varepsilon} = 0$ because of $\neg G(\boldsymbol{a}) = 0$, and moreover we can obtain $G(\overline{\boldsymbol{a}}^{\varepsilon}) = 0$ since $\overline{G(\boldsymbol{a})}^{\varepsilon} = G(\overline{\boldsymbol{a}}^{\varepsilon}) > 0$. Therefore, $(\neg G)$ satisfies this theorem. Next, we show that $(G \vee H)$ and $(G \cdot H)$ also satisfy this theorem. If $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ then $\overline{G(\boldsymbol{a})}^{\varepsilon} \leq \overline{H(\boldsymbol{a})}^{\varepsilon}$. This leads to $G(\overline{\boldsymbol{a}}^{\varepsilon}) \leq H(\overline{\boldsymbol{a}}^{\varepsilon})$. Therefore, it holds that $\overline{(G \vee H)(\boldsymbol{a})}^{\varepsilon} = \overline{G(\boldsymbol{a}) \vee H(\boldsymbol{a})}^{\varepsilon} = \overline{H(\boldsymbol{a})}^{\varepsilon} = H(\overline{\boldsymbol{a}}^{\varepsilon}) = G(\overline{\boldsymbol{a}}^{\varepsilon}) \vee H(\overline{\boldsymbol{a}}^{\varepsilon}) = (G \vee H)(\overline{\boldsymbol{a}}^{\varepsilon})$. The converse case $(G(\boldsymbol{a}) \geq H(\boldsymbol{a}))$ is proved in the similar manner. Therefore, $(G \vee H)$ satisfies this theorem. $(G \cdot H)$ is evident since $(G \cdot H) = \sim(\sim G \vee \sim H)$. Thus it has been shown that any Kleene-Stone logic function satisfies the equation $F(\overline{\boldsymbol{a}}^{\varepsilon}) = \overline{F(\boldsymbol{a})}^{\varepsilon}$. ∎

Example 4 *Suppose 2-variable Kleene-Stone logic function $F = \sim(\sim x_1 \vee \neg x_1 \cdot x_2)$, $\boldsymbol{a} = (0.1, 0.4)$, and $\varepsilon = 0.3$, then $\overline{F(\boldsymbol{a})}^{\varepsilon} = \overline{\sim(\sim 0.1 \vee \neg 0.1 \cdot 0.4)}^{\varepsilon} = \overline{0.1}^{\varepsilon} = 1/4$ and $F(\overline{\boldsymbol{a}}^{\varepsilon}) = \sim(\sim \overline{0.1}^{\varepsilon} \vee \neg \overline{0.1}^{\varepsilon} \cdot \sim \overline{0.4}^{\varepsilon}) = \sim(\sim 1/4 \vee \neg 1/4 \cdot \sim 1/2) = 1/4$.*

Theorem 3 *Let G and H be Kleene-Stone logic functions. $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_5^n if and only if $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V^n.*

Proof: Let us suppose that $G(\boldsymbol{a}) = H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_5^n and the theorem does not hold. Then we can assume without loss of generality that there is at least one element $\boldsymbol{a}$ of V^n such that $G(\boldsymbol{a}) \neq H(\boldsymbol{a})$. This means one of $G(\boldsymbol{a}) > H(\boldsymbol{a})$ or $G(\boldsymbol{a}) < H(\boldsymbol{a})$. If $G(\boldsymbol{a}) > H(\boldsymbol{a})$ then we can obtain $\overline{G(\boldsymbol{a})}^{\varepsilon} = G(\overline{\boldsymbol{a}}^{\varepsilon}) > H(\overline{\boldsymbol{a}}^{\varepsilon}) = \overline{H(\boldsymbol{a})}^{\varepsilon}$ for $\varepsilon = \min(\varepsilon', 1-\varepsilon')$ such that $\varepsilon' = (G(\boldsymbol{a}) + H(\boldsymbol{a}))/2$ by Theorem 2. This contradicts to the assumption since $\overline{\boldsymbol{a}}^{\varepsilon} \in V_5^n$. We can prove the result in the similar manner when $G(\boldsymbol{a}) < H(\boldsymbol{a})$. Therefore, we have been shown the first part of the theorem. The converse case is trivial. Thus, it has been shown that Theorem 2 holds. ■

From Theorem 2, it is guaranteed that all outputs of a Kleene-Stone logic function are determined uniquely by inputs of $V_5^n = \{0, 1/4, 1/2, 3/4, 1\}^n$.

Corollary 1 *Let G and H be Kleene-Stone logic functions.* $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ *for any element* $\boldsymbol{a}$ *of* V^n *if and only if* $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ *for any element* $\boldsymbol{a}$ *of* V_5^n.
(The proof is omitted)

Definition 5 *Let G and H be Kleene-Stone logic functions. It is said to be that H includes G (or G is included in H) if and only if* $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ *for any element* $\boldsymbol{a}$ *of* V^n, *and we denote it as* $G \sqsubseteq H$ *(or* $H \sqsupseteq G$*).*

In accordance with Corollary 1, $G \sqsubseteq H$ if and only if $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_5^n.

6.3 Canonical Disjunctive Forms of Kleene-Stone Logic Functions

6.3.1 Minterms of Type 1, Type 2 and Type 3

The logic formulas obtained by applying $\sim$ and $\neg$ to a variable x represent only the following six different kinds of Kleene-Stone logic functions, since $\sim\sim x = x$, $\neg\neg\neg x = \neg x$ and $\sim \neg x = \neg\neg x$ stand always true as discussed in Section 6.2

$$x,\ \sim x,\ \neg x,\ \neg\sim x,\ \neg\neg x \text{ and } \neg\neg\sim x.$$

For example, $\sim \neg\neg x = \neg x$ since $\sim \neg\neg x = \neg\neg\neg x = \neg x$. Hereafter, we call each these six logic formulas a *literal*, and the truth table of each literal appears in Table 6.1.

A product term on the variables $x_1, \ldots, x_n$ is defined as a conjunction (AND) of some of literals, where $x_i' \not\sqsubseteq x_i''$ for any two literals x_i' and x_i'' appearing in the product term. For example, $x \sim xy\neg\neg \sim y$ and $\sim x\neg\neg x\neg\neg z\neg\neg \sim z$ are product terms, but not $xxy\neg\neg \sim y$ and $x\neg\neg x\neg\neg z\neg\neg \sim z$ since there are two same literals x in the first product and $x \sqsubseteq \neg\neg x$ in the second product.

Next consider conjunctions of some of literals for a variable x. Then we can easily obtain four different kinds of Kleene-Stone logic functions, and each of them can be represented by $x \sim x$, $x\neg\neg \sim x$, $\sim x\neg\neg x$ and $\neg\neg x\neg\neg \sim x$, respectively. Because, $\neg x^* \sim x^* = 0$, $\neg x^*\neg \sim x^* = 0$ and $\neg x^*\neg\neg \sim x^* = 0$ and moreover $\neg x^* \sqsubseteq x^* \sqsubseteq \neg\neg x^*$ stand always true, where x^* denotes one of x or $\sim x$ (refer to Table 6.1). Hereafter, as a matter of convenience, each of the literals and the above four product terms is called an *atom*, and Table 6.1 shows truth tables of atoms.

Among the literals except for $\neg x$ and $\neg \sim x$ the following equations stand always true.

$$\begin{aligned} x &= \neg \sim x \vee x\neg\neg \sim x \\ \sim x &= \neg x \vee \sim x\neg\neg x \\ \neg\neg x &= \neg \sim x \vee \neg\neg x\neg\neg \sim x \\ \neg\neg \sim x &= \neg x \vee \neg\neg x\neg\neg \sim x \end{aligned}$$

Therefore, any product term can be expanded into a disjunction of product terms consisting of only the following six atoms.

Table 6.1: Truth Table of Atoms

x	0	1/4	1/2	3/4	1
$\sim x$	1	3/4	1/2	1/4	0
$\neg x$ (X^0)	1	0	0	0	0
$\neg \sim x$ (X^1)	0	0	0	0	1
$\neg\neg x$	0	1	1	1	1
$\neg\neg \sim x$	1	1	1	1	0
$x \sim x$ (X^{00})	0	1/4	1/2	1/4	0
$x\neg\neg \sim x$ (X^{01})	0	1/4	1/2	3/4	0
$\sim x\neg\neg x$ (X^{10})	0	3/4	1/2	1/4	0
$\neg\neg x\neg\neg \sim x$ (X^{11})	0	1	1	1	0

$$\neg x,\ \neg \sim x,\ x \sim x,\ x\neg\neg \sim x,\ \sim x\neg\neg x \text{ and } \neg\neg x\neg\neg \sim x.$$

Hereafter, we denote the above six atoms as the symbols X^0, X^1, X^{00}, X^{01}, X^{10} and X^{11}, respectively.

If a variable x_i does not appear in a product term α, then $\alpha = \alpha(X_i^0 \vee X_i^1 \vee X_i^{11}) = \alpha X_i^0 \vee \alpha X_i^1 \vee \alpha X_i^{11}$ stands always true, since $X_i^0 \vee X_i^1 \vee X_i^{11} = 1$ always holds. Therefore, a product term, which does not appear a variable x_i, is expanded into the disjunction (OR) of product terms which exist all variables. From the above discussions, we can always expand a product term α into a disjunction of product terms appearing all variables and represented by only at most one of the following six special atoms for each variable x_i.

$$X_i^0,\ X_i^1,\ X_i^{00},\ X_i^{01},\ X_i^{10} \text{ and } X_i^{11}.$$

Hereafter, we call a product term, which is finally obtained by expansion denoted above, a *minterm*.

Example 5 *3-variable product term $x_2\neg x_3$ is expanded into a disjunction of product terms appearing all variables x_1, x_2 and x_3 as follows. $x_2\neg x_3 = x_2X_3^0 = (X_1^0 \vee X_1^1 \vee X_1^{11})x_2X_3^0 = X_1^0x_2X_3^0 \vee X_1^1x_2X_3^0 \vee X_1^{11}x_2X_3^0$. For the above logic formula, the first product term $X_1^0x_2X_3^0$ is expanded into a disjunction of minterms as follows. $X_1^0x_2X_3^0 = X_1^0(X_2^1 \vee X_2^{01})X_3^0 = X_1^0X_2^1X_3^0 \vee X_1^0X_2^{01}X_3^0$.*

Now, a logic formula representing a Kleene-Stone logic function F can be expanded into a disjunctive form. $F = \alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_m$, where α_i $(i = 1, \ldots, m)$ is a minterm, by the distributive laws, the absorption laws, De Morgan's laws and so on as discussed in Section 6.2. Each minterm α_i can be classified into one of the following three types.

type 1: A minterm α consisting only of X^0, X^1 and X^{11} for any variable x.

type 2: A minterm α consisting only of X^0, X^1, X^{11}, X^{01} and X^{10} for any variable x and appearing at least one of Y^{01} or Y^{10} for some variable y.

type 3': A minterm α appearing X^{00} for some variable x.

Let α be a minterm of type 3'. If X^{11} is appeared in α for some variable x, then we always have the expansion $Y^{00}X^{11} = Y^{00}(X^{01} \vee X^{10}) = Y^{00}X01 \vee Y^{00}X^{10}$. Because let $\boldsymbol{a}$ be an element of V_5^n, then $(X^{01} \vee X^{10})(\boldsymbol{a}) \neq 0$ if and only if $X^{11}(\boldsymbol{a}) \neq 0$. Therefore, $(Y^{00}X^{01} \vee Y^{00}X^{10})(\boldsymbol{a}) \neq 0$

if and only if $(Y^{00}X^{11})(\boldsymbol{a}) \neq 0$, and in such the element $\boldsymbol{a}$ $Y^{00}(\boldsymbol{a})$ is always smaller than any one of $X^{11}(\boldsymbol{a})$, $X^{10}(\boldsymbol{a})$ and $X^{01}(\boldsymbol{a})$, that is, $(Y^{00}X^{01} \vee Y^{00}X^{10})(\boldsymbol{a}) = (Y^{00}X^{11})(\boldsymbol{a}) = Y^{00}(\boldsymbol{a})$. Accordingly, $Y^{00}X^{11} = Y^{00}(X^{01} \vee X^{10}) = Y^{00}X^{01} \vee Y^{00}X^{10}$ stands always true. From the above, a type 3' minterm α can be expanded into a disjunction of minterms in which X^{11} never appear for any variable x. Consequently, any Kleene-Stone logic function can always be expanded into a disjunction of the following three types of minterms.

type 1, type 2 and

type 3: A minterm α appearing X^{00} for some variable x, and never existing Y^{11} for any variable y.

6.3.2 Some Properties of Type 1 ~ Type 3 Minterms

Next, let us examine the relationship of each type of minterm. Hereafter, the sets $\{0,1\}$, $\{0,1/2,1\}$ and $\{0,1/4,3/4,1\}$ are denoted by V_2, V_3 and V_4, respectively.

Definition 6 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V_3^n. Then the element $\boldsymbol{a}$ corresponds to a minterm $\alpha = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ of type 1 if one of the following relations holds for every i.*

$$x_i^{a_i} = \begin{cases} X_i^0 & \text{if} \quad a_i = 0 \\ X_i^{11} & \text{if} \quad a_i = 1/2 \\ X_i^1 & \text{if} \quad a_i = 1 \end{cases}$$

Then obviously $\alpha(\boldsymbol{a}) = 1$ and there is one-to-one correspondence between minterms of type 1 and elements of V_3^n.

Definition 7 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of $V_5^n - V_3^n$. Then the element $\boldsymbol{a}$ corresponds to a minterm $\alpha = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ of type 2 if one of the following relations holds for every i.*

$$x_i^{a_i} = \begin{cases} X_i^0 & \text{if} \quad a_i = 0 \\ X_i^{10} & \text{if} \quad a_i = 1/4 \\ X_i^{11} & \text{if} \quad a_i = 1/2 \\ X_i^{01} & \text{if} \quad a_i = 3/4 \\ X_i^1 & \text{if} \quad a_i = 1 \end{cases}$$

Then obviously $\alpha(\boldsymbol{a}) = 3/4$ and there is one-to-one correspondence between minterms of type 2 and elements of $V_5^n - V_3^n$.

Definition 8 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of $V_5^n - V_4^n$. Then the element $\boldsymbol{a}$ corresponds to a minterm $\alpha = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ of type 3 if one of the following relations holds for every i.*

$$x_i^{a_i} = \begin{cases} X_i^0 & \text{if} \quad a_i = 0 \\ X_i^{10} & \text{if} \quad a_i = 1/4 \\ X_i^{00} & \text{if} \quad a_i = 1/2 \\ X_i^{01} & \text{if} \quad a_i = 3/4 \\ X_i^1 & \text{if} \quad a_i = 1 \end{cases}$$

Then obviously $\alpha(\boldsymbol{a}) = 1/2$ and there is one-to-one correspondence between minterms of type 3 and elements of $V_5^n - V_4^n$.

Example 6 $(0, 1/2, 1)$ *corresponds to a minterm of type 1,* $X_1^0 X_2^{11} X_3^1$*, and* $(0, 1/4, 1/2)$ *corresponds to that of type 2,* $X_1^0 X_2^{10} X_3^{11}$*, and it corresponds to that of type 3,* $X_1^0 X_2^{10} X_3^{00}$*.*

In the partial order relation $\leq_{\mathrm{KS}}$ discussed in Section 6.2, if the elements a and b of V_5 are comparable to each other, then we can find the greatest lower bound of a and b concerning with $\leq_{\mathrm{KS}}$, otherwise we can not. We will write a greatest lower bound of a and b as $a \Delta_{\mathrm{KS}} b$, and if the greatest lower bound of a and b does not exist, then we will write it as $a \Delta_{\mathrm{KS}} b = \emptyset$. This can be extended among V_5^n as follows. For two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of V_5^n, we will define $\boldsymbol{a} \Delta_{\mathrm{KS}} \boldsymbol{b}$ as $(a_1 \Delta_{\mathrm{KS}} b_1, \ldots, a_n \Delta_{\mathrm{KS}} b_n)$ and if $a_i \Delta_{\mathrm{KS}} b_i = \emptyset$ for some i $(i = 1, \ldots, n)$, then we will define it as $\boldsymbol{a} \Delta_{\mathrm{KS}} \boldsymbol{b} = \emptyset$.

Example 7 *Suppose* $\boldsymbol{a}_1 = (0, 1/4, 1/2)$*,* $\boldsymbol{a}_2 = (0, 1/4, 3/4)$ *and* $\boldsymbol{a}_3 = (0, 1/2, 3/4)$*, then* $\boldsymbol{a}_2 \leq_{\mathrm{KS}} \boldsymbol{a}_1$ *and* $\boldsymbol{a}_3 \leq_{\mathrm{KS}} \boldsymbol{a}_2$*, whereas* $\boldsymbol{a}_1$ *and* $\boldsymbol{a}_3$ *are not comparable, while* $\boldsymbol{a}_1 \Delta_{\mathrm{KS}} \boldsymbol{a}_3 = (0, 1/4, 3/4)$*.*

Lemma 1 *Let* $\boldsymbol{a}$ *be any element of* V_3^n *and* α *be the corresponding minterm of type 1. If* $\boldsymbol{b}$ *is an element of* V_5^n *then*

(1) $\boldsymbol{b} \leq_{\mathrm{KS}} \boldsymbol{a}$ *if and only if* $\alpha(\boldsymbol{b}) = 1$*,*

(2) $\boldsymbol{b} \not\leq_{\mathrm{KS}} \boldsymbol{a}$ *if and only if* $\alpha(\boldsymbol{b}) = 0$*.*

Proof: (1) is evident from Definition 5. (2) is derived directly from (1), because $\alpha(\boldsymbol{b}) = 1$ or 0 for all elements $\boldsymbol{b}$ of V_5^n from Definition 5. ■

Lemma 2 *Let* $\boldsymbol{a}$ *be any element of* $V_5^n - V_3^n$ *and* α *be the corresponding minterm of type 2. If* $\boldsymbol{b}$ *is an element of* V_5^n *then*

(1) $\boldsymbol{b} \leq_{\mathrm{KS}} \boldsymbol{a}$ *if and only if* $\alpha(\boldsymbol{b}) = 3/4$*,*

(2) $\boldsymbol{b} \not\leq_{\mathrm{KS}} \boldsymbol{a}$ *and* $\boldsymbol{a} \Delta_{\mathrm{KS}} \boldsymbol{b} \neq \emptyset$ *if and only if* $\alpha(\boldsymbol{b}) = 1/2$*,*

(3) $\boldsymbol{a} \Delta_{\mathrm{KS}} \boldsymbol{b} = \emptyset$ *if and only if* $\alpha(\boldsymbol{b}) = 0$*.*

(The proof is omitted)

Lemma 3 *Let* $\boldsymbol{a}$ *be any element of* $V_5^n - V_4^n$ *and* α *be the corresponding minterm of type 3. If* $\boldsymbol{b}$ *is an element of* V_5^n *then*

(1) $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ *if and only if* $\alpha(\boldsymbol{b}) = 1/4$*,*

(2) $\boldsymbol{a} \not\leq_{\mathrm{KS}} \boldsymbol{b}$ *if and only if* $\alpha(\boldsymbol{b}) = 0$*.*

(The proof is omitted)

Let α, β and γ be minterms of type 1, type 2 and type 3, respectively. Then $\alpha \not\sqsubseteq \beta$ $\alpha \not\sqsubseteq \gamma$ and $\beta \not\sqsubseteq \alpha$, that is, $\alpha \vee \beta \neq \beta$, $\alpha \vee \gamma \neq \gamma$ and $\beta \vee \gamma \neq \gamma$, from each definition of minterms.

Lemma 4 *Let* α *and* β *be minterms of type 1* $\sim$ *type 3 and,* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be the corresponding elements of* V_5^n*. Then*

(1) $\beta \sqsubseteq \alpha$ if and only if $\boldsymbol{a} = \boldsymbol{b}$, when α and β are minterms of type 1,

(2) $\beta \sqsubseteq \alpha$ if and only if $\boldsymbol{b} \leq_{\mathrm{KS}} \boldsymbol{a}$, when α is a minterm of type 1 and β is a minterm of type 2,

(3) $\beta \sqsubseteq \alpha$ if and only if $\boldsymbol{b} \leq_{\mathrm{KS}} \boldsymbol{a}$, when α is a minterm of type 1 and β is a minterm of type 3,

(4) $\beta \sqsubseteq \alpha$ if and only if $\boldsymbol{b} \leq_{\mathrm{KS}} \boldsymbol{a}$, when α and β are minterms of type 2,

(5) $\beta \sqsubseteq \alpha$ if and only if $\boldsymbol{a}\Delta_{\mathrm{KS}}\boldsymbol{b} \neq \emptyset$, when α is a minterm of type 2 and β is a minterm of type 3,

(6) $\beta \sqsubseteq \alpha$ if and only if $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$, when α and β are minterms of type 3.

Proof: (1) It is evident $\alpha \vee \beta = \alpha$ when $\boldsymbol{a} = \boldsymbol{b}$. Let us suppose $\alpha \vee \beta = \alpha$. This implies that $\alpha(\boldsymbol{a}') \leq \beta(\boldsymbol{a}')$ for all elements $\boldsymbol{a}'$ of V_5^n, that is, $\alpha(\boldsymbol{b}) \leq \beta(\boldsymbol{b})$. Therefore, $\boldsymbol{b} \leq_{\mathrm{KS}} \boldsymbol{a}$ holds from $\beta(\boldsymbol{b}) = 1$ and Lemma 1.1. Thus, we have $\boldsymbol{a} = \boldsymbol{b}$, since $\boldsymbol{a}$ and $\boldsymbol{b}$ are elements of V_3^n. The proofs of (2) $\sim$ (6) are similar to that of (1). ■

Definition 9 *If a Kleene-Stone logic function F is represented by a logic formula $F = \alpha_1 \vee \ldots \vee \alpha_m$, then it is said that F is in the canonical disjunctive form where α_i $(i = 1, \ldots, m)$ is each one of type 1, type 2 or type 3 minterm and $\alpha_i \not\sqsubseteq \alpha_j$ for any i and j $(i \neq j)$.*

Example 8 *The canonical disjunctive form of 3-variable Kleene-Stone logic function $F = (X_1^1 \vee x_1 X_2^0)X_3^{00}$ is obtained as follows.*

$$\begin{aligned}
F &= (X_1^1 \vee x_1 X_2^0)X_3^{00} \\
&= X_1^1 X_3^{00} \vee x_1 X_2^0 X_3^{00} \\
&= X_1^1 (X_2^0 \vee X_2^1 \vee X_2^{11}) X_3^{00} \vee x_1 X_2^0 X_3^{00} \\
&= X_1^1 X_2^0 X_3^{00} \vee X_1^1 X_2^1 X_3^{00} \vee X_1^1 X_2^{11} X_3^{00} \vee x_1 X_2^0 X_3^{00} \\
&= X_1^1 X_2^0 X_3^{00} \vee X_1^1 X_2^1 X_3^{00} \vee X_1^1 X_2^{11} X_3^{00} \vee (X_1^1 \vee X_1^{01}) X_2^0 X_3^{00} \\
&= X_1^1 X_2^0 X_3^{00} \vee X_1^1 X_2^1 X_3^{00} \vee X_1^1 X_2^{11} X_3^{00} \vee X_1^1 X_2^0 X_3^{00} \vee X_1^{01} X_2^0 X_3^{00} \\
&= X_1^1 X_2^0 X_3^{00} \vee X_1^1 X_2^1 X_3^{00} \vee X_1^1 (X_2^{01} \vee X_2^{10}) X_3^{00} \vee X_1^1 X_2^0 X_3^{00} \vee X_1^{01} X_2^0 X_3^{00} \\
&= X_1^1 X_2^0 X_3^{00} \vee X_1^1 X_2^1 X_3^{00} \vee X_1^1 X_2^{01} X_3^{00} \vee X_1^1 X_2^{10} X_3^{00} \vee X_1^1 X_2^0 X_3^{00} \vee X_1^{01} X_2^0 X_3^{00} \\
&= X_1^1 X_2^0 X_3^{00} \vee X_1^1 X_2^1 X_3^{00} \vee X_1^1 X_2^{01} X_3^{00} \vee X_1^1 X_2^{10} X_3^{00} \vee X_1^{01} X_2^0 X_3^{00}
\end{aligned}$$

Here, the elements $(1, 0, 1/2)$, $(1, 1, 1/2)$, $(1, 3/4, 1/2)$, $(1, 1/4, 1/2)$ and $(3/4, 0, 1/2)$ are corresponding to the minterms $X_1^1 X_2^0 X_3^{00}$, $X_1^1 X_2^1 X_3^{00}$, $X_1^1 X_2^{01} X_3^{00}$ $X_1^1 X_2^{10} X_3^{00}$ and $X_1^{01} X_2^0 X_3^{00}$, respectively. Therefore, every minterm is not omitted by another minterms from Lemma 4. Thus, the canonical disjunctive form of F is

$$X_1^1 X_2^0 X_3^{00} \vee X_1^1 X_2^1 X_3^{00} \vee X_1^1 X_2^{01} X_3^{00} \vee X_1^1 X_2^{10} X_3^{00} \vee X_1^{01} X_2^0 X_3^{00}$$

Lemma 5 *Let F be a Kleene-Stone logic function and $\alpha_1 \vee \ldots \vee \alpha_m$ the canonical disjunctive form of F. Then $F(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$ for the corresponding element $\boldsymbol{a}$ to α_i $(i = 1, \ldots, m)$.*

Proof: $F(\boldsymbol{a}) = 1$ since $\alpha_i(\boldsymbol{a}) = 1$ when α_i is a type 1 minterm. Next, suppose α_i is a type 2 minterm, then $F(\boldsymbol{a}) = 1$ never hold. Because if $F(\boldsymbol{a}) = 1$, then there is a type 1 minterm α_j $(j \neq i)$ such that $\alpha_j(\boldsymbol{a}) = 1$. Accordingly by Lemma 1.1 $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{a}'$ where $\boldsymbol{a}'$ is the corresponding element to α_j, and therefore by Lemma 4.2 $\alpha_i \sqsubseteq \alpha_j$. This contradicts to $\alpha_1 \vee \ldots \vee \alpha_m$ is the canonical disjunctive form. Therefore, we also have $F(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$ when α_i is a type 2 minterm. We have $F(\boldsymbol{a}) = \alpha_i(\boldsymbol{a})$ in the similar manner when α_i is a type 3 minterm. ■

Theorem 4 *Any Kleene-Stone logic function can be represented uniquely (ignoring the order of the minterms) by the canonical disjunctive form.*

Proof: Let us suppose $F_1 = \alpha_1 \vee \ldots \vee \alpha_s$ and $F_2 = \alpha'_1 \vee \ldots \vee \alpha'_t$ are two different canonical disjunctive forms of a Kleene-Stone logic function F (It is evident from the above discussion that there is at least one canonical disjunctive form of F). Now, we can suppose that a minterm α exists in F_1 but in F_2 without loss of generality. In the following proof, $\boldsymbol{a}$ is assumed to be an element corresponding to α.

First, assume α is a type 1 minterm. Then by Lemma 5 $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a}) = 1$. Therefore, there is a type 1 minterm α' in F_2 such that $\alpha'(\boldsymbol{a}) = 1$, and accordingly by Lemma 1.1 $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{a}'$ where $\boldsymbol{a}'$ is the corresponding element to α'. Since both α and α' are type 1 minterms, $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{a}'$ implies $\boldsymbol{a} = \boldsymbol{a}'$, that is, $\alpha = \alpha'$. Thus, any type 1 minterm appearing in F_1 also appear in F_2.

Second assume α is a type 2 minterm. Then by Lemma 5 $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a}) = 3/4$. Therefore, there is a type 2 minterm α' in F_2 such that $\alpha'(\boldsymbol{a}) = 3/4$, and accordingly $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{a}'$ by Lemma 2.1 where $\boldsymbol{a}'$ is the corresponding element to α'. In the similar manner, there is a type 2 minterm α'' in F_2 such that $\alpha''(\boldsymbol{a}) = 3/4$, and by Lemma 2.1 $\boldsymbol{a}' \leq_{\mathrm{KS}} \boldsymbol{a}''$ for the corresponding element $\boldsymbol{a}''$ to α''. By the assumption $\alpha \neq \alpha'$, we have $\alpha \neq \alpha''$, and therefore by Lemma 4.4 $\alpha \sqsubseteq \alpha''$. This contradicts that F_1 is the canonical disjunctive form.

Finally, assume α is a type 3 minterm. Then by Lemma 5 $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a}) = 1/2$. This implies that there is a type 2 or type 3 minterm α' in F_2 such that $\alpha'(\boldsymbol{a}) = 1/2$. If α' is a type 2, then by Lemma 2.3 $\boldsymbol{a}\Delta_{\mathrm{KS}}\boldsymbol{a}' \neq \emptyset$ for the corresponding element $\boldsymbol{a}'$ to α'. Since $F_1(\boldsymbol{a}') = F_2(\boldsymbol{a}') = 3/4$ by Lemma 5, there is a type 2 minterm α'' in F_2 such that $\alpha''(\boldsymbol{a}') = 3/4$, and by Lemma 2.1 $\boldsymbol{a}' \leq_{\mathrm{KS}} \boldsymbol{a}''$ for the corresponding element $\boldsymbol{a}''$ to α''. $\boldsymbol{a}\Delta_{\mathrm{KS}}\boldsymbol{a}' \neq \emptyset$ and $\boldsymbol{a}' \leq_{\mathrm{KS}} \boldsymbol{a}''$ imply $\boldsymbol{a}\Delta_{\mathrm{KS}}\boldsymbol{a}'' \neq \emptyset$, and accordingly $\alpha \sqsubseteq \alpha''$ by Lemma 4.5. This contradicts that F_1 is the canonical disjunctive form. If α' is a type 3, then by Lemma 3.1 $\boldsymbol{a}' \leq_{\mathrm{KS}} \boldsymbol{a}$ for the corresponding element $\boldsymbol{a}'$ to α'. Since $F_1(\boldsymbol{a}') = F_2(\boldsymbol{a}') = 1/2$ by Lemma 5, there is a type 2 or type 3 minterm α'' in F_1 such that $\alpha''(\boldsymbol{a}') = 1/2$. α'' is never type 2 since by Lemma 2.3 $\boldsymbol{a}'\Delta_{\mathrm{KS}}\boldsymbol{a}'' \neq \emptyset$ for the corresponding element $\boldsymbol{a}''$ to α'', and therefore we have $\boldsymbol{a}\Delta_{\mathrm{KS}}\boldsymbol{a}'' \neq \emptyset$. Accordingly by Lemma 4.5 $\alpha \sqsubseteq \alpha''$, and this contradicts that F_1 is the canonical disjunctive form. If α'' is a type 3, then by Lemma 3.1 $\boldsymbol{a}'' \leq_{\mathrm{KS}} \boldsymbol{a}'$ for the corresponding element $\boldsymbol{a}''$ to α'', and accordingly $\boldsymbol{a}'' \leq_{\mathrm{KS}} \boldsymbol{a}$. Therefore by Lemma 4.6 $\alpha \sqsubseteq \alpha''$, and this contradicts that F_1 is the canonical disjunctive form.

Therefore, any minterm appearing in F_1 also appear in F_2. In the similar manner, any minterm appearing in F_2 also appear in F_1. This completes the proof of the theorem. ■

6.4 A Characterization of Kleene-Stone Logic Functions

By Theorem 3, any Kleene-Stone logic function is determined uniquely only for every input of V_5^n, and it is clear that V_5 is the range of Kleene-Stone logic functions whose domain is restricted to the set V_5^n. Hereafter, we call a Kleene-Stone logic function whose domain is restricted to V_5^n a 5-valued Kleene-Stone logic function, that is, a 5-valued Kleene-Stone logic function is a function $F : V_5^n \to V_5$ (called a 5-valued function below) represented by a logic formula.

Obviously, it is not true that every 5-valued function can obtain by means of a logic formula, that is, 5-valued Kleene-Stone logic functions are not functionally complete. Of course, Kleene-Stone logic functions are also not functionally complete by Theorem 1. Therefore, first, we discuss a necessary and sufficient condition for a 5-valued function to be a 5-valued Kleene-Stone logic function. Then, we will show a necessary and sufficient condition for a function $F : V^n \to V$, called an infinite-valued function below, to be a Kleene-Stone logic functions, and it can be proved by using the result of 5-valued Kleene-Stone logic functions.

6.4.1 Necessary and Sufficient Condition for 5-Valued Kleene-Stone Logic Functions

The following set of four conditions is a necessary and sufficient condition for a 5-valued function to be a 5-valued Kleene-Stone logic function.

(*a*) $\boldsymbol{a} \in \{0,1\}^n$ implies $F(\boldsymbol{a}) \in \{0,1\}$,

(*b*) $\boldsymbol{a} \in \{0,1/2,1\}^n$ implies $F(\boldsymbol{a}) \in \{0,1/2,1\}$,

(*c*) $\boldsymbol{a} \in \{0,1/4,3/4,1\}^n$ implies $F(\boldsymbol{a}) \in \{0,1/4,3/4,1\}$,

(*d*) $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ implies $F(\boldsymbol{a}) \leq_{\mathrm{KS}} F(\boldsymbol{b})$,

where $\boldsymbol{a}, \boldsymbol{b} \in V_5^n$.

First, we will discuss 5-valued functions satisfying Conditions $(a\ ) \sim (d\ )$. Let F be a 5-valued function satisfying Conditions $(a) \sim (d)$. Then, we consider specific five subsets $F^{-1}(0)$, $F^{-1}(1/4)$, $F^{-1}(1/2)$, $F^{-1}(3/4)$ and $F^{-1}(1)$ of V_5^n as follows.

$$F^{-1}(i) = \{\boldsymbol{a} \in V_5^n \mid F(\boldsymbol{a}) = i\},$$

where $i \in V_5$.

It is clear that $F^{-1}(i) \cap F^{-1}(j)$ $(i \neq j$ and $i, j \in V_5)$ and $\bigcup_{i \in V_5} F^{-1}(i) = V_5^n$. Suppose $\boldsymbol{a}$ be an element of $F^{-1}(3/4)$. Then, all elements $\boldsymbol{b}$ such as $\boldsymbol{b} \leq_{\mathrm{KS}} \boldsymbol{a}$ are also elements of $F^{-1}(3/4)$ from Condition (d). If $\boldsymbol{a}$ is an element of $F^{-1}(1/2)$, then all elements $\boldsymbol{b}$ such as $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ are also elements of $F^{-1}(1/2)$. Moreover, if $\boldsymbol{a}$ is an element of $F^{-1}(1)$, then all elements $\boldsymbol{b}$ such as $\boldsymbol{b} \leq_{\mathrm{KS}} \boldsymbol{a}$ or $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ are also elements of $F^{-1}(1)$. Therefore, the subsets $F^{-1}(1/2)$, $F^{-1}(3/4)$ and $F^{-1}(1)$ each form a partial order finite sets concerning with the relation $\leq_{\mathrm{KS}}$. Therefore, for any given 5-valued function F satisfying Conditions $(a) \sim (d)$ the set of maximal elements of $F^{-1}(3/4)$ and $F^{-1}(1)$, denoted by $\partial F^{-1}(3/4)$ and $\partial F^{-1}(1)$, and the set of minimal elements of $F^{-1}(1/2)$, denoted by $\partial F^{-1}(1/2)$, are uniquely determined, respectively.

Lemma 6 *Let F be any 5-valued function satisfying the conditions $(a) \sim (d)$, then*

(1) $\partial F^{-1}(1) \subseteq V_3^n$,

(2) Let $\boldsymbol{a}$ be an element of $F^{-1}(1/4)$, then there exists an element $\boldsymbol{b}$ of $\partial F^{-1}(1/2)$ such that $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$.

(Proof is omitted)

Theorem 5 *If F is a 5-valued Kleene-Stone logic function, then, F also satisfies Condition $(a) \sim (d)$.*

Proof: It is evident form the definition of each operation and Theorem 1 and 3. ■

Theorem 6 *If F is a 5-valued function satisfying Conditions $(a) \sim (d)$, then there is a 5-valued Kleene-Stone logic function F_{KS} such that $F(\boldsymbol{a}) = F_{KS}(\boldsymbol{a})$ for any element $\boldsymbol{a} \in V_5^n$.*

Proof: For any given 5-valued function F satisfying Conditions $(a) \sim (d)$, we will construct a 5-valued Kleene-Stone logic function F_{KS} such that $F(\boldsymbol{a}) = F_{KS}(\boldsymbol{a})$ for any element $\boldsymbol{a} \in V_5^n$. Let F_i (i = 1, 2, 3) be the logic formula constructed by a disjunction of all minterms of type i (i = 1, 2, 3) corresponding to the elements of $\partial F^{-1}(1)$, $\partial F^{-1}(3/4)$ and $\partial F^{-1}(1/2)$, respectively We can always construct each F_i (i = 1, 2, 3) since $\partial F^{-1}(1) \subseteq V_3^n$ by Lemma 6.1 and by Condition (b) $\partial F^{-1}(3/4) \subseteq V_5^n - V_3^n$ and also by Condition (c) $\partial F^{-1}(1/2) \subseteq V_5^n - V_4^n$. Then, we can show that $F_f(\boldsymbol{a}) = F(\boldsymbol{a})$ for any element $\boldsymbol{a} \in V_5^n$ as follows, where $F_f = F_1 \vee F_2 \vee F_3$.

(1) Suppose $F(\boldsymbol{a}) = 1$, then there exists an element $\boldsymbol{b}$ in $\partial F^{-1}(1)$ such as $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$. Therefore, for the type 1 minterm α corresponding to $\boldsymbol{b}$ in F_1, we obtain $\alpha(\boldsymbol{a}) = 1$ from Lemma 1.1. Therefore, we have $F_f(\boldsymbol{a}) = 1$. Conversely, suppose $F_f(\boldsymbol{a}) = 1$, then $F_1(\boldsymbol{a}) = 1$. Therefore, there exists a minterm α corresponding to $\boldsymbol{b}$ in F_1 such as $\alpha(\boldsymbol{a}) = 1$, and $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ from Lemma 1.1. Thus, $F(\boldsymbol{a}) = 1$ holds since $\boldsymbol{b} \in \partial F^{-1}(1)$.

(2) Suppose $F(\boldsymbol{a}) = 3/4$, then there exists an element $\boldsymbol{b}$ in $\partial F^{-1}(3/4)$ such as $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$. Therefore, for the minterm α corresponding to $\boldsymbol{b}$ in F_2, we obtain $\alpha(\boldsymbol{a}) = 3/4$ from Lemma 2.1. Therefore, we have $F_f(\boldsymbol{a}) = 3/4$. Conversely, suppose $F_f(\boldsymbol{a}) = 3/4$, then $F_2(\boldsymbol{a}) = 3/4$. Therefore, there exists a minterm α corresponding to $\boldsymbol{b}$ in F_2 such as $\alpha(\boldsymbol{a}) = 3/4$, and $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ from Lemma 2.1. Thus, $F(\boldsymbol{a}) = 3/4$ holds since $\boldsymbol{b} \in \partial F^{-1}(3/4)$.

(3) Suppose $F(\boldsymbol{a}) = 1/2$, then there exists an element $\boldsymbol{b}$ in $\partial F^{-1}(1/2)$ such as $\boldsymbol{b} \leq_{\mathrm{KS}} \boldsymbol{a}$. Therefore, for the minterm α corresponding to $\boldsymbol{b}$ in F_3, we obtain $\alpha(\boldsymbol{a}) = 1/2$ from Lemma 3.1. Therefore, $F_3(\boldsymbol{a}) = 1/2$ holds, that is, we have $F_f(\boldsymbol{a}) = 1/2$. Conversely, suppose $F_f(\boldsymbol{a}) = 1/2$, then $F_2(\boldsymbol{a}) = 1/2$ or $F_3(\boldsymbol{a}) = 1/2$. First assume $F_2(\boldsymbol{a}) = 1/2$ then there exists a minterm α corresponding to $\boldsymbol{b}$ in F_2 such as $\alpha(\boldsymbol{a}) = 1/2$, and therefore $\boldsymbol{a}\Delta_{\mathrm{KS}}\boldsymbol{b} \neq \emptyset$ form Lemma 2.3. Let $\boldsymbol{c} = \boldsymbol{a}\Delta_{\mathrm{KS}}\boldsymbol{b}$, that is, $\boldsymbol{c} \leq_{\mathrm{KS}} \boldsymbol{a}$ and $\boldsymbol{c} \leq_{\mathrm{KS}} \boldsymbol{b}$. Then, we have $F(\boldsymbol{a}) = 3/4$ or $1/2$ since $F(\boldsymbol{c}) = 3/4$ holds from $\boldsymbol{b} \in \partial F^{-1}(3/4)$ $F(\boldsymbol{a}) = 3/4$, however, does not hold from the discussion (2). Therefore, we have $F(\boldsymbol{a}) = 1/2$ when $F_2(\boldsymbol{a}) = 1/2$. Next assume $F_3(\boldsymbol{a}) = 1/2$, then there exists a minterm α corresponding to $\boldsymbol{b}$ in F_3 such as $\alpha(\boldsymbol{a}) = 1/2$, and $\boldsymbol{b} \leq_{\mathrm{KS}} \boldsymbol{a}$ holds from Lemma 3.1. Therefore, we have $F(\boldsymbol{a}) = 1/2$ since $\boldsymbol{b} \in \partial F^{-1}(1/2)$, when $F_3(\boldsymbol{a}) = 1/2$.

(4) Suppose $F(\boldsymbol{a}) = 1/4$, then there exists an element $\boldsymbol{b}$ of $\partial F^{-1}(1/2)$ such as $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ from Lemma 6.2. Therefore, we have $F_f(\boldsymbol{a}) = 1/4$ since $\alpha(\boldsymbol{a}) = 1/4$ from Lemma 3.1 where α is the minterm corresponding to $\boldsymbol{b}$. Conversely, suppose $F_f(\boldsymbol{a}) = 1/4$, then $F_2(\boldsymbol{a}) = 1/4$ or $F_3(\boldsymbol{a}) = 1/4$. First assume $F_2(\boldsymbol{a}) = 1/4$, then there exists a minterm α corresponding to $\boldsymbol{b}$ in F_2 such as $\alpha(\boldsymbol{a})$ $= 1/4$, and therefore $\boldsymbol{a}\Delta_{\mathrm{KS}}\boldsymbol{b} \neq \emptyset$ from Lemma 2.3. Therefore, by $\boldsymbol{a}\Delta_{\mathrm{KS}}\boldsymbol{b} \neq \emptyset$, $\boldsymbol{b} \in \partial F^{-1}(1/2)$ and Condition (d) $F(\boldsymbol{a}) = 0$ or 1 never occurs, that is, $F(\boldsymbol{a}) = 1/4$, $1/2$ or $3/4$, however, $F(\boldsymbol{a})$ $= 1/2$ or $3/4$ dose not hold from the discussions (2) and (3). Thus, we have $F(\boldsymbol{a}) = 1/4$ when $F_2(\boldsymbol{a}) = 1/4$. Next assume $F_3(\boldsymbol{a}) = 1/2$, then there exists a minterm α corresponding to $\boldsymbol{b}$ in F_3 such as $\alpha(\boldsymbol{a}) = 1/4$, and therefore $\boldsymbol{a}\Delta_{\mathrm{KS}}\boldsymbol{b} \neq \emptyset$ from Lemma 3.3. In the similar manner we obtain $F(\boldsymbol{a}) = 1/4$, $1/2$ or $3/4$, and however, $F(\boldsymbol{a}) =1/2$ or $3/4$ does not hold from the discussions (2) and (3). Therefore, $F(\boldsymbol{a}) = 1/4$

(5) $F(\boldsymbol{a}) = 0$ if and only if $F_f(\boldsymbol{a}) = 0$ is derived directly from (1), (2), (3) and (4). ■

Among the conditions $(a) \sim (d)$, we can derive (a) from (b) and (c), because if a function F satisfies (b) and (c), then for any $\boldsymbol{a} \in \{0,1\}^n = \{0,1/2,1\}^n \cap \{0,1/4,3/4,1\}^n$ $F(\boldsymbol{a}) \in \{0,1/2,1\} \cap \{0,1/4,3/4,1\} = \{0,1\}$, and we have (a). The remaining conditions (b), (c) and (d) are independent to each other. Because, the function F_x (x is one of b, c or d) in Table 6.2

satisfies the remaining two conditions, but not the condition (x), and therefore, F_x is an example that the condition (x) can not derived from the remaining two conditions. Thus, the conditions (b), (c) and (d) are independent to each other.

Table 6.2: Truth Tables of F_b, F_c and F_d

x	0	1/4	1/2	3/4	1
F_b	1/4	0	0	0	0
F_c	1/2	1/2	1/2	1/2	1/2
F_d	0	1/4	0	1/4	0

6.4.2 Necessary and Sufficient Condition for Kleene-Stone Logic Functions

Here, we prove that the following five conditions is a necessary and sufficient condition for an infinite-valued function to be a Kleene-Stone logic function. Obviously, among the following five conditions, (A) is necessary by (B), (C) and the discussions of the end of the previous section.

(A) $\boldsymbol{a} \in \{0,1\}^n$ implies $F(\boldsymbol{a}) \in \{0,1\}$

(B) $\boldsymbol{a} \in \{0,1/2,1\}^n$ implies $F(\boldsymbol{a}) \in \{0,1/2,1\}$

(C) $\boldsymbol{a} \in \{0,1/4,3/4,1\}^n$ implies $F(\boldsymbol{a}) \in \{0,1/4,3/4,1\}$

(D) $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ implies $F(\boldsymbol{a}) \leq_{\mathrm{KS}} F(\boldsymbol{b})$

(E) $\overline{F(\boldsymbol{a})}^{\varepsilon} = F(\overline{\boldsymbol{a}}^{\varepsilon})$ for any ε such that $0 < \varepsilon \leq 1/2$

where $\boldsymbol{a}, \boldsymbol{b} \in V^n$.

Lemma 7 *Let F be an infinite-valued function. If F satisfies Condition (E), then F also satisfies that $F(\boldsymbol{a})$ is an element of V_5 whenever $\boldsymbol{a} \in V_5^n$.*

Proof: Suppose that $\boldsymbol{a}$ is an element of V_5^n such that $F(\boldsymbol{a}) \notin V_5$. Then, $\boldsymbol{a} = \overline{\boldsymbol{a}}^{\varepsilon}$ holds for $\varepsilon = 1/2$. Therefore, we obtain $F(\boldsymbol{a}) = F(\overline{\boldsymbol{a}}^{\varepsilon}) = \overline{F(\boldsymbol{a})}^{\varepsilon} \in V_5$ for $\varepsilon = 1/2$ from Condition (E). This contradicts to $F(\boldsymbol{a}) \notin V_5$. ■

Theorem 7 *If F is a Kleene-Stone logic function, then F is an infinite-valued function satisfying Conditions (A) $\sim$ (E).*

Proof: It is evident from the definition of each operations $(\cdot, \vee, \sim, \neg)$ and Theorem 1, Theorem 2 and the definition of logic formulas. ■

Theorem 8 *If F is an infinite-valued function satisfying Conditions (A) $\sim$ (E), then F is a Kleene-Stone logic function.*

Proof: It stands always true from Lemma 7 and Condition (E) that $\boldsymbol{a} \in V_5^n$ implies $F(\boldsymbol{a}) \in V_5$. Therefore, there is a Kleene-Stone logic function F_{KS} such that $F(\boldsymbol{a}) = F_{KS}(\boldsymbol{a})$ for any element $\boldsymbol{a} \in V_5^n$ from Theorem 6. Then, we can prove that $F(\boldsymbol{a}) = F_{KS}(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V^n as follows. Suppose that $F(\boldsymbol{b}) \neq F_{KS}(\boldsymbol{b})$ for an element $\boldsymbol{b}$ of $V^n - V_5^n$. This implies each one of

$$(1)\ F(\boldsymbol{b}) > F_{KS}(\boldsymbol{b}) \text{ or } \quad (2)\ F(\boldsymbol{b}) < F_{KS}(\boldsymbol{b}).$$

If (1) holds then we obtain $\overline{F(\boldsymbol{b})}^{\varepsilon} = F(\overline{\boldsymbol{b}}^{\varepsilon}) > F_{KS}(\overline{\boldsymbol{b}}^{\varepsilon}) = \overline{F_{KS}(\boldsymbol{b})}^{\varepsilon}$ for $\varepsilon = \min(\varepsilon', 1-\varepsilon')$ and $\varepsilon' = (F(\boldsymbol{b}) + F_{KS}(\boldsymbol{b}))/2$. This contradicts that $F(\boldsymbol{a}) = F_{KS}(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_5^n since $\overline{\boldsymbol{b}}^{\varepsilon} \in V_5^n$. This completes the proof of the theorem. ■

6.4.3 Relationship between Conditions $(A) \sim (E)$

Conditions $(A) \sim (E)$ are not independent to each other, since (A) is derived from (B) and (C), and (B) and (D) are derived from (E). In this section, we will show the above relationships.

Lemma 8 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V_5^n such that $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$. Then, there exist an element $\boldsymbol{t} = (t_1, \ldots, t_n)$ of V^n and constants ε_1, ε_2 such that $\boldsymbol{a} = \overline{\boldsymbol{t}}^{\varepsilon_1}$, $\boldsymbol{b} = \overline{\boldsymbol{t}}^{\varepsilon_2}$ and $0 < \varepsilon_2 < \varepsilon_1 \leq 1/2$.*

Proof: $a_i \leq_{\mathrm{KS}} b_i$ holds for any $i = 1, \ldots, n$ from $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$. This implies that one of the following three relations holds for each $i = 1, \ldots, n$

$$a_i = b_i, \quad a_i = 1/4 \text{ and } b_i = 1/2, \text{ or } \quad a_i = 3/4 \text{ and } b_i = 1/2.$$

First, suppose $a_i = 1/4$ and $b_i = 1/2$. In order to exist t_i and ε_1, ε_2 such that $a_i = \overline{t_i}^{\varepsilon}_1$ and $b_i = \overline{t_i}^{\varepsilon}_2$, the following relations must be held.

$$0 < t_i < \varepsilon_1 \text{ and } \varepsilon_1 \geq t_i \geq 1 - \varepsilon_2.$$

Therefore, we obtain $a_i = \overline{t_i}^{\varepsilon}$ and $b_i = \overline{t_i}^{\varepsilon}$ for any t_i *suchthat*$\varepsilon_2 < t_i \leq \varepsilon_2$, and accordingly $0 < \varepsilon_1 < \varepsilon_2 \leq 1/2$. In the remaining cases, we can prove in the similar manner. ■

Lemma 9 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V^n. Then $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$ if and only if $\overline{\boldsymbol{a}}^{\varepsilon} \leq_{\mathrm{KS}} \overline{\boldsymbol{b}}^{\varepsilon}$ for any ε $(0 < \varepsilon \leq 1/2)$.*

Proof: First, suppose $\boldsymbol{a} \leq_{\mathrm{KS}} \boldsymbol{b}$. This implies that $a_i \leq_{\mathrm{KS}} b_i$ for any $i = 1, \ldots, n$. Moreover, we obtain $\overline{b_i}^{\varepsilon} = 0$, $1/4$, $1/2$, $3/4$ or 1 for any ε $(0 < \varepsilon \leq 1/2)$. If $\overline{b_i}^{\varepsilon} = 1/4$ holds, then this implies $0 < b_i < \varepsilon$. Therefore, we obtain $0 < a_i \leq b_i < \varepsilon$ since $a_i \leq_{\mathrm{KS}} b_i$ and $0 < b_i < \varepsilon$. This implies $\overline{a_i}^{\varepsilon} = 1/4$. Thus, we obtain $\overline{a_i}^{\varepsilon} \leq_{\mathrm{KS}} \overline{b_i}^{\varepsilon}$. For the remaining cases, we can prove in the similar manner. Therefore, it has been proved the first part of the lemma. Next, suppose $\overline{\boldsymbol{a}}^{\varepsilon} \leq_{\mathrm{KS}} \overline{\boldsymbol{b}}^{\varepsilon}$ for any ε $(0 < \varepsilon \leq 1/2)$. We can assume without loss of generality that $\boldsymbol{a} \not\leq_{\mathrm{KS}} \boldsymbol{b}$, that is, there is at least $i = 1, \ldots, n$ such that $a_i \not\leq_{\mathrm{KS}} b_i$. This implies one of

(1) a_i and b_i are not comparable to each other, or

(2) $b_i \leq_{\mathrm{KS}} a_i$ and $a_i \neq b_i$.

If (1) holds, then $\overline{a_i}^{1/2}$ and $\overline{b_i}^{1/2}$ are not comparable to each other. Therefore, $\overline{\boldsymbol{a}}^{1/2}$ and $\overline{\boldsymbol{b}}^{1/2}$ are not comparable to each other, and this contradicts to the assumption. If (2) holds, then $\overline{b_i}^{\varepsilon'} \leq_{\mathrm{KS}} \overline{a_i}^{\varepsilon'}$ and $\overline{a_i}^{\varepsilon'} \neq \overline{b_i}^{\varepsilon'}$ for $\varepsilon' = \min(\varepsilon'', 1-\varepsilon'')$ and $\varepsilon'' = (a_i + b_i)/2$. Therefore, $\overline{\boldsymbol{a}}^{\varepsilon'} \leq_{\mathrm{KS}} \overline{\boldsymbol{b}}^{\varepsilon'}$ does not hold, and this contradicts to the assumption. This completes the proof of the lemma. ■

Lemma 10 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V^n and ε_1, ε_2 be constants such that $0 < \varepsilon_1 \leq 1/2$ and $0 < \varepsilon_2 \leq 1/2$. If $\varepsilon_1 \leq_{\mathrm{KS}} \varepsilon_2$, then $\overline{\boldsymbol{a}}^{\varepsilon_1} \leq_{\mathrm{KS}} \overline{\boldsymbol{a}}^{\varepsilon_2}$.*

Proof: It is a trivial problem when $\varepsilon_1 = \varepsilon_2$. Suppose that $\varepsilon_1 \leq_{KS} \varepsilon_2$ and $\varepsilon_1 \neq \varepsilon_2$. This implies $0 < \varepsilon_1 < \varepsilon_2 \leq 1/2$. Then, we can prove that one of the following relations holds for any a_i $(0 \leq a_i \leq 1,\ i = 1, \ldots, n)$.

$$\overline{a_i}^{\varepsilon_1} = \overline{a_i}^{\varepsilon_2}, \text{ or } \quad \overline{a_i}^{\varepsilon_1} = 1/2 \text{ and } \overline{a_i}^{\varepsilon_2} = 1/4, \text{ or } 3/4$$

Therefore, we obtain $\overline{a_i}^{\varepsilon_1} \leq_{KS} \overline{a_i}^{\varepsilon_2}$ for any $i = 1, \ldots, n$, that is, $\overline{\boldsymbol{a}}^{\varepsilon_2} \leq_{KS} \overline{\boldsymbol{a}}^{\varepsilon_1}$. ■

Theorem 9 *Let F be an infinite-valued function. If F satisfies Condition (E), then F also satisfies Condition (B).*

Proof: Let $\boldsymbol{a}$ be an element of V_5^n. Then, $F(\boldsymbol{a}) \in V_5$ holds from Condition (e) and Lemma 6. Suppose that $\boldsymbol{b}$ is an element of V_3^n such that $F(\boldsymbol{b}) \notin V_3$. This implies that $F(\boldsymbol{b}) \in V_5 - V_3$ $= \{1/4, 3/4\}$ and $\boldsymbol{b} = \overline{\boldsymbol{b}}^{\varepsilon}$ for any ε $(0 < \varepsilon \leq 1/2)$. Therefore, $F(\boldsymbol{b}) = F(\overline{\boldsymbol{b}}^{\varepsilon}) = \overline{F(\boldsymbol{b})}^{\varepsilon}$ holds for any ε $(0 < \varepsilon \leq 1/2)$ from Condition (E). On the other hand, $\overline{F(\boldsymbol{b})}^{\varepsilon} = 1/2$ holds for any ε $(0 < \varepsilon \leq 1/4)$ from $F(\boldsymbol{b}) \in \{1/4, 3/4\}$. This contradicts to $F(\boldsymbol{b}) = \overline{F(\boldsymbol{b})}^{\varepsilon}$ for any ε $(0 < \varepsilon \leq 1/2)$. This completes the proof of the theorem. ■

Theorem 10 *Let F be an infinite-valued function. If F satisfies Condition (E), then F also satisfies Condition (D).*

Proof: Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ be elements of V^n such that $\boldsymbol{a} \leq_{KS} \boldsymbol{b}$. Then, $\overline{\boldsymbol{a}}^{\varepsilon} \leq_{KS} \overline{\boldsymbol{b}}^{\varepsilon}$ holds for any ε $(0 < \varepsilon \leq 1/2)$ from Lemma 8. Therefore, there exist an element $\boldsymbol{t}$ of V^n and ε_1, ε_2 such that $\overline{\boldsymbol{a}}^{\varepsilon} = \overline{\boldsymbol{t}}^{\varepsilon_1}$, $\overline{\boldsymbol{b}} = \overline{\boldsymbol{t}}^{\varepsilon_2}$ and $0 < \varepsilon_2 < \varepsilon_1 \leq 1/2$ from Lemma 7. Then, we obtain the following relations from Condition (E).

$$\overline{F(\boldsymbol{a})}^{\varepsilon} = F(\overline{\boldsymbol{a}}^{\varepsilon}) = F(\overline{\boldsymbol{t}}^{\varepsilon_1}) = \overline{F(\boldsymbol{t})}^{\varepsilon_1},$$

$$\overline{F(\boldsymbol{b})}^{\varepsilon} = F(\overline{\boldsymbol{b}}^{\varepsilon}) = F(\overline{\boldsymbol{t}}^{\varepsilon_2}) = \overline{F(\boldsymbol{t})}^{\varepsilon_2},$$

where $\varepsilon_2 \leq_{KS} \varepsilon_1$ holds since $0 < \varepsilon_2 < \varepsilon_1 \leq 1/2$. Therefore, $\overline{F(\boldsymbol{t})}^{\varepsilon_1} \leq_{KS} \overline{F(\boldsymbol{t})}^{\varepsilon_2}$ from Lemma 9, that is, we obtain $\overline{F(\boldsymbol{t})}^{\varepsilon_1} = \overline{F(\boldsymbol{a})}^{\varepsilon} \leq_{KS} \overline{F(\boldsymbol{b})}^{\varepsilon} = \overline{F(\boldsymbol{t})}^{\varepsilon_2}$ for any ε $(0 < \varepsilon \leq 1/2)$. Thus, $F(\boldsymbol{a}) \leq_{KS} F(\boldsymbol{b})$ holds from Lemma 8. Therefore, it has been shown that this theorem holds. ■

We can easily show that (A) is derived from (B) and (C). Because, let $\boldsymbol{a}$ be an element of V_3^n and V_4^n, that is, $\boldsymbol{a} \in V_2^n$. Then, $F(\boldsymbol{a}) \in V_3$ and $F(\boldsymbol{a}) \in V_4$ from (B) and (C). Therefore, $F(\boldsymbol{a}) \in V_2$.

The conditions (E) and (C) are independent of each other. Because, it is easily to show that F_E of Figure 6.3 satisfies Condition (E), but not (C), and F_C of Figure 6.4 satisfies Condition (C), but not (E). From the above, the set of Conditions (E) and (C) is a necessary and sufficient condition, which is independent of each other, for an infinite multiple-valued function to be a Kleene-Stone logic function.

6.5 Minimization for Kleene-Stone Logic Functions

In this section, we describe minimization for Kleene-Stone logic functions. A minimal form described in this section is motivated by Boolean functions and fuzzy logic functions.

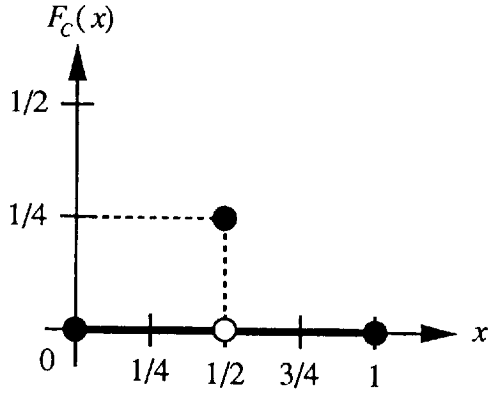

Figure 6.3: An Example Satisfying Only Condition (C)

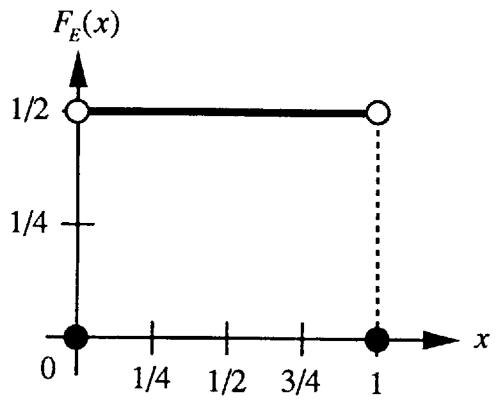

Figure 6.4: An Example Satisfying Only Condition (E)

6.5.1 Definitions

First, we give some definitions concerning with minimization for Kleene-Stone logic functions.

Definition 10 *Let F be a disjunctive form of a Kleene-Stone logic function. Then, F is said to be a minimal form if and only if no other equivalent disjunctive form involving a smaller total number of literals.*

Note that a minimal form of any Kleene-Stone logic function does not determined uniquely as well as Boolean functions and fuzzy logic functions.

Definition 11 *Let F be a Kleene-Stone logic function and α be a product term. Then α is said to be an implicant of F if and only if $F \sqsupseteq \alpha$. Moreover, an implicant α of F is said to be prime if and only if there is no product term β such that $F \sqsupseteq \beta \sqsupseteq \alpha$ and $\alpha \neq \beta$, and especially an implicant α of F is said to be an essential if α exists in any minimal form of F.*

Theorem 11 *A minimal form of a Kleene-Stone logic function F can be represented by a disjunction of some of the prime implicants of F.*

Note that the above theorem does not claim any minimal form of a given Kleene-Stone logic function F is represented by a disjunction of only the prime implicants of F, and this is different from Boolean functions and fuzzy logic functions. We will show such an example in latter half of this section.

From Theorem 3, in order to find a minimal form of a Kleene-Stone logic function F, we need to get all prime implicants of F. Therefore, next we consider how to get all prime implicants of F.

In the following definition, type A product term means a product term such that $x_i \sim x_i$ does not appear in it for any variable x_i, otherwise we call it type B product term. It is evident that every product term can be classified one of type A or type B product term.

Definition 12 *Let α and β be product terms satisfying at least one of the following two conditions (a) and (b).*

(a) For some variable x_i, α and β can be represented as $\alpha' x_i'$ and $\beta' x_i''$ respectively, such that $x_i' \vee x_i'' = 1$, $\alpha' \not\sqsubseteq x_i''$ and $\beta' \not\sqsubseteq x_i'$, where x_i' and x_i'' each denotes a literal of x_i.

(b) For some variable x_i, α and β can be represented as $\alpha' x_i^*$ ($\alpha' \not\sqsubseteq \sim x_i^*$) and $\beta' \sim x_i^*$ ($\beta' \not\sqsubseteq x_i^*$) respectively, where x_i^* denotes one of x_i or $\sim x_i$.

Then, a consensus γ of α and β is defined as a product term below, where the symbol $C_{\alpha\beta}$ means a set of all consensus of α and β, and especially $\alpha'\beta'$ is called a base of consensus of α and β.

(1) When they satisfy (a), then $\alpha'\beta' \in C_{\alpha\beta}$.

(2) When they satisfy (b), then for some variable x_j ($i \neq j$), $\alpha'\beta' x_j \sim x_j \in C_{\alpha\beta}$ if $\alpha'\beta'$ is a type A product term. If $\alpha'\beta'$ is a type B product term, then $\alpha'\beta' \in C_{\alpha\beta}$.

(3) The only consensus of α and β are given by (1) and (2).

Any repeated literals are removed from the consensus of α and β.

Example 9 *Let α and β be product terms on x_1, x_2 and x_3.*

(1) When $\alpha = \neg x_1 \sim x_2$ and $\beta = \sim x_1 \neg\neg x_1 x_3$, we have $C_{\alpha\beta} = \{\sim x_1 \sim x_2 x_3\}$

(2) When $\alpha = \sim x_2 \neg\neg \sim x_3$ and $\beta = \sim x_2 \neg\neg x_3$,
we have $C_{\alpha\beta} = \{x_1 \sim x_1 \neg\neg x_3 \neg\neg \sim x_3, x_2 \sim x_2\}$

(3) When $\alpha = x_1$ and $\beta = \sim x_1$, we have $C_{\alpha\beta} = \{x_2 \sim x_2, x_3 \sim x_3\}$

6.5.2 Lemmas

In this section, we show some lemmas before deriving an algorithm to find a minimal form of a given Kleene-Stone logic function F

Lemma 11 *Let $\alpha = x_1^\alpha \cdot \ldots \cdot x_n^\alpha$ and $\beta = x_1^\beta \cdot \ldots \cdot x_n^\beta$ be product terms, where each x_i^α and x_i^β denotes an atom of x_i ($i = 1, \ldots, n$). Then $\alpha \sqsubseteq \beta$ implies that $x_i^\alpha \sqsubseteq x_i^\beta$ for any $i = 1, \ldots, n$.*

Proof: It is clear from the truth tables of atoms that we always have an element $\boldsymbol{a} = (a_1, \ldots, a_n) \in V_5^n$ such that $\alpha(\boldsymbol{a}) \geq 1/2$. Therefore, by $\alpha \sqsubseteq \beta$ we have $\beta(\boldsymbol{a}) \geq 1/2$. Now let assume that there are two atoms x_i^α and x_i^β such that $x_i^\alpha \not\sqsubseteq x_i^\beta$. In this case, it is always possible from the truth tables of atoms that there is a least an element $a' \in V_5$ such that $x_i^\alpha(a') \geq 1/2$ and $x_i^\beta(a') < 1/2$, and therefore, we obtain $\alpha(\boldsymbol{a}') \geq 1/2$ and $\beta(\boldsymbol{a}') < 1/2$ where $\boldsymbol{a}' = (a_1, \ldots, a_{i-1}, a', a_{i+1}, \ldots, a_n)$. This contradicts to $\alpha \sqsubseteq \beta$, thus $x_i^\alpha \sqsubseteq x_i^\beta$ holds for any $i = 1, \ldots, n$. This completes the proof of the lemma. ■

Lemma 12 *Let α and β be product terms. If γ is a consensus of α and β, then $\gamma \sqsubseteq \alpha \vee \beta$, that is, $\alpha \vee \beta \vee \gamma = \alpha \vee \beta$.*

Proof: First assume α and β satisfy the condition (a) of Definition 12. Then α and β can be represent that $\alpha = \alpha' x_i'$ and $\beta = \beta' x_i''$ such that $x_i' \vee x_i'' = 1$, where x_i' and x_i'' each denotes a literal. In this case $\gamma = \alpha'\beta'$. For any element $\boldsymbol{a} \in V_5^n$, $\gamma(\boldsymbol{a}) = \alpha'(\boldsymbol{a})$ or $\beta'(\boldsymbol{a})$ holds. When $\gamma(\boldsymbol{a}) = \alpha'(\boldsymbol{a})$ that is, $\alpha'(\boldsymbol{a}) \leq \beta'(\boldsymbol{a})$, then let assume that

$$(1)\ (\alpha' x_i')(\boldsymbol{a}) < \alpha'(\boldsymbol{a}) \quad \text{and} \quad (2)\ (\beta' x_i'')(\boldsymbol{a}) < \alpha'(\boldsymbol{a}).$$

By $x_i' \vee x_i'' = 1$, $x_i'(\boldsymbol{a}) = 1$ or $x_i''(\boldsymbol{a}) = 1$ always satisfies. $x_i'(\boldsymbol{a}) = 1$, however, does not hold because if so, then we obtain $\alpha'(\boldsymbol{a}) < \alpha'(\boldsymbol{a})$ from (1), and this is a contradiction. Thus, $x_i''(\boldsymbol{a}) = 1$ must be held. By $x_i''(\boldsymbol{a}) = 1$ and (2) we can derive $\beta'(\boldsymbol{a}) < \alpha'(\boldsymbol{a})$, and this contradicts to $\alpha'(\boldsymbol{a}) \leq \beta'(\boldsymbol{a})$. Therefore, $\gamma(\boldsymbol{a}) = \alpha'(\boldsymbol{a}) \leq (\alpha' x_i')(\boldsymbol{a}) \vee (\beta' x_i'')(\boldsymbol{a}) = (\alpha \vee \beta)(\boldsymbol{a})$. We can also show the same result when $\beta'(\boldsymbol{a}) \leq \alpha'(\boldsymbol{a})$.

Next assume α and β satisfy the condition (b) of Definition 12. Then α and β are represent that $\alpha = \alpha' x_i^*$ and $\beta = \beta' \sim x_i^*$ where x_i^* denotes one of x_i or $\sim x_i$. When $\alpha'\beta'$ is a type A product term, $\gamma = \alpha'\beta' x_j \sim x_j$ $(i \neq j)$. Therefore, for any element $\boldsymbol{a} \in V_5^n$, $\gamma(\boldsymbol{a}) \leq 1/2$. If $\gamma(\boldsymbol{a}) = 1/2$ then $(x_j \sim x_j)(\boldsymbol{a}) = 1/2$ and $(\alpha'\beta')(\boldsymbol{a}) \geq 1/2$, that is, $\alpha'(\boldsymbol{a}) \geq 1/2$ and $\beta'(\boldsymbol{a}) \geq 1/2$. Thus, we obtain $(\alpha \vee \beta)(\boldsymbol{a}) \geq 1/2$ since $x_i^*(\boldsymbol{a}) \geq 1/2$ or $\sim x_i^*(\boldsymbol{a}) \geq 1/2$ stands always true. If $\gamma(\boldsymbol{a}) = 1/4$ then the least $(\alpha'\beta')(\boldsymbol{a}) \geq 1/4$ must be held. This implies $\alpha'(\boldsymbol{a}) \geq 1/4$ and $\beta'(\boldsymbol{a}) \geq 1/4$, and therefore we obtain $(\alpha \vee \beta)(\boldsymbol{a}) \geq 1/4$. When $\alpha'\beta'$ is a type B product term, $\gamma = \alpha'\beta'$. Thus $\gamma(\boldsymbol{a}) = \alpha'(\boldsymbol{a})$ or $\beta'(\boldsymbol{a})$ for any element $\boldsymbol{a} \in V_5^n$. When $\gamma(\boldsymbol{a}) = \alpha'(\boldsymbol{a})$, that is, $\alpha'(\boldsymbol{a}) \leq \beta'(\boldsymbol{a})$, let assume that

$$(3)\ (\alpha' x_i^*)(\boldsymbol{a}) < \alpha'(\boldsymbol{a}) \quad \text{and} \quad (4)\ (\beta' \sim x_i^*)(\boldsymbol{a}) < \alpha'(\boldsymbol{a}).$$

By $\alpha'(\boldsymbol{a}) \leq 1/2$ and (3) we have $x_i^*(\boldsymbol{a}) < 1/2$, and therefore $\sim x_i^*(\boldsymbol{a}) > 1/2$ must be held. On the other hand, by (4) $\beta'(\boldsymbol{a}) < \alpha'(\boldsymbol{a})$ or $\sim x_i^*(\boldsymbol{a}) < \alpha'(\boldsymbol{a})$ hold, however, both two contradict to $\alpha'(\boldsymbol{a}) \leq \beta'(\boldsymbol{a})$ and $\sim x_i^*(\boldsymbol{a}) > 1/2$, respectively. Therefore, $\gamma(\boldsymbol{a}) = \alpha'(\boldsymbol{a}) \leq (\alpha' x_i^* \vee \beta' \sim x_i^*)(\boldsymbol{a}) = (\alpha \vee \beta)(\boldsymbol{a})$. We can prove some result when $\beta'(\boldsymbol{a}) \leq \alpha'(\boldsymbol{a})$. This completes the proof of the lemma. ■

Lemma 13 *Let α be a minterm of type 1, 2 or 3 and $\boldsymbol{a}$ be the corresponding element to α. If β is an arbitrary product term, then*

(1) when α is a type 1 minterm, $\alpha \sqsubseteq \beta$ if and only if $\beta(\boldsymbol{a}) = 1$

(2) when α is a type 2 minterm, $\alpha \sqsubseteq \beta$ if and only if $\beta(\boldsymbol{a}) \geq 3/4$

(3) when α is a type 3 minterm, $\alpha \sqsubseteq \beta$ if and only if $\beta(\boldsymbol{a}) \geq 1/2$.

Proof: Let $\alpha = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$, $\beta = x_1^* \cdot \ldots \cdot x_n^*$ where each $x_i^{a_i}$ and x_i^* denote atoms or constant 1. Suppose $\beta(\boldsymbol{a}) = 1$, then $x_i^*(a_i) = 1$ for any i $(i = 1, \ldots, n)$. Therefore, the following relations must be held.

$$\begin{aligned} x_i^* &= \neg x_i, \sim x_i, \neg\neg \sim x_i \text{ or } 1 \text{ when } a_i = 0 \\ x_i^* &= \neg\neg x_i, \neg\neg \sim x_i, \neg\neg x_i \neg\neg \sim x_i \text{ or } 1 \text{ when } a_i = 1/2 \\ x_i^* &= \neg \sim x_i, x_i, \neg\neg x_i, \text{ or } 1 \text{ when } a_i = 1. \end{aligned}$$

Thus, we have $\alpha \sqsubseteq \beta$ since $x_i^{a_i} \sqsubseteq x_i^*$ for any i $(i = 1, \ldots, n)$. Conversely suppose $\alpha \sqsubseteq \beta$, then $\beta(\boldsymbol{a}) = 1$ since $\alpha(\boldsymbol{a}) = 1$. This completes the proof of the lemma (1). The proof of (2) and (3) are similar to that of (1). ■

Lemma 14 *Let α be a product term and $\alpha = \alpha_1 \vee \ldots \vee \alpha_s$ be the canonical disjunctive form of α. Moreover, let F be a Kleene-Stone logic function and $F = \beta_1 \vee \ldots \vee \beta_t$ be a disjunctive form of F. Then, $\alpha \sqsubseteq F$ implies that there is at least one product term β_j $(j = 1, \ldots, t)$ for any product term α_i $(i = 1, \ldots, s)$ such that $\alpha_i \sqsubseteq \beta_j$.*

Proof: By $\alpha \sqsubseteq F$ we have $\alpha_1 \vee \ldots \vee \alpha_s \sqsubseteq \beta_1 \vee \ldots \vee \beta_t$. Suppose α_i $(i = 1, \ldots, s)$ is a type 1 minterm, then $\alpha_i(\boldsymbol{a}) = 1$ for the corresponding element $\boldsymbol{a}$ to α_i. Therefore, there is at least one j $(j = 1, \ldots, t)$ such that $\beta_j(\boldsymbol{a}) = 1$. Thus, $\alpha_i \sqsubseteq \beta_j$ from Lemma 1.2.1. Next suppose α_i is a type 2 minterm, then $\alpha_i(\boldsymbol{a}) = 3/4$ for the corresponding element $\boldsymbol{a}$ to α_i. Therefore, $\beta_j(\boldsymbol{a}) \geq 3/4$ should be hold for some j $(j = 1, \ldots, t)$. Thus, $\alpha_i \sqsubseteq \beta_j$ from Lemma 1.2.2. We can prove in the similar manner that $\alpha_i \sqsubseteq \beta_j$ for some j even if α_i is a type 3 minterm. This completes the proof of the lemma. ■

Corollary 2 *Let α be a type 1, 2 or 3 minterm, F be a Kleene-Stone logic function and $\beta_1 \vee \ldots \vee \beta_t$ be an arbitrary disjunctive form of F. Then $\alpha \sqsubseteq F$ implies that $\alpha \sqsubseteq \beta_j$ for some product term β_j $(j = 1, \ldots, t)$.*

Lemma 15 *Let α be one of type 1, 2 or 3 minterm. If α is a prime implicant of a Kleene-Stone logic function F, then α is an essential one.*

Proof: We can assume without loss of generality that $\alpha_1 \vee \ldots \vee \alpha_s$ is a minimal form of F and $\alpha \neq \alpha_i$ for any i $(i = 1, \ldots, s)$. First suppose α is a type 1 minterm. Then $\alpha(\boldsymbol{a}) = 1$ where $\boldsymbol{a}$ is the corresponding element to α. Therefore, $F(\boldsymbol{a}) = 1$ holds, and so there is at least one α_i such that $\alpha_i(\boldsymbol{a}) = 1$. Thus, $\alpha \sqsubseteq \alpha_i$ ($\alpha \neq \alpha_i$ from the assumption) from Lemma 1.1, and this contradicts that α is a prime implicant of F. Next suppose α is a type 2 minterm. Then $\alpha(\boldsymbol{a}) = 3/4$ for the corresponding element $\boldsymbol{a}$ to α, and therefore $\alpha_i(\boldsymbol{a}) \geq 3/4$ has to be held for some i $(i = 1, \ldots, s)$. Accordingly by Lemma 1.2.2 we obtain $\alpha \sqsubseteq \alpha_i$ ($\alpha \neq \alpha_i$ from the assumption). This contradicts that α is a prime implicant of F. We can also show that α is never type 3 minterm in the similar manner. From the above, if a type 1, 2 or 3 minterm α is a prime implicant of F, then it has to appear in any minimal form of F, that is, α is an essential one. ■

Lemma 16 *Let $\alpha = x_1^\alpha \cdot \ldots \cdot x_n^\alpha$, $\beta = x_1^\beta \cdot \ldots \cdot x_n^\beta$ and $\gamma = x_1^\gamma \cdot \ldots \cdot x_n^\gamma$ be product terms such that $\alpha \not\sqsubseteq \beta$ and $\alpha \not\sqsubseteq \gamma$, where each x_i^ξ (ξ denotes one of α, β or γ) is an atom of x_i. Then $\alpha\neg\neg x_i^* \sqsubseteq \beta$ and $\alpha\neg x_i^* \sqsubseteq \gamma$ imply that it is possible to represent that $\beta = \beta' x_i'$ and $\gamma = \gamma' x_i''$, where x_i' and x_i'' each denotes a literal of x_i, and $\beta'\gamma'$ is a consensus of β and γ by applying the condition (a) of Definition 12. Moreover we obtain $\alpha \sqsubseteq \beta'\gamma'$.*

Proof: By $\alpha\neg\neg x_i^* \sqsubseteq \beta$, $\alpha\neg x_i^* \sqsubseteq \gamma$ and Lemma 11 we have $x_j^\alpha \sqsubseteq x_j^\beta$ and $x_j^\alpha \sqsubseteq x_j^\gamma$ for any j $(j = 1, \ldots, i-1, i+1, \ldots, n)$, and $x_i^\alpha\neg\neg x_i^* \sqsubseteq x_i^\beta$ and $x_i^\alpha\neg x_i^* \sqsubseteq x_i^\gamma$. Here, let assume $x_i^\alpha \sqsubseteq \neg\neg x_i^*$, then we obtain $x_i^\alpha = x_i^\alpha\neg\neg x_i^* \sqsubseteq x_i^\beta$. This contradicts to $x_i^\alpha \not\sqsubseteq x_i^\beta$, and therefore $x_i^\alpha \not\sqsubseteq \neg\neg x_i^*$. We also obtain $x_i^\alpha \not\sqsubseteq \neg x_i^*$ in the similar manner from $x_i^\alpha\neg x_i^* \sqsubseteq x_i^\gamma$. Thus, x_i^α is one of $\sim x_i^*$, $\neg\neg\sim x_i^*$ or 1. On the other hand, $x_i^\beta = 1$ never holds because if $x_i^\beta = 1$, then $x_i^\alpha \sqsubseteq 1$ stands always true, and therefore we have $\alpha \sqsubseteq \beta$. This, however, contradicts to $\alpha \not\sqsubseteq \beta$. Accordingly we can represent x_i^β as $x_i^{\beta'}x_i^{\beta''}$, where $x_i^{\beta'}$ and $x_i^{\beta''}$ each denotes a literal of x_i or constant 1. Thus we have $x_i^\alpha \sqsubseteq x_i^{\beta'}$ and $\neg\neg x_i^* \sqsubseteq x_i^{\beta''}$. However, $x_i^{\beta''} = 1$ does not hold because if $x_i^{\beta''} = 1$ then we have $x_i^\alpha \sqsubseteq x_i^{\beta'} \cdot 1 = x_i^\beta$, and this contradicts to $\alpha \not\sqsubseteq \beta$. Therefore $x_i^{\beta''} = \neg\neg x_i^*$. In the similar manner, we can represent x_i^γ as $x_i^{\gamma'}x_i^{\gamma''}$ such that $x_i^\alpha \sqsubseteq x_i^{\gamma'}$ and $x_i^{\gamma''} = \neg x_i^*$, where $x_i^{\gamma'}$ and $x_i^{\gamma''}$ each denotes a literal of x_i or constant 1. From the above, if we represent $\beta = \beta' x_i^{\beta''}$ and $\gamma = \gamma' x_i^{\gamma''}$, then $\beta'\gamma'$ is a consensus of β and γ by applying the condition (a) of Definition 12 since $x_i^{\beta''} \vee x_i^{\gamma''} = 1$, $\beta' \not\sqsubseteq x_i^{\gamma''}$ and $\gamma' \not\sqsubseteq x_i^{\beta''}$ stand true. Moreover, we have $\alpha \sqsubseteq \beta'\gamma'$ since $\alpha = x_1^\alpha \cdot \ldots \cdot x_n^\alpha \sqsubseteq x_1^\beta \cdot \ldots \cdot x_{i-1}^\beta x_i^{\beta'} x_{i+1}^\beta \cdot \ldots \cdot x_n^\beta = \beta'$ and $\alpha = x_1^\alpha \cdot \ldots \cdot x_n^\alpha \sqsubseteq x_1^\gamma \cdot \ldots \cdot x_{i-1}^\gamma x_i^{\gamma'} x_{i+1}^\gamma \cdot \ldots \cdot x_n^\gamma = \gamma'$ stand true. In this case, we can not find another consensus of β and γ by removing a literal of x_i. ■

Lemma 17 *Let $\alpha = x_1^\alpha \cdot \ldots \cdot x_n^\alpha$, $\beta = x_1^\beta \cdot \ldots \cdot x_n^\beta$ and $\gamma = x_1^\gamma \cdot \ldots \cdot x_n^\gamma$ be product terms such that $\alpha \not\sqsubseteq \beta$ and $\alpha \not\sqsubseteq \gamma$, where each x_i^ξ (ξ denotes one of α, β or γ) is an atom of x_i. Then $\alpha x_i^* \sqsubseteq \beta$ and $\alpha \sim x_i^* \sqsubseteq \gamma$ imply that it is possible to represent that $\beta = \beta' x_i'$ and $\gamma = \gamma' x_i''$, where x_i^* denotes one of x_i or $\sim x_i$ and x_i', x_i'' are literals of x_i, and we have $\beta'\gamma'$ is a base of consensus of β and γ such that $\alpha \sqsubseteq \beta'\gamma'$.*

Proof: By $\alpha x_i^* \sqsubseteq \beta$, $\alpha \sim x_i^* \sqsubseteq \gamma$ and Lemma 11 we have $x_j^\alpha \sqsubseteq x_j^\beta$ and $x_j^\alpha \sqsubseteq x_j^\gamma$ for any j $(j = 1, \ldots, i-1, i+1, \ldots, n)$ and $x_i^\alpha x_i^* \sqsubseteq x_i^\beta$ and $x_i^\alpha \sim x_i^* \sqsubseteq x_i^\gamma$. Here, $x_i^\beta = 1$ never holds because if $x_i^\beta = 1$, then $x_i^\alpha \sqsubseteq 1$ stands always true, and therefore we have $\alpha \sqsubseteq \beta$. This, however, contradicts to $\alpha \not\sqsubseteq \beta$. Accordingly we can represent x_i^β as $x_i^{\beta'}x_i^{\beta''}$, where $x_i^{\beta'}$ and $x_i^{\beta'}$ each denotes a literal of x_i. On the other hand, $x_i^\alpha \not\sqsubseteq x_i^*$ is always true because if $x_i^\alpha \sqsubseteq x_i^*$, then $x_i^\alpha = x_i^\alpha x_i^* \sqsubseteq x_i^\beta$ stands true, and this leads $\alpha \sqsubseteq \beta$, and contradicts to $\alpha \not\sqsubseteq \beta$. In the similar manner, we derive $x_i^\alpha \not\sqsubseteq \sim x_i^*$ form $x_i^\alpha \sim x_i^* \sqsubseteq x_i^\gamma$. Therefore, x_i^α is one of $\neg\neg x_i$, $\neg\neg\sim x_i$, $\neg\neg x_i \neg\neg \sim x_i$ or 1. Thus, we can represent that $x_i^\alpha x_i^* = x_i^{\alpha'} x_i^{\alpha''}$ where $x_i^{\alpha'}$ and $x_i^{\alpha''}$ each denotes a literal of x_i, and accordingly $x_i^{\alpha'} = x_i^*$ and $x_i^{\alpha''} = \neg\neg \sim x_i^*$ or 1 stand true. Since $x_i^\alpha x_i^* = x_i^{\alpha'} x_i^{\alpha''} \subseteq x_i^{\beta'} x_i^{\beta''} = x_i^\beta$ holds we obtain $x_i^{\alpha'} \sqsubseteq x_i^{\beta'}$ and $x_i^{\alpha''} \sqsubseteq x_i^{\beta''}$, that is, $x_i^* \sqsubseteq x_i^{\beta'}$ and $\neg\neg \sim x_i^* \sqsubseteq x_i^{\beta''}$, where $x_i^{\beta'} = 1$ and $x_i^{\beta''} = 1$ never hold from $x_i^\beta \neq 1$. In the similar manner, we can represent that $x_i^\gamma = x_i^{\gamma'} x_i^{\gamma''}$, where $x_i^{\gamma'}$ and $x_i^{\gamma''}$ each denotes a literal of x_i. such that $\sim x_i^* \sqsubseteq x_i^{\gamma'}$ and $\neq \neg x_i^* \sqsubseteq x_i^{\gamma''}$, and moreover $x_i^{\gamma'} = 1$ and $x_i^{\gamma''} = 1$ never hold from $x_i^\gamma \neq 1$. Here, it is impossible that $x_i^{\beta'} = x_i^{\gamma''} = 1$ or $x_i^{\beta''} = x_i^{\gamma'} = 1$. Because if $x_i^{\beta'} = 1$ and $x_i^{\gamma''} = 1$ hold, then it has to be that $x_i^{\beta''} = \neg\neg \sim x_i^*$ and $x_i^{\gamma'} = \sim x_i^*$ or $\neg\neg \sim x_i^*$. In order to satisfy the relation $x_i^\alpha x_i^* \sqsubseteq x_i^\beta = \neg\neg \sim x_i^*$, $x_i^\alpha = \neg\neg x_i^*$ or $\neg\neg x_i \neg\neg \sim x_i$ has to be held. This, however, contradicts to $\alpha \not\sqsubseteq \beta$. Therefore, $x_i^{\beta'} = 1$ and $x_i^{\gamma''} = 1$ never occur. We can also derive a contradiction to $\alpha \not\sqsubseteq \beta$ when $x_i^{\beta''} = 1$ and $x_i^{\gamma'} = 1$ in the similar manner, and therefore $x_i^{\beta''} = 1$ and $x_i^{\gamma'} = 1$ never hold. Moreover, it is also impossible that $x_i^{\beta'} \neq 1$ and $x_i^{\beta''} \neq 1$. Because if $x_i^{\beta'} \neq 1$ and $x_i^{\beta''} \neq 1$ hold, that is, $x_i^{\beta'} = x_i^*$ or $\neg\neg x_i^*$ and $x_i^{\beta''} = \neg\neg \sim x_i^*$, then it has to be held that $x_i^\beta = x_i^* \neg\neg \sim x_i^*$ or $\neg\neg x_i \neg\neg \sim x_i$. This, however, contradicts to $\alpha \not\sqsubseteq \beta$, and therefore we never hold $x_i^{\beta'} \neq 1$ and $x_i^{\beta''} \neq 1$. We can show that $x_i^{\gamma'} \neq 1$ and $x_i^{\gamma''} \neq 1$ never occur in the similar manner. From the above we have the following.

(1) $x_i^\beta = x_i^{\beta'} = x_i^*$ or $\neg\neg x_i^*$, and $x_i^\gamma = x_i^{\gamma'} = \sim x_i^*$ or $\neg\neg \sim x_i^*$, or

(2) $x_i^\beta = x_i^{\beta''} = \neg\neg \sim x_i^*$, and $x_i^\gamma = x_i^{\gamma''} = \neg\neg x_i^*$.

In any combination of x_i^β and x_i^γ, we can represent that $\beta = \beta' x_i^\beta$ and $\gamma = \gamma' x_i^\gamma$, and $\beta'\gamma'$ is a base of a consensus of β and γ. Moreover, it is evident from the above discussion that $\alpha \sqsubseteq \beta'\gamma'$ since $\alpha = x_1^\alpha \cdot \ldots \cdot x_n^\alpha \sqsubseteq x_1^\beta \cdot \ldots \cdot x_{i-1}^\beta x_{i+1}^\beta \cdot \ldots \cdot x_n^\beta = \beta'$ and $\alpha = x_1^\alpha \cdot \ldots \cdot x_n^\alpha \sqsubseteq x_1^\gamma \cdot \ldots \cdot x_{i-1}^\gamma x_{i+1}^\gamma \cdot \ldots \cdot x_n^\gamma = \gamma'$ stand true. This completes the proof of the lemma. ■

6.5.3 Algorithm for Deriving a Minimal Form

Theorem 12 *Let $F = \alpha_1 \vee \ldots \vee \alpha_s$ be a disjunctive form of a Kleene-Stone logic function. Then, F is the disjunction of all prime implicants of F if and only if*

(1) no term includes any other term, that is, $\alpha_i \not\sqsubseteq \alpha_j$ for any i and j (i, j=1, …, s and $i \neq j$), and

(2) the consensus of any two product terms does not exist or is included in other term α_i (i=1, …, s).

Proof: It is evident that (1) and (2) hold when $F = \alpha_1 \vee \ldots \vee \alpha_s$ is the disjunction of all prime implicants of F. Conversely, let assume $F = \alpha_1 \vee \ldots \vee \alpha_s$ is not a disjunction of all prime implicants of F when it holds (1) and (2). This means one of the following relations holds.

(a) A prime implicant α of F does not appear in $\alpha_1, \ldots, \alpha_s$, or

(b) A product term α_i ($i = 1, \ldots, s$) is not a prime implicant of F.

In the following x_i^* denotes one of x_i or $\sim x_i$, and x_i' and x_i'' each represents a literal of x_i.

First assume (a) holds, and suppose α is a type A product term. If $\alpha \sqsubseteq \neg\neg x_i$ and $\alpha \sqsubseteq \neg\neg \sim x_i$ hold for any i ($i = 1, \ldots, n$), then α has to be a type 1 or a type 2 minterm. Accordingly by Lemma 15 α is an essential one of F, and this contradicts to the assumption (a). Moreover, if $\alpha \sqsubseteq \neg x_i^*$ for any i ($i = 1, \ldots, s$) such that $\alpha \not\sqsubseteq \neg\neg x_i^*$, then in the similar manner α is a type 1 or 2 minterm, and this also contradicts to (a). Therefore, there is some variable x_i such that $\alpha \not\sqsubseteq \neg\neg x_i^*$ and $\alpha \not\sqsubseteq \neg x_i^*$. Thus, it is possible to add some literals $\neg\neg x_i^*$ or $\neg x_i^*$ into α if $\alpha \not\sqsubseteq \neg\neg x_i^*$ and $\alpha \not\sqsubseteq \neg x_i^*$ hold, and then we can construct a product term α' satisfying the following two conditions.

Condition 1: $\alpha' \not\sqsubseteq \alpha_i$ for any i ($i = 1, \ldots, s$).

Condition 2: For any possible x_i, that is, $\alpha' \not\sqsubseteq \neg\neg x_i^*$ and $\alpha' \not\sqsubseteq \neg x_i^*$, there are α_j and α_k such that $\alpha' \neg\neg x_i^* \sqsubseteq \alpha_j$ and $\alpha' \neg x_i^* \sqsubseteq \alpha_k$ where $j, k = 1, \ldots, s$ and $j \neq k$.

By Lemma 14 and Condition 1 it is impossible that α' is a type 1 or a type 2 minterm. Thus, in the similar manner, $\alpha \not\sqsubseteq \neg\neg x_m^*$ and $\alpha \not\sqsubseteq \neg x_m^*$ hold for some variable x_m, and by Condition 2 there exist α_j and α_k such that $\alpha' \neg\neg x_m^* \sqsubseteq \alpha_j$ and $\alpha' \neg x_m^* \sqsubseteq \alpha_k$. From Lemma 16 it is possible to represent that $\alpha_j = x_m' \alpha_j'$ and $\alpha_k = x_q'' \alpha_k'$ where $q = 1, \ldots, n$, and $\alpha_j' \alpha_k'$ is a consensus of α_j and α_k, and we have $\alpha' \sqsubseteq \alpha_j' \alpha_k'$. By the theorem (2) $\alpha_j' \alpha_k'$ has to be included in other product term α_l ($l = 1, \ldots, s$), that is, $\alpha' \sqsubseteq \alpha_j' \alpha_k' \sqsubseteq \alpha_l$, and this contradicts to Condition 1. Therefore, a consensus $\alpha_j' \alpha_k'$ is never included in any other product term α_l ($l = 1, \ldots, s$), and this contradicts to the theorem (2). Accordingly every prime implicant of F which is a type A product term has to appear among $\alpha_1, \ldots, \alpha_s$.

Next suppose α is a type B product term, that is, there is a product $x_q \sim x_q$ in α for some variable x_q ($q = 1, \ldots, n$). Then one of the following relations holds for some variable x_i.

$$(i)\ \alpha \not\sqsubseteq x_i \text{ and } \alpha \not\sqsubseteq\sim x_i, \text{ or } \quad (ii)\ \alpha \sqsubseteq x_i^* \text{ and } \alpha \not\sqsubseteq\sim x_i^*.$$

Because, if $\alpha \sqsubseteq x_i$ and $\alpha \sqsubseteq\sim x_i$ hold for any i, that is, $\alpha \sqsubseteq x_i \sim x_i$, then α is a type 3 minterm constructed by only $x_i \sim x_i$ for any i, and this contradicts to Lemma 12.4. Let $\alpha = x_1^\alpha \cdot \ldots \cdot x_n^\alpha$ where each x_i^α denotes an atom of x_i. In the case (ii), if $x_i^\alpha \neq x_i^*$ holds for any i $(i \neq q)$, then α has to be a type 3 minterm. This contradicts to Lemma 12.4. Therefore, we can modify (ii) below.

(ii) $x_i^\alpha = x_i^*$ for some variable x_i.

Accordingly, it is possible to add some literals x_i or $\sim x_i$ for the variable x_i satisfying (i) and also add some literal $\neg\neg \sim x_i^*$ or $\neg \sim x_i^*$ for the variable x_i satisfying (ii). Then we can construct a product term α' holding the following conditions.

Condition 3: $\alpha \not\sqsubseteq \alpha'$ for any i $(i = 1, \ldots, s)$.

Condition 4-1: In the case (i), for any possible x_i, that is, $\alpha \not\sqsubseteq x_i$ and $\alpha \not\sqsubseteq\sim x_i$, there are α_j and α_k such that $\alpha' x_i \sqsubseteq \alpha_j$ and $\alpha' \sim x_i \sqsubseteq \alpha_k$ where $j, k = 1, \ldots, s$ and $j \neq k$.

Condition 4-2: In the case (ii), for any possible x_i, that is, $x_i^\alpha = x_i^*$, there are α_j and α_k such that $\alpha' \neg\neg \sim x_i^* \sqsubseteq \alpha_j$ and $\alpha' \neg \sim x_i^* \sqsubseteq \alpha_k$ where $j, k = 1, \ldots, s$ and $j \neq k$.

By Corollary 2 and Condition 3 it is impossible α' is a type 3 minterm, and therefore there is some variable x_i satisfying at least one of (i) or (ii). When x_i satisfies (i), then by Condition 4-1 there exist α_j and α_k such that $\alpha' x_i \sqsubseteq \alpha_j$ and $\alpha' \sim x_i \sqsubseteq \alpha_k$. Moreover by Lemma 12.6 α_j and α_k have to be represented as $\alpha'_j x'_i$ and $\alpha'_k x''_i$, respectively, and also $\alpha' \sqsubseteq \alpha'_j \alpha'_k$. Therefore, $\alpha'_j \alpha'_k$ or $\alpha'_j \alpha'_k x_q \sim x_q$ are consensus of α_j and α_k. By the theorem (2) $\alpha'_j \alpha'_k$ or $\alpha'_j \alpha'_k x_q \sim x_q$ has to be included in other product term α_l $(l = 1, \ldots, s)$, that is, $\alpha' \sqsubseteq \alpha'_j \alpha'_k \sqsubseteq \alpha_l$ or $\alpha' \sqsubseteq \alpha'_j \alpha'_k x_q \sim x_q \sqsubseteq \alpha_l$. This contradicts to Condition 3. Next, when x_i satisfies (ii), then by Condition 4-2 there exist α_j and α_k such that $\alpha \neg\neg \sim x_i^* \sqsubseteq \alpha_j$ and $\alpha' \neg \sim x_i^* \sqsubseteq \alpha_k$. Moreover, α_j and α_k have to be represented as $\alpha'_j x'_i$ and $\alpha'_k x''_i$, respectively, $\alpha' \sqsubseteq \alpha'_j \alpha'_k$ from Lemma 12.5. Therefore, $\alpha'_j \alpha'_k$ is a consensus of α_j and α_k. By the theorem (2) $\alpha'_j \alpha'_k$ has to be included in other product term α_l $(l = 1, \ldots, s)$, that is, $\alpha' \sqsubseteq \alpha'_j \alpha'_k \sqsubseteq \alpha_l$. This contradicts to Condition 3.

From the above, if α does not appear in $\alpha_1, \ldots, \alpha_s$, then it is possible to construct a consensus from some product terms appearing in $\alpha_1, \ldots, \alpha_s$ and the consensus never include in any other product term α_i $(i = 1, \ldots, s)$. This, however, contradicts to the theorem (2), and accordingly all of the prime implicants of F have to be appear among $\alpha_1, \ldots, \alpha_s$.

Next assume (b) holds, then from the above discussion all prime implicants of F appear in $\alpha_1, \ldots, \alpha_s$, and therefore there is a prime implicant α_j such that $\alpha_i \sqsubseteq \alpha_j$ and $i \neq j$. This, however, contradicts to the theorem (1). This completes the proof of the theorem. ■

From the above theorem we have the following algorithm to find all prime implicants of a given Kleene-Stone logic function F.

Algorithm A

Step 1: Expand F into a disjunctive form.

Step 2: Remove any product term included in other product term, and let $\alpha_1, \ldots \alpha_s$ be the remaining product terms.

Step 3: Find all of the consensus of any two product terms α_i and α_j $(i, j = 1, \ldots, s$ and $i \neq j)$. If no consensus exists, then in Step 5, otherwise in Step 4.

Step 4: Construct the disjunctive form from $\alpha_1, \ldots \alpha_s$ and all of the consensus getting in Step 3. If any consensus is included in one of $\alpha_1, \ldots \alpha_s$ then in Step 5, otherwise in Step 2.

Step 5: The remaining product terms are all of the prime implicants of F.

Theorem 13 *Let $\alpha_1 \vee \ldots \vee \alpha_s$ be the canonical disjunctive form of a Kleene-Stone logic function F, and let $\beta_1 \vee \ldots \vee \beta_t$ be a minimal form of F. Then, each minterm α_i $(i = 1, \ldots, s)$ is included in a product term β_j $(j = 1, \ldots, t)$.*

Proof: First suppose α_i is a type 1 minterm. Then $\alpha_i(\boldsymbol{a}) = 1$ for the corresponding element $\boldsymbol{a}$ to α_i, and therefore $F(\boldsymbol{a}) = 1$. This implies that there is a product term β_j such that $\beta_j(\boldsymbol{a}) = 1$. Accordingly by Lemma 2.1 we obtain $\alpha_i \sqsubseteq \beta_j$. Next suppose α_i is a type 2 minterm. Then $\alpha_i(\boldsymbol{a}) = 3/4$ for the corresponding element $\boldsymbol{a}$ to α_i, and therefore by Lemma 7 $F(\boldsymbol{a}) = 3/4$. This implies that there is a product term β_j such that $\beta_j(\boldsymbol{a}) = 3/4$, and accordingly by Lemma 2.2 we have $\alpha_i \sqsubseteq \beta_j$. Finally, suppose α_i is a type 3 minterm. Then $\alpha_i(\boldsymbol{a}) = 1/2$ for the corresponding element $\boldsymbol{a}$ to α_i, and therefore by Lemma 7 $F(\boldsymbol{a}) = 1/2$. This implies that there is a product term β_j such that $\beta_j(\boldsymbol{a}) = 1/2$, and thus we have $\alpha_i \sqsubseteq \beta_j$ by Lemma 2.3. From the above, we have been shown the proof of the theorem. ■

From the above theorem we have the algorithm to find a minimal form of a Kleene-Stone logic function F.

Algorithm B

Step 1: Expand F into the canonical disjunctive form, and let $\alpha_1, \ldots, \alpha_s$ be minterms appearing in the canonical disjunctive form.

Step 2: Find all prime implicants $\beta_1, \ldots, \beta_t$ of F by applying Algorithm A.

Step 3: Find a minimal group $\beta_{i_1}, \ldots, \beta_{i_k}$ $(i_j = 1, \ldots, t$ and $1 \leq j \leq k \leq t)$ of prime implicants such that each α_i $(i = 1, \ldots, s)$ is included in a prime implicant β_{i_j}, then $\beta_{i_1} \vee \ldots \vee \beta_{i_k}$ is a minimal form of F.

Step 3 of Algorithm B corresponds to the minimum covering problem for Boolean functions. Therefore, Step 3 is solved by using a prime implicants table like Boolean functions.

Example 10 *Let F be a 2-variable Kleene-Stone logic function such that*

$$\begin{aligned} F &= x_1\neg\neg \sim x_1 \sim x_2\neg\neg x_2 \vee \neg x_1 x_2\neg\neg \sim x_2 \vee \neg x_1\neg \sim x_2 \vee \neg \sim x_1\neg\neg x_2\neg\neg \sim x_2 \vee \\ &\quad \neg\neg x_1\neg\neg \sim x_1\neg x_2 \vee \neg\neg x_1\neg\neg \sim x_1 x_2\neg\neg \sim x_2 \end{aligned}$$

which is the canonical disjunctive form. By Algorithm A, *we have all of the prime implicants of F denoted below.*

$$x_1 \sim x_2\neg\neg x_2,\ \neg x_1 x_2,\ \neg \sim x_1\neg\neg x_2\neg\neg \sim x_2,\ \neg\neg x_1\neg\neg \sim x_1\neg x_2,$$
$$x_1\neg\neg \sim x_1 \sim x_2,\ x_2\neg\neg \sim x_2,\ x_1 \sim x_1\neg\neg \sim x_2$$

Table 6.3: Prime Implicant Table of Example 10

	α_1	α_2	α_3	α_4	α_5	α_6	α_7
β_1	$\surd$				$\surd$		
β_2		$\surd$				$\surd$	
β_3		$\surd$					
β_4			$\surd$				
β_5				$\surd$			
β_6						$\surd$	

Table 6.4: The Correspondence between Each α_i and Prime Implicant of F, and between Each β_j and Minterm of the Canonical Disjunctive Form of Example 10

α_1	$x_1 \sim x_2\neg\neg x_2$	β_1	$x_1\neg\neg x_1 \sim x_2\neg\neg x_2$
α_2	$\neg x_1 x_2$	β_2	$\neg x_1 x_2 \neg\neg \sim x_2$
α_3	$\neg \sim x_1\neg\neg x_2\neg\neg \sim x_2$	β_3	$\neg x_1 \neg \sim x_2$
α_4	$\neg\neg x_1\neg\neg \sim x_1\neg x_2$	β_4	$\neg \sim x_1\neg\neg x_2\neg\neg \sim x_2$
α_5	$x_1\neg\neg \sim x_1 \sim x_2$	β_5	$\neg\neg x_1\neg\neg \sim x_1\neg x_2$
α_6	$x_2\neg\neg \sim x_2$	β_6	$\neg\neg x_1\neg\neg \sim x_1 x_2\neg\neg \sim x_2$
α_7	$x_1 \sim x_1\neg\neg \sim x_2$		

Then we have the following two minimal forms of F by finding minimal group from Table 6.2 and Table 6.3.

$$F = \neg x_1 x_2 \vee \neg \sim x_1\neg\neg x_2\neg\neg \sim x_2 \vee \\ x_2\neg\neg \sim x_2 \vee \left\{ \begin{array}{c} x_1 \sim x_2\neg\neg x_2 \\ x_1\neg\neg \sim x_1 \sim x_2 \end{array} \right\}$$

The following example shows that an existence of a minimal form appearing a product term besides the prime implicants of a given Kleene-Stone logic function F.

Example 11 *Let consider the following 2-variable Kleene-Stone logic function F.*

$$F = \neg x_1\neg x_2 \vee \sim x_1\neg\neg x_1\neg\neg \sim x_2$$

Then we get the following disjunctive form F' appearing every prime implicant of F.

$$F' = \sim x_1\neg x_2 \vee \sim x_1\neg\neg x_1\neg\neg \sim x_2$$

Here F' is a minimal form of F and you can easily verify that F also is minimal. However, the first product term $\neg x_1\neg x_2$ of F is not a prime implicant, since $\neg x_1\neg x_2 \sqsubseteq \sim x_1\neg x_2$ and $\neg x_1\neg x_2 \neq \sim x_1\neg x_2$. Therefore, this example shows the existence of a minimal form appearing besides the prime implicants of F.

6.6 Number of n-Variable Kleene-Stone Logic Functions

The number of n-variable Kleene-Stone logic functions is same as that of n-variable 5-valued Kleene-Stone logic functions from Theorem 3. Therefore, we discuss the number of n-variable 5-valued Kleene-Stone logic functions in the section.

Definition 13 *A relation $\equiv_{KS}$ on the set V_5 is defined as follows.*

$a \equiv_{KS} b$ whenever $a, b \in \{1/4, 1/2, 3/4\}$ and

$i \equiv_{KS} i$ where $i \in V_5$.

The relation $\equiv_{KS}$ is expanded into V_5^n as follows. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V_5^n. Then $\boldsymbol{a} \equiv_{KS} \boldsymbol{b}$ if and only if $a_i \equiv_{KS} b_i$ for every $i = 1, \ldots n$.

It is clear that the relation $\equiv_{KS}$ is an equivalence relation. Therefore, we can define the equivalence class $[\boldsymbol{c}]_{\equiv_{KS}}$ of $\boldsymbol{c} \in V_5^n$ by $[\boldsymbol{c}]_{\equiv_{KS}} = \{\boldsymbol{c}' \in V_5^n \mid \boldsymbol{c} \equiv_{KS} \boldsymbol{c}'\}$, and then obviously $a_i = 0(1)$ if and only if $b_i = 0(1)$ for any $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of $[\boldsymbol{c}]_{\equiv_{KS}}$. Moreover, each $[\boldsymbol{c}]_{\equiv_{KS}}$ forms a finite partial order set under $\leq_{KS}$, and it is sure there exists a maximum element in $[\boldsymbol{c}]_{\equiv_{KS}}$, which is also an element of V_3^n. Thus, the relation $\bigcup_{\boldsymbol{c} \in V_3^n} [\boldsymbol{c}]_{\equiv_{KS}} = V_5^n$ holds. Therefore, we can choose the element of V_3^n as the representative of each equivalence class $[\boldsymbol{c}]_{\equiv_{KS}}$. Obviously, if $\boldsymbol{c}_1, \boldsymbol{c}_2 \in V_3^n$ such that $\boldsymbol{c}_1 \neq \boldsymbol{c}_2$, then $[\boldsymbol{c}_1]_{\equiv_{KS}} \cap [\boldsymbol{c}_2]_{\equiv_{KS}} = \emptyset$.

Example 12 *The set V_5^2 is classified nine different kinds of equivalence classes, and some of them, for example $[(0,1)] = \{(0,1)\}$, $[(0,1/2)] = \{(0,1/4), (0,1/2), (0,3/4)\}$ and $[(1/2,1/2)] = \{(1/4,1/4), (1/4,1/2), (1/4,3/4), (1/2,1/4), (1/2,1/2), (1/2,3/4), (3/4,1/4), (3/4,1/2), (3/4,3/4)\}$. Figure 6.5 and 6.6 each shows Hasse diagram of $[(0,1/2)]$ and $[(1/2,1/2)]$, respectively.*

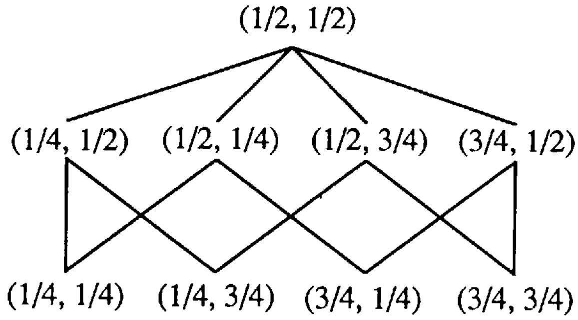

Figure 6.5: Hasse Diagram of $[(0,1/2)]$ of Example 12

Recall that the set of Conditions $(a) \sim (d)$ in Section 6.4 is a necessary and sufficient condition for a 5-valued function to be a 5-valued Kleene-Stone logic function. Among Conditions $(a) \sim (d)$, only Condition (d) describes the relation between the outputs of a 5-valued Kleene-Stone logic function for different kinds of inputs. Now, let $[\boldsymbol{c}_1]_{\equiv_{KS}}$ and $[\boldsymbol{c}_2]_{\equiv_{KS}}$ be equivalence classes such that $[\boldsymbol{c}_1]_{\equiv_{KS}} \neq [\boldsymbol{c}_2]_{\equiv_{KS}}$. Then, it is clear the definitions of each relation $\leq_K$ and $\equiv_{KS}$ that any $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, which are respectively elements of $[\boldsymbol{c}_1]_{\equiv_{KS}}$ and $[\boldsymbol{c}_2]_{\equiv_{KS}}$, are not comparable to each other under the relation $\leq_{KS}$. Then the antecedent of Condition (d) stands always false for

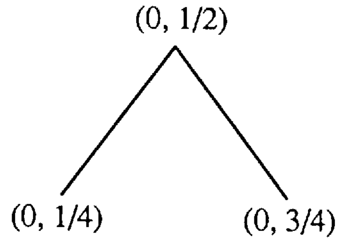

Figure 6.6: Hasse Diagram of $[(1/2, 1/2)]$ of Example 12

such two elements $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, and thus Condition (d) stands always true whether the consequence of (d) is true or false. Therefore, for any 5-valued Kleene-Stone logic function f_{KS}, there is not any relation between $f_{KS}(\boldsymbol{a}_1)$ and $f_{KS}(\boldsymbol{a}_2)$ if $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are respectively elements of $[\boldsymbol{c}_1]_{\equiv_{KS}}$ and $[\boldsymbol{c}_2]_{\equiv_{KS}}$. From the above, we have the following formula concerning with the number of n-variable 5-valued Kleene-Stone logic functions.

$$|F_{KS}(n)| = \prod_{\boldsymbol{c} \in V_3^n} |F_{KS}([\boldsymbol{c}]_{\equiv_{KS}})| \tag{6.1}$$

In the equation (6.1), $F_{KS}(n)$ and $F_{KS}([\boldsymbol{c}]_{\equiv_{KS}})$ are defined below.

$$\begin{aligned} F_{KS}(n) &= \{f_{KS} \mid f_{KS} \text{ is an } n\text{-variable 5-valued Kleene-Stone logic function}\}, \\ F_{KS}([\boldsymbol{c}]_{\equiv_{KS}}) &= \{f_{KS}|[\boldsymbol{c}]_{\equiv_{KS}} \mid f_{KS} \in F_{KS}(n)\}, \end{aligned}$$

where $f_{KS}|[\boldsymbol{c}]_{\equiv_{KS}}$ is the restriction of f_{KS} to the equivalence class $[\boldsymbol{c}]_{\equiv_{KS}}$.

The set $F_{KS}([\boldsymbol{c}]_{\equiv_{KS}})$ has a connection with B-ternary logic functions discussed in Chapter 3. Now, we show again the definition of B-ternary logic functions in the terms of the relation $\prec$.

An n-variable B-ternary logic function f_B is a mapping $f_B : V_3^n \to V_3$ satisfying the following two conditions.

(B1) $\boldsymbol{a} \in V_2^n$ implies $f_B(\boldsymbol{a}) \in V_2$

(B2) $\boldsymbol{a}, \boldsymbol{b} \in V_3^n$ and $\boldsymbol{a} \prec \boldsymbol{b}$ imply $f_B(\boldsymbol{a}) \prec f_B(\boldsymbol{b})$

where the notation $\prec$ implies a partial order relation on V_3 defined as follows.

$$0 \prec 1/2,\ 1 \prec 1/2 \text{ and } i \prec i,$$

where $i \in V_3$. The relation $\prec$ is available among V_3^n by defining as follows. For two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of V_3^n, $\boldsymbol{a} \prec \boldsymbol{b}$ if and only if $a_i \prec b_i$ for every $i = 1, \ldots, n$.

In the following, we discuss the relationship between 5-valued Kleene-Stone logic functions and B-ternary logic functions. In the following, $F_B(n)$ denotes the set of all n-variable B-ternary logic functions, that is, $F_B(n) = \{f_B \mid f_B \text{ is an } n\text{-variable B-ternary logic function}\}$

Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V_3^n. Then a mapping $\lambda_i : V_3^n \to V_2$ is defined by the following manner.

$$\lambda_i(\boldsymbol{a}) = \begin{cases} 1 & \text{if } a_i = 1/2 \\ 0 & \text{if } a_i = 0 \text{ or } 1 \end{cases}$$

Obviously, $|[\mathbf{c}]_{\equiv_{KS}}| = 3^k$ for any element $\mathbf{c} \in V_3^n$ where $k = \sum_{i=1}^{n} \lambda_i(\mathbf{c})$. Therefore, the number of all elements of the set $[\mathbf{c}]_{\equiv_{KS}}$ and that of V_3^k is equivalent, that is, $|[\mathbf{c}]_{\equiv_{KS}}| = |V_3^k|$. Then, next, we define a mapping φ_c as follows.

Definition 14 *Let* $\mathbf{c}$ *be an element of* $V_3^n - V_2^n$ *and* $k = \sum_{i=1}^{n} \lambda_i(\mathbf{c})$. *Then a mapping* $\varphi_c : [\mathbf{c}]_{\equiv_{KS}} \to V_3^k$ *is defined as* $\varphi_c(\mathbf{a}) = (a'_1, \ldots, a'_k) \in V_3^k$ *for every* $\mathbf{a} \in [\mathbf{c}]_{\equiv_{KS}}$, *which satisfies the following condition for any* $j = 1, \ldots, k$,

$$a'_j = \begin{cases} 0 & \textit{if and only if } a_{i_j} = 1/4 \\ 1/2 & \textit{if and only if } a_{i_j} = 1/2 \\ 1 & \textit{if and only if } a_{i_j} = 3/4 \end{cases}$$

where the set $\{i_1, \ldots, i_k\}$ $(1 \leq i_1 < i_2 < \ldots < i_k \leq n)$ *is a subset of* $\{1, \ldots, n\}$ *such that* $a_{i_j} \neq 0$ *and* 1.

It is evident that φ_c is one-to-one and onto mapping, and $\mathbf{a} \leq_{\mathrm{KS}} \mathbf{b}$ if and only if $\varphi_c(\mathbf{a}) \prec \varphi_c(\mathbf{b})$. Therefore, the finite order set $[\mathbf{c}]_{\equiv_{KS}}$ with $\leq_{\mathrm{KS}}$ and the finite order set V_3^k with $\prec$ are isomorphic, that is, φ_c is an order isomorphism between $[\mathbf{c}]_{\equiv_{KS}}$ and V_3^k.

Example 13 *Let* $\mathbf{c} = (0, 1/2, 1)$, *then* $[\mathbf{c}]_{\equiv_{KS}} = \{(0, 1/4, 1), (0, 1/2, 1), (0, 3/4, 1)\}$. *Table 6.5 shows the mapping* φ_c, *and Figure 6.7 and 6.8 show Hasse diagrams of* $[\mathbf{c}]_{\equiv_{KS}}$ *under* $\leq_{\mathrm{KS}}$ *and* $\varphi_c([\mathbf{c}]_{\equiv_{KS}}) = \{\varphi_c(\mathbf{a}) \mid \mathbf{a} \in [\mathbf{c}]_{\equiv_{KS}}\}$, *under* $\prec$, *respectively.*

Table 6.5: Mapping φ_c of Example 13

$\mathbf{a} \in [\mathbf{c}]_{\equiv_{KS}}$	$\varphi_c(\mathbf{a})$
$(0, 1/4, 1)$	0
$(0, 1/2, 1)$	1/2
$(0, 3/4, 1)$	1

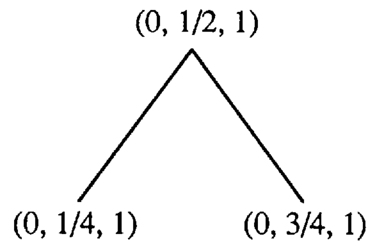

Figure 6.7: Hasse Diagram of $[c]_{\equiv_{KS}}$ in Example 13

Definition 15 *Let* $\mathbf{c}$ *be an element* $V_3^n - V_2^n$ *and* $k = \sum_{i=1}^{n} \lambda_i(\mathbf{c})$. *Then* $\xi_c : F_{KS}([\mathbf{c}]_{\equiv_{KS}}) \to F_3(k)$, *where* $F_3(k) = \{f_3 \mid f_3 : V_3^k \to V_3\}$, *is defined such a mapping that it satisfies the following conditions (1) and (2), where* $f_{KS} \in F_{KS}([\mathbf{c}]_{\equiv_{KS}})$ *and* $f_3 = \xi_c(f_{KS})$ *below.*

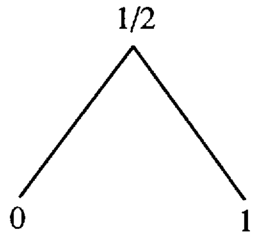

Figure 6.8: Hasse Diagram of $\varphi_c([c]_{\equiv_{KS}})$in Example 13

(1) $f_{KS}(\boldsymbol{a}) = 0(1)$ *for some* $\boldsymbol{a} \in [c]_{\equiv_{KS}}$ *if and only if* $f_3(\boldsymbol{a}') = 0(1)$ *for every element* $\boldsymbol{a}' \in V_3^k$*, or*

(2) If $f_{KS}(\boldsymbol{a}) \neq 0$ *and* 1 *for some* $\boldsymbol{a} \in [c]_{\equiv_{KS}}$*, then for every* $\boldsymbol{a} \in [c]_{\equiv_{KS}}$

(a) $f_3(\boldsymbol{a}') = 0$ *if and only if* $f_{KS}(\boldsymbol{a}) = 1/4$*,*

(b) $f_3(\boldsymbol{a}') = 1/2$ *if and only if* $f_{KS}(\boldsymbol{a}) = 1/2$ *and*

(c) $f_3(\boldsymbol{a}') = 1$ *if and only if* $f_{KS}(\boldsymbol{a}) = 3/4$*,*

where $\boldsymbol{a}' = \varphi_c(\boldsymbol{a})$.

Note that let f_{KS} be 5-valued Kleene-Stone logic function, then $f_{KS}(\boldsymbol{a}) = 0(1)$ for some $\boldsymbol{a} \in [c]_{\equiv_{KS}}$ if and only if $f_{KS}(\boldsymbol{a}) = 0(1)$ for all elements $\boldsymbol{a} \in [c]_{\equiv_{KS}}$ from Condition (d).

The above definition is well defined and it is easy verification that ξ_c is a one-to-one mapping. Let $f_{KS} \in F_{KS}([c]_{\equiv_{KS}})$ and $\boldsymbol{a}, \boldsymbol{b} \in [c]_{\equiv_{KS}}$, then it is also evident that $f_{KS}(\boldsymbol{a}) \leq_{\text{KS}} f_{KS}(\boldsymbol{b})$ if and only if $\xi_c(\boldsymbol{a}') \prec \xi_c(\boldsymbol{b}')$, where $\boldsymbol{a}' = \varphi(\boldsymbol{a})$ and $\boldsymbol{b}' = \varphi(\boldsymbol{b})$. Therefore, ξ_c preserves the monotonicity relation. Each associated function $\xi_c(f_{KS})$ is a B-ternary logic function since φ_c is an order isomorphism between $[c]_{\equiv_{KS}}$ and V_3^k and

(a') $\boldsymbol{a} \in V_2^n \cap [c]_{\equiv_{KS}}$ implies $f_{KS}(\boldsymbol{a}) \in V_2$

(b') $\boldsymbol{a} \in V_3^n \cap [c]_{\equiv_{KS}}$ implies $f_{KS}(\boldsymbol{a}) \in V_3$

(c') $\boldsymbol{a} \in V_4^n \cap [c]_{\equiv_{KS}}$ implies $f_{KS}(\boldsymbol{a}) \in V_4$

(d') $\boldsymbol{a}, \boldsymbol{b} \in [c]_{\equiv_{KS}}$ and $\boldsymbol{a} \leq_{\text{KS}} \boldsymbol{b}$ imply $f_{KS}(\boldsymbol{a}) \leq_{\text{KS}} f_{KS}(\boldsymbol{b})$

are satisfied for every $f_{KS} \in F_{KS}([c]_{\equiv_{KS}})$. Moreover, obviously there is a function $f_{KS} \in F_{KS}([c]_{\equiv_{KS}})$ for every B-ternary logic function f_B such that $\xi_c(f_{KS}) = f_B$. From the above, the image of ξ_c, denoted by $\xi_c(F_{KS}([c]_{\equiv_{KS}}))$, is equivalent to $F_B(k)$, i.e., $\xi_c(F_{KS}([c]_{\equiv_{KS}})) = F_B(k)$. Since ξ_c is one-to-one, the relation $|F_{KS}([c]_{\equiv_{KS}})| = |\xi_c(F_{KS}([c]_{\equiv_{KS}}))|$ stands always true, and therefore, we can conclude

$$\left|F_{KS}([c]_{\equiv_{KS}})\right| = \left|F_B(k)\right| \tag{6.2}$$

(refer to Figure 6.9)

Example 14 *Let f be a 2-variable 5-valued Kleene-Stone logic function represented by* $f = x_1\neg\neg \sim x_1x_2 \sim x_2 \vee \sim x_1\neg\neg \sim x_1x_2\neg\neg \sim x_2$ *and let* $\mathbf{c} = (1/2, 1/2)$*. Then, Table 6.6 shows truth table of f, and Table 6.7 shows truth table of 2-variable B-ternary logic function* $\xi_c(f \mid [c]_{\equiv_{KS}})$.

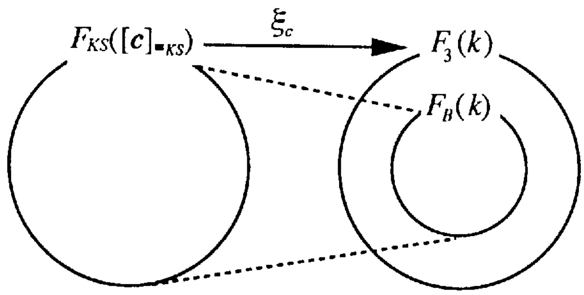

Figure 6.9: Relationship between $F_{KS}([c]_{\equiv_{KS}})$ and $F_B(k)$

Table 6.6: Truth Table of 5-Valued Kleene-Stone Logic Function of Example 14

x_1 / x_2	0	1/4	1/2	3/4	1
0	0	0	0	0	0
1/4	0	1/4	1/2	3/4	0
1/2	0	1/4	1/2	1/2	0
3/4	0	1/4	1/2	1/4	0
1	0	0	0	0	0

Finally, from the equations (6.1), (6.2) and

$$\left|\left\{c \in V_3^n \mid \sum_{i=1}^{n} \lambda_i(c) = 1/2\right\}\right| =_n C_k \times 2^{n-k} \tag{6.3}$$

we have the following equation concerning with the number of n-variable 5-valued Kleene-Stone logic functions.

$$\begin{aligned} \left|F_{KS}(n)\right| &= \prod_{c \in V_2^n} \left|F_{KS}([c]_{\equiv_{KS}})\right| \times \prod_{c \in V_3^n - V_2^n} \left|F_{KS}([c]_{\equiv_{KS}})\right| \\ &= \prod_{c \in V_2^n} \left|F_{KS}([c]_{\equiv_{KS}})\right| \times \prod_{k=1}^{n} \left|F_B(k)\right|^{{}_nC_k \times 2^{n-k}} \end{aligned} \tag{6.4}$$

In the above equation (6.4), for any $c \in V_2^n$, $F_{KS}([c]_{\equiv_{KS}}) = \{0, 1\}$ because the constants 0 and 1 are only possible functions for the restriction $f_{KS}|[c]_{\equiv_{KS}}$, where $f_{KS} \in F_{KS}(n)$ by Condition

Table 6.7: Truth Table of B-Ternary Logic Function of Example 14

x_1 / x_2	0	1/2	1
0	0	1/2	1
1/2	0	1/2	1/2
1	0	1/2	0

$(a') \sim (d')$. Therefore, we have $\prod_{c \in V_2^n} \left|F_{KS}([c]_{\equiv_{KS}})\right| = 2^{2^n}$. The number 2^{2^n} is equivalent to the number of all n-variable Boolean functions, and therefore, the equation (4) can be modified as

$$\left|F_{KS}(n)\right| = \left|B(n)\right| \times \prod_{k=1}^{n} \left|F_B(k)\right|^{{}_nC_k \times 2^{n-k}}, \tag{6.5}$$

where $B(n)$ denotes the set of all n-variable Boolean functions. Also in the equation (6.4), $\left|F_{KS}([c]_{\equiv_{KS}})\right| = \left|\{0,1\}\right| = \left|F_B(0)\right|$, where $c \in V_2^n$, because 0-variable B-ternary logic functions are nothing but the constants 0 and 1 from the definition of B-ternary logic functions (refer to Chapter 3). Therefore, we also have the following equation instead of the equation (6.4),

$$\begin{aligned} \left|F_{KS}(n)\right| &= \left|F_B(0)\right|^{2^n} \times \prod_{k=1}^{n} \left|F_B(k)\right|^{{}_nC_k \times 2^{n-k}} \\ &= \prod_{k=0}^{n} \left|F_B(k)\right|^{{}_nC_k \times 2^{n-k}} \end{aligned} \tag{6.6}$$

Especially, we obtained $\left|F_{KS}(1)\right| = \left|B(1)\right| \times \left|F_B(1)\right| = 2^{2^1} \times 6 = 24$. Table 6.8, 6.9 and 6.10 show truth tables of 1-variable Boolean functions, 1-variable B-ternary logic functions and 1-variable 5-valued Kleene-Stone logic functions, and Figure 6.10, 6.11 and 6.12 show the lattice structures of them, respectively.

From the equation (6.6), we have the accurate number of n-variable 5-valued Kleene-Stone logic functions until $n \leq 4$ by using the number of n-variable B-ternary logic functions indicated in Table 6.11. This table is referred by [3]. Therefore, we have the number of n-variable Kleene-Stone logic functions in the equation (6.4) (or (6.5), (6.6)) since Theorem 3 enables us to lead that the number of n-variable Kleene-Stone logic functions is equivalent to that of n-variable 5-valued Kleene-Stone logic functions.

Table 6.8: Truth Table of 1-Variable Boolean Functions

x	0	1
f_{b1}	1	1
f_{b2}	1	0
f_{b3}	0	1
f_{b4}	0	0

6.7 Conclusions

In the chapter, we defined the new class of infinite-valued functions called Kleene-Stone logic functions, and almost of the basic properties of them have been cleared as the results of the chapter.

In the discussions of minimization for previous multiple-valued logic functions, that is, fuzzy logic functions and Stone logic function etc., all minimal forms of a give function can determine by using the concept of prime implicants. That is, any minimal form of a given function can be represented by sum of only its prime implicants. However, in the case of Kleene-Stone logic

Table 6.9: Truth Table of 1-Variable B-Ternary Logic Functions

x	0	1/2	1
f_{k1}	1	1	1
f_{k2}	1	1/2	1
f_{k3}	1	1/2	0
f_{k4}	0	1/2	1
f_{k5}	0	1/2	0
f_{k6}	0	0	0

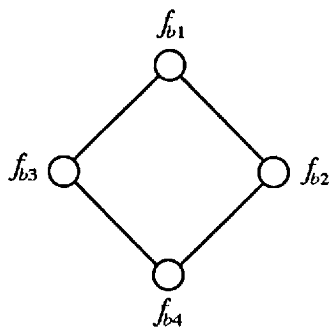

Figure 6.10: Lattice Structure of 1-Variable Boolean Functions

functions, it is not true that any minimal form of a given Kleene-Stone logic function can represent by sum of only prime implicants, that is, there is an example of minimal form involving a product term which is not prime implicant. Therefore, it is not true that we can find all of minimal forms for any given Kleene-Stone logic function every time in our algorithm. Finding a new algorithm which enables us to take all of minimal forms of any given Kleene-Stone logic functions is a future problem.

Table 6.10: Truth Table of 1-Variable 5-Valued Kleene-Stone Logic Functions

x	0	1/4	1/2	3/4	1	Logic Formula
f_1	1	1	1	1	1	$f_6 \vee f_{19}$
f_2	1	3/4	1/2	3/4	1	$f_6 \vee f_{20}$
f_3	1	3/4	1/2	1/4	1	$f_6 \vee f_{21}$
f_4	1	1/4	1/2	3/4	1	$f_6 \vee f_{22}$
f_5	1	1/4	1/2	1/4	1	$f_6 \vee f_{23}$
f_6	1	0	0	0	1	$\neg \sim x \vee \neg x$
f_7	1	1	1	1	0	$f_{12} \vee f_{19}$
f_8	1	3/4	1/2	3/4	0	$f_{12} \vee f_{20}$
f_9	1	3/4	1/2	1/4	0	$f_{12} \vee f_{21}$
f_{10}	1	1/4	1/2	3/4	0	$f_{12} \vee f_{22}$
f_{11}	1	1/4	1/2	1/4	0	$f_{12} \vee f_{23}$
f_{12}	1	0	0	0	0	$\neg x$
f_{13}	0	1	1	1	1	$f_{18} \vee f_{19}$
f_{14}	0	3/4	1/2	3/4	1	$f_{18} \vee f_{20}$
f_{15}	0	3/4	1/2	1/4	1	$f_{18} \vee f_{21}$
f_{16}	0	1/4	1/2	3/4	1	$f_{18} \vee f_{22}$
f_{17}	0	1/4	1/2	1/4	1	$f_{18} \vee f_{23}$
f_{18}	0	0	0	0	1	$\neg \sim x$
f_{19}	0	1	1	1	0	$\neg\neg \sim x \neg\neg x$
f_{20}	0	3/4	1/2	3/4	0	$(x \neg\neg x) \vee (\sim x \neg\neg x)$
f_{21}	0	3/4	1/2	1/4	0	$\sim x \neg\neg x$
f_{22}	0	1/4	1/2	3/4	0	$x \neg\neg \sim x$
f_{23}	0	1/4	1/2	1/4	0	$x \sim x$
f_{24}	0	0	0	0	0	$x \neg x$

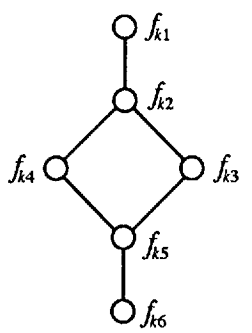

Figure 6.11: Lattice Structure of 1-Variable B-Ternary Functions

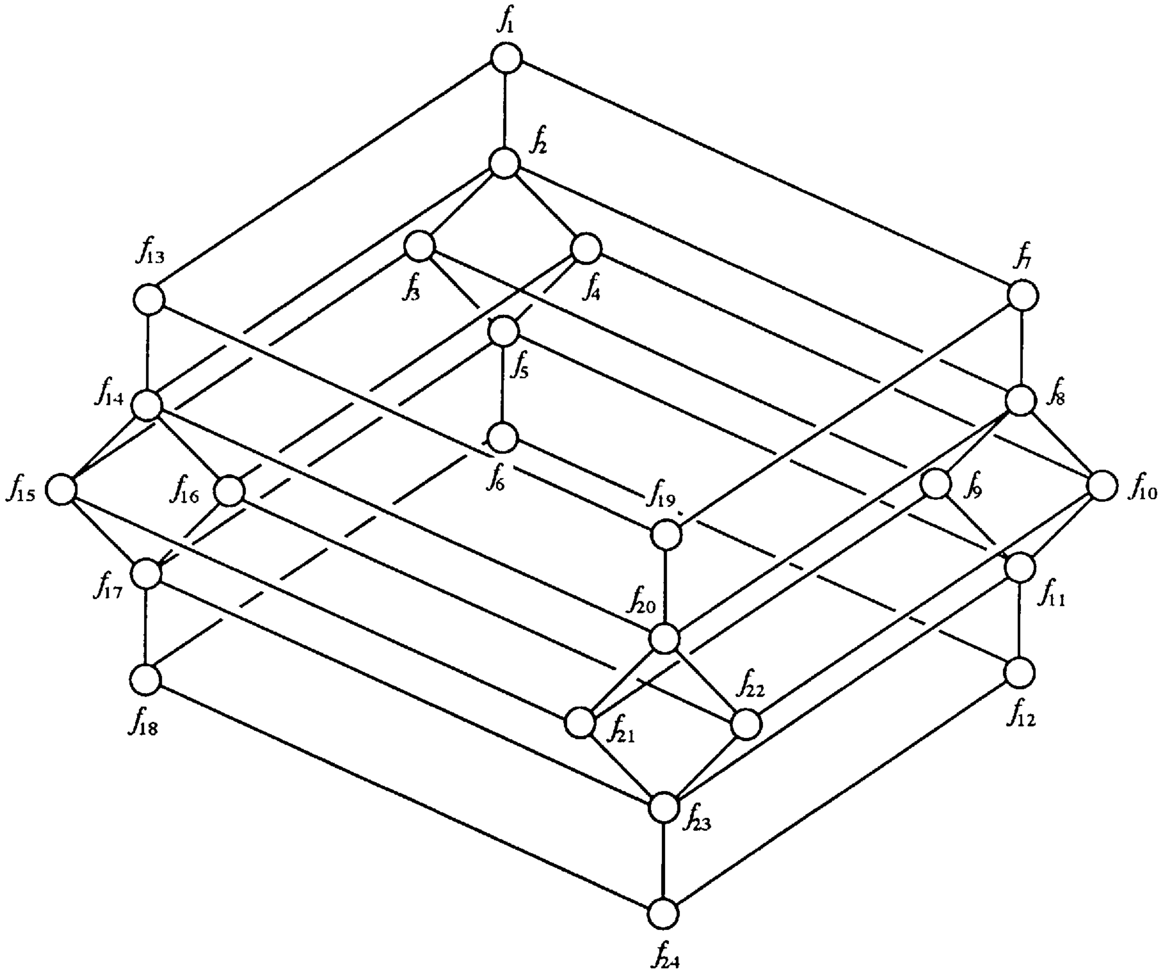

Figure 6.12: Lattice Structure of 1-Variable 5-Valued Kleene-Stone Functions

Table 6.11: The Number of B-Ternary Logic Functions

n	The number of B-ternary logic functions
0	2
1	6
2	84
3	43,918
4	160,297,985,276

Chapter 7

α-KS Logic Functions

7.1 Introduction

Kleene-Stone logic functions was discussed in Chapter 6, and they are employed two different kinds of unary operations $\sim$ and $\neg$. In the chapter, the unary operation $\neg$ is expanded into the following manner.

$$\neg x = \begin{cases} 1 & \text{if } x \leq \alpha \\ 0 & \text{if } x > \alpha \end{cases}$$

where the threshold α is in the range $0 < \alpha < 1/2$. This expanded Kleene-Stone logic function is called an α-Kleene-Stone logic function (α-KS logic function for short) in the chapter.

The set of each previous multiple-valued logic functions forms an algebraic system, that is, the set of B-ternary logic functions or fuzzy logic functions etc. each forms a Kleene-Stone algebra, and that of Stone logic functions and Kleene-Stone logic functions each forms a Stone algebra and a Kleene-Stone algebra, respectively. However, unfortunately we do not know an algebraic system which the set of α-KS logic functions forms, and obviously the system is different form a Kleene-Stone algebra, and of course, is different form a Kleene algebra and a Stone algebra.

We want to know the general properties do not depend on the range of the threshold α, that is, the properties of fuzzy logic functions with the unary operation $\neg$ defined below.

$$\neg x = \begin{cases} 1 & \text{if } x \leq \alpha \\ 0 & \text{if } x > \alpha \end{cases}$$

where $\alpha \in [0,1]$. However, for the present time, we have to classify the threshold α into the following five cases:

$$\alpha = 0,\ \ 0 < \alpha < 1/2,\ \ \alpha = 1/2,\ \ 1/2 < \alpha < 1 \text{ and } \alpha = 1.$$

If $\alpha = 0$, the functions become to Kleene-Stone logic functions, and if $0 < \alpha < 1$, then they become to α-KS logic functions. We think that if some properties of Kleene-Stone logic functions and α-KS logic functions are cleared, then it is easy to clear the properties of functions when $\alpha = 1/2$, $1/2 < \alpha < 1$ and $\alpha = 1$. These kinds of investigations can easily lead us to the studies of fuzzy logic functions with α-cut operator $cut_\alpha : [0,1] \to \{0,1\}$ defined below.

$$cut_\alpha(x) = \begin{cases} 1 & \text{if } x \geq \alpha \\ 0 & \text{if } x < \alpha \end{cases}$$

where $\alpha \in [0,1]$. The studies of fuzzy logic functions with α-cut operator cut_α is the future problem.

The chapter describes some of the following properties of α-KS logic functions. First, in Section 7.2, we give a definition of α-KS logic functions in the term of logic formulas. Then we define a partial order relation $\leq_\alpha$ and more three different types of partial order relations, denoted by the symbols $\leq_{\mathrm{B}}$, $\leq_{\mathrm{K}}$ and $\preceq_\alpha$, are defined for the aim of convenience of the discussions, then it is shown that any α-KS logic function is monotone for these four partial order relations. Also in the section, it is proved that any α-KS logic function is uniquely determined only 7-valued inputs. In Section 7.3, we show a canonical disjunctive form which allows for the unique representation of any α-KS logic function. α-KS logic functions are not functionally complete, that is, any function can not obtain by means of a logic formula. Finally, in Section 7.4, a necessary and sufficient condition for α-KS logic function is cleared in the term of the partial order relation $\leq_\alpha$.

7.2 α-KS Logic Functions and Their Properties

In this section, we define α-KS logic functions and show some of their properties.

7.2.1 Definition of α-KS Logic Functions

First, we give the definition of α-KS logic functions. Let V be the closed interval [0, 1]. An n-variable α-KS logic function is defined to be a mapping from V^n into V; $F : V^n \to V$, which can be represented by constants 0 and 1, and logic operations AND($\cdot$), OR($\vee$) and NOT($\sim$) which are identical with operations of fuzzy logic functions, and a unary operation($\neg$) controlled by a threshold α $(0 < \alpha < 1/2)$. These logic operations are defined as follows.

$$x \cdot y = \min(x,y), \qquad x \vee y = \max(x,y),$$
$$\sim x = 1 - x, \qquad \neg x = \begin{cases} 1 & \text{if } x \leq \alpha \\ 0 & \text{if } x > \alpha \end{cases}$$

where x, $y \in V$. Then, an n-variable α-KS logic function is defined strictly as follows.

Definition 1 *Logic formulas are defined inductively as follows.*

(1) Constants 0 and 1, and variables $x_1, \ldots, x_n$ are logic formulas.

(2) If G and H are logic formulas, then $(G \cdot H)$, $(G \vee H)$, $(\sim G)$ and $(\neg G)$ are also logic formulas.

(3) The only logic formulas are given by (1) and (2).

Definition 2 *A mapping $F : V^n \to V$ represented by a logic formula is called an n-variable α-KS logic function.*

Hereafter, we will call an n-variable α-KS logic function an α-KS logic function, for simplicity, and identify an α-KS logic function with the logic formula which represents it. In writing logic formulas, we assume that symbols $\sim$ and $\neg$ are stronger than $\cdot$, and $\cdot$ is stronger than $\vee$.

Example 1 *Let $F(x_1,x_2) = \neg \sim x_1 \vee \sim x_1 \cdot \neg x_2$, $\boldsymbol{a} = (0.1, 0.2)$ and $\alpha = 0.2$. Then, $F(\boldsymbol{a}) = \neg \sim 0.1 \vee \sim 0.1 \cdot \neg 0.2 = \neg 0.9 \vee 0.9 \cdot 1 = 0 \vee 0.9 = 0.9$.*

The following equations are the properties of α-KS logic functions, however, we do not know the axioms of an algebraic system which the set of α-KS logic functions forms.

(1)	$a \cdot a = a, \quad a \vee a = a$	(the idempotent laws)
(2)	$a \cdot b = b \cdot a, \quad a \vee b = b \vee a$	(the commutative laws)
(3)	$a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad a \vee (b \vee c) = (a \vee b) \vee c$	(the associative laws)
(4)	$a \cdot (a \vee b) = a, \quad a \vee a \cdot b = a$	(the absorption laws)
(5)	$a \cdot (b \vee c) = a \cdot b \vee a \cdot c,$ $a \vee b \cdot c = (a \vee b) \cdot (a \vee b)$	(the distributive laws)
(6)	$\sim (a \cdot b) = \sim a \vee \sim b, \quad \sim (a \vee b) = \sim a \cdot \sim b,$ $\neg(a \cdot b) = \neg a \vee \neg b, \quad \neg(a \vee b) = \neg a \cdot \neg b$	(De Morgan's laws)
(7)	$\sim\sim a = a$	(the double negation law)
(8)	$0 \cdot a = 0, \quad 0 \vee a = a$	(the least element)
(9)	$1 \cdot a = a, \quad 1 \vee a = 1$	(the greatest element)
(10)	$a \cdot \sim a \cdot (b \vee \sim b) = a \cdot \sim a,$ $a \cdot \sim a \vee (b \vee \sim b) = b \vee \sim b$	(Kleene's laws)
(11)	$\neg\neg\neg a = \neg a$	
(12)	$a \cdot \neg a = 0, \quad \neg a \vee \neg\neg a = 1$	

7.2.2 Partial Order Relations

In this section, first we define a partial order relation $\leq_\alpha$ on V.

Definition 3 *Let a and b be elements of V. Then $a \leq_\alpha b$ holds if and only if one of the following relations holds.*

(1) $0 \leq b \leq a \leq \alpha$, *(2)* $1 - \alpha \leq a \leq b \leq 1$,
(3) $\alpha < a \leq b \leq 1/2$, *or* *(4)* $1 - \alpha < a \leq b \leq 1/2$.

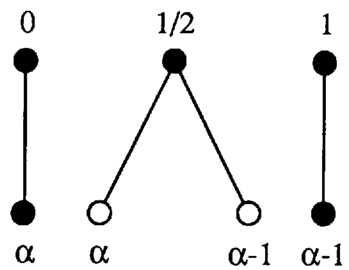

Figure 7.1: Partial Order Relation $\leq_\alpha$

Next, we define more three partial order relations $\leq_{\mathrm{B}}$, $\leq_{\mathrm{K}}$ and $\preceq_\alpha$ on V for the aim of convenience of the following discussions.

Definition 4 *Let a and b be elements of V. Then $a \leq_{\mathrm{B}} b$ holds if and only if one of the following relations holds.*

(1) $0 \leq b \leq a \leq \alpha$, *(2)* $1 - \alpha \leq a \leq b \leq 1$, *or* *(3)* $a = b$.

Definition 5 *Let a and b be elements of V. Then $a \leq_{\mathrm{K}} b$ holds if and only if one of the following relations holds.*

$$(1)\ \alpha < a \leq b \leq 1/2, \quad (2)\ 1-\alpha < a \leq b \leq 1/2, \text{ or } \quad (3)\ a = b.$$

Definition 6 *Let a and b be elements of V. Then $a \preceq_\alpha b$ holds if and only if one of the following relations holds.*

$$\begin{array}{ll} (1)\ 0 \leq a \leq b \leq \alpha, & (2)\ 1-\alpha \leq b \leq a \leq 1, \\ (3)\ \alpha < a \leq b \leq 1/2, \text{ or} & (4)\ 1-\alpha < a \leq b \leq 1/2. \end{array}$$

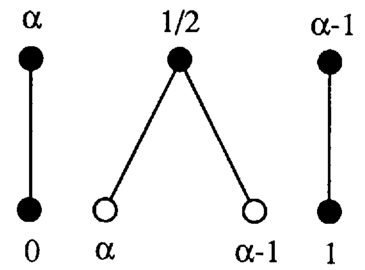

Figure 7.2: Partial Order Relation $\preceq_\alpha$

The relations $\leq_\alpha$, $\leq_{\mathrm{B}}$, $\leq_{\mathrm{K}}$ and $\preceq_\alpha$ can be expanded among V^n as follows. For two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of V^n, $\boldsymbol{a}\mathrm{R}\boldsymbol{b}$ if and only if $a_i \mathrm{R} b_i$ for all i, where R is one of $\leq_\alpha$, $\leq_{\mathrm{B}}$, $\leq_{\mathrm{K}}$ or $\preceq_\alpha$. Here, $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be comparable to each other under the partial order relation R when $\boldsymbol{a}\mathrm{R}\boldsymbol{b}$ or $\boldsymbol{b}\mathrm{R}\boldsymbol{a}$ hold, otherwise they are not comparable.

Theorem 1 *Let F be an α-KS logic function and $\boldsymbol{a}$, $\boldsymbol{b}$ be elements of V^n. Then, the following four hold.*

$$\begin{array}{ll} (1)\ \boldsymbol{a} \leq_\alpha \boldsymbol{b} \text{ implies } F(\boldsymbol{a}) \leq_\alpha F(\boldsymbol{b}), & (2)\ \boldsymbol{a} \leq_{\mathrm{B}} \boldsymbol{b} \text{ implies } F(\boldsymbol{a}) \leq_{\mathrm{B}} F(\boldsymbol{b}), \\ (3)\ \boldsymbol{a} \leq_{\mathrm{K}} \boldsymbol{b} \text{ implies } F(\boldsymbol{a}) \leq_{\mathrm{K}} F(\boldsymbol{b}), & (4)\ \boldsymbol{a} \preceq_\alpha \boldsymbol{b} \text{ implies } F(\boldsymbol{a}) \preceq_\alpha F(\boldsymbol{b}). \end{array}$$

Proof: Theorem is proved by induction concerning the number of operations ($\cdot$, $\vee$, $\sim$, $\neg$). The results trivially hold for 0, 1 and each variable $x_1, \ldots, x_n$. Now it is easy to show that if it holds for G and H which are α-KS logic functions, then it also holds for $G \cdot H$, $G \vee H$, $\sim G$ and $\neg G$ in any case (1), (2), (3) and (4). ■

Example 2 $F(x_1, x_2) = \neg \sim x_1 \vee \sim x_1 \cdot \neg x_2$, $\boldsymbol{a}_1 = (\alpha, \beta)$, $\boldsymbol{a}_2 = (0, \beta)$, $\boldsymbol{a}_3 = (\alpha, 1/2)$ *and* $\boldsymbol{a}_4 = (0, 1/2)$. *Then,* $\boldsymbol{a}_1 \leq_{\mathrm{B}} \boldsymbol{a}_2$ *and* $F(\boldsymbol{a}_1) = \beta \leq_{\mathrm{B}} \beta = F(\boldsymbol{a}_2)$. $\boldsymbol{a}_1 \leq_{\mathrm{K}} \boldsymbol{a}_3$ *and* $F(\boldsymbol{a}_1) = \beta \leq_{\mathrm{K}} 1/2 = F(\boldsymbol{a}_3)$. $\boldsymbol{a}_1 \leq_\alpha \boldsymbol{a}_4$ *and* $F(\boldsymbol{a}_1) = \beta \leq_\alpha 1/2 = F(\boldsymbol{a}_4)$. $\boldsymbol{a}_2 \preceq_\alpha \boldsymbol{a}_1$ *and* $F(\boldsymbol{a}_2) = \beta \preceq_\alpha F(\boldsymbol{a}_1)$.

7.2.3 Set $V_7 = \{\mathbf{0},\ \alpha,\ \beta,\ \mathbf{1/2},\ \beta',\ \alpha',\ \mathbf{1}\}$

Let V_7 be the set $\{0, \alpha, \beta, 1/2, \beta', \alpha', 1\}$, and we will define the following mapping from V to V_7, where $0 < \alpha < \beta < 1/2$, $\alpha' = 1 - \alpha$ and $\beta' = 1 - \beta$.

Definition 7 *Let a be an element of V. $\overline{a}^{\varepsilon\delta}$ is defined as follows, where $0 < \varepsilon \leq \alpha < \delta \leq 1/2$, $\varepsilon' = 1 - \varepsilon$ and $\delta' = 1 - \delta$ (refer to Figure 7.3).*

$$\overline{a}^{\varepsilon\delta} = \begin{cases} 0 & \text{if } 0 \leq a < \varepsilon, \\ \alpha & \text{if } \varepsilon \leq a \leq \alpha, \\ \beta & \text{if } \alpha < a < \delta, \\ 1/2 & \text{if } \delta \leq a \leq \delta', \\ \beta' & \text{if } \delta' < a < \alpha', \\ \alpha' & \text{if } \alpha' \leq a \leq \varepsilon', \\ 1 & \text{if } \varepsilon' < a \leq 1 \end{cases}$$

Moreover, let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V^n, then an element $\overline{\boldsymbol{a}}^{\varepsilon\delta}$ of V_7^n is defined by $(\overline{a_1}^{\varepsilon\delta}, \ldots, \overline{a_n}^{\varepsilon\delta})$.

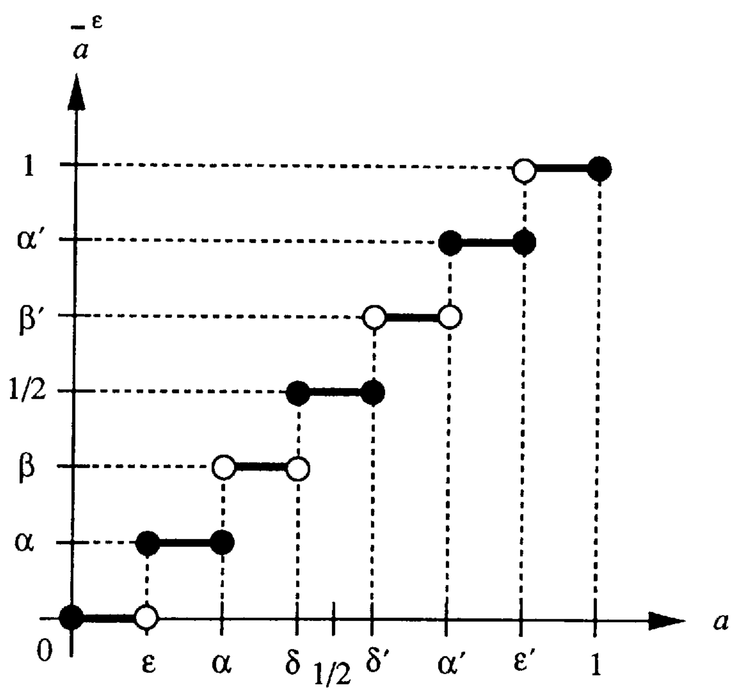

Figure 7.3: Mapping $\overline{a}^{\varepsilon\delta}$ $(0 < \varepsilon \leq \alpha < \delta \leq 1/2)$

Example 3 *Let $\boldsymbol{a} = (0.1,\ 0.2,\ 0.3,\ 0.4)$, $\alpha = \varepsilon = 0.2$ and $\delta = 0.4$. Then, $\overline{\boldsymbol{a}}^{\varepsilon\delta} = (\overline{0.1}^{\varepsilon\delta}, \overline{0.2}^{\varepsilon\delta}, \overline{0.3}^{\varepsilon\delta}, \overline{0.4}^{\varepsilon\delta}) = (0,\ \alpha,\ \beta,\ 1/2)$, where $\alpha < \beta < 1/2$.*

Theorem 2 *Let F be an α-KS logic function and $\boldsymbol{a}$ be an element of V^n. Then, $\overline{F(\boldsymbol{a})}^{\varepsilon\delta} = F(\overline{\boldsymbol{a}}^{\varepsilon\delta})$ for any value ε and δ $(0 < \varepsilon \leq \alpha < \delta \leq 1/2)$.*

Proof: Theorem is proved by induction concerning the number of operations ($\cdot$, $\vee$, $\sim$, $\neg$). The result trivially holds for 0, 1 and each variable $x_1, \ldots, x_n$. Now it is easy to show that if it holds for G and H, which are α-KS logic functions, then it holds for $G \cdot H$, $G \vee H$, $\sim G$ and $\neg G$. ■

Theorem 3 *Let F_1 and F_2 be α-KS logic functions. $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V^n if and only if $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_7^n.*

Proof: Suppose that $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a})$ for all elements $\boldsymbol{a}$ of V_7^n and the theorem does not hold. Then, we can assume that there exists at least one element $\boldsymbol{a}$ of V^n such that $F_1(\boldsymbol{a}) \neq F_2(\boldsymbol{a})$. This means either $F_1(\boldsymbol{a}) > F_2(\boldsymbol{a})$ or $F_1(\boldsymbol{a}) < F_2(\boldsymbol{a})$. First, suppose $F_1(\boldsymbol{a}) > F_2(\boldsymbol{a})$. Let $\lambda = (F_1(\boldsymbol{a}) + F_2(\boldsymbol{a}))/2$. When $\min(1, 1-\lambda) \leq \alpha$, by Theorem 1 we obtain $F_1(\overline{\boldsymbol{a}}^{\varepsilon\delta}) = \overline{F_1(\boldsymbol{a})}^{\varepsilon\delta} > F_2(\overline{\boldsymbol{a}}^{\varepsilon\delta}) = \overline{F_2(\boldsymbol{a})}^{\varepsilon\delta}$ for $\varepsilon = \min(1, 1-\lambda)$ and all δ such as $\alpha < \delta \leq 1/2$. When $\min(1, 1-\lambda) > \alpha$, we also obtain same result for $\delta = \min(1, 1-\lambda)$ and all ε such as $0 < \varepsilon \leq \alpha$. This contradicts the assumption, since $\overline{\boldsymbol{a}}^{\varepsilon\delta}$ is an element of V_7^n. We can prove the result in the similar manner when $F_1(\boldsymbol{a}) < F_2(\boldsymbol{a})$. Therefore, we can prove the first part of this theorem. The converse is trivial. Thus, it has been shown that Theorem 2 holds. ■

From Theorem 2, it is guaranteed that all outputs of an α-KS logic function are determined uniquely by all inputs of $V_7^n = \{0, \alpha, \beta, 1/2, \beta', \alpha', 1\}$

Corollary 1 *Let G and H be α-KS logic functions. $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V^n if and only if $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_7^n.*
(The proof is omitted)

Definition 8 *Let G and H be α-KS logic functions. It is said to be that H includes G (or G is included in H) if and only if $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V^n, and we denote it as $G \sqsubseteq H$ (or $H \sqsupseteq G$).*

In accordance with Corollary 1, $G \sqsubseteq H$ if and only if $G(\boldsymbol{a}) \leq H(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_7^n.

7.3 Canonical Disjunctive Forms of α-KS Logic Functions

7.3.1 Minterms of Type 1 $\sim$ Type 4

The logic formulas obtained by applying $\sim$ and $\neg$ to a variable x represent only the following six different kinds of α-KS logic functions, since $\sim\sim x = x$, $\neg\neg\neg x = \neg x$ and $\sim \neg x = \neg\neg x$ stand always true as shown in Section 7.2.

$$x, \quad \sim x, \quad \neg x, \quad \neg\neg x, \quad \neg \sim x, \text{ and } \neg\neg \sim x.$$

Hereafter, we call each these six logic formulas a *literal*. The truth tables of literals appear in Table 7.1.

A product term on the variables $x_1, \ldots, x_n$ is defined as a conjunction (AND) of literals, where $x_i' \not\sqsubseteq x_i''$ for any two literals x_i' and x_i'' appearing in the product term. For example, $x \sim xy\neg\neg \sim y$ and $\sim x\neg\neg x\neg\neg z\neg\neg \sim z$ are product terms, but not $xxy\neg\neg \sim y$ and $x\neg\neg x\neg\neg z\neg\neg \sim z$ since there are two same literals x in the first product and $x \sqsubseteq \neg\neg x$ in the second product.

Table 7.1: Truth Tables of Literals

x	0	α	β	1/2	β'	α'	1
$\sim x$	1	α'	β'	1/2	β	α	0
$\neg x$	1	1	0	0	0	0	0
$\neg\sim x$	0	0	0	0	0	1	1
$\neg\neg x$	0	0	1	1	1	1	1
$\neg\neg\sim x$	1	1	1	1	1	0	0

Among the literals except for $\neg x$ and $\neg \sim x$ the following equations hold.

$$\begin{aligned}
x &= x\neg x \vee x\neg\neg x\neg\neg \sim x \vee x\neg \sim x \\
\sim x &= \sim x\neg x \vee \sim x\neg\neg x\neg\neg \sim x \vee \sim x\neg \sim x \\
\neg\neg x &= \neg \sim x \vee \neg\neg x\neg\neg \sim x \\
\neg\neg \sim x &= \neg x \vee \neg\neg x\neg\neg \sim x
\end{aligned}$$

Therefore, any literal except for $\neg x$ and $\neg \sim x$ can be expanded into the disjunction of some of the following product terms.

$$x\neg x, \quad \sim x\neg \sim x, \quad \sim x\neg x, \quad x\neg \sim x,$$
$$\neg\neg x\neg\neg \sim x, \quad x\neg\neg x\neg\neg \sim x, \quad \sim x\neg\neg x\neg\neg \sim x,$$

Moreover, a conjunction of any two x' and x'' of the above product terms and literals $\neg x$ and $\neg \sim x$ is equivalent to x' (or x'') or the constant 0 except for the pair of $x\neg\neg x\neg\neg \sim x$ and $\sim x\neg\neg x\neg\neg \sim x$, and these two $x\neg\neg x\neg\neg \sim x$ and $\sim x\neg\neg x\neg\neg \sim x$ yields a new product term $x \sim x\neg\neg x\neg\neg \sim x$ which is not identical with any one of product terms listing the above seven and literals $\neg x$ and $\neg \sim x$. Therefore, we conclude that any literal can be expanded into a disjunction of some of ten product terms appearing in Table 7.2, which shows their truth tables.

Table 7.2: Truth Tables of Atoms

x	0	α	β	1/2	β'	α'	1
$\neg x$ (X^0)	1	1	0	0	0	0	0
$\sim x\neg x$ ($X^{1\alpha}$)	1	α'	0	0	0	0	0
$x\neg x$ ($X^{0\alpha}$)	0	α	0	0	0	0	0
$\neg \sim x$ (X^1)	0	0	0	0	0	1	1
$x\neg \sim x$ ($X^{\alpha 1}$)	0	0	0	0	0	α'	1
$\sim x\neg \sim x$ ($X^{\alpha 0}$)	0	0	0	0	0	α	0
$\neg\neg x\neg\neg \sim x$ (X^{11})	0	0	1	1	1	0	0
$x\neg\neg x\neg\neg \sim x$ (X^{01})	0	0	β	1/2	β'	0	0
$\sim x\neg\neg x\neg\neg \sim x$ (X^{10})	0	0	β'	1/2	β	0	0
$x \sim x\neg\neg x\neg\neg \sim x$ (X^{00})	0	0	β	1/2	β	0	0

Moreover, any one of ten product terms in Table 7.2 never be represented by a disjunction of some of another product terms of Table 7.2. Hereafter, for the aim of convenience, we call each ten product terms of Table 7.2 an *atom*, and represent them as the symbols X^0, $X^{1\alpha}$, $X^{0\alpha}$

X^1, $X^{\alpha 1}$, $X^{\alpha 0}$, X^{11}, X^{01}, X^{10} and X^{00}, respectively. In accordance with the above discussions we can conclude that any product term γ can be expanded into a disjunction of product terms constructed by only atoms, that is, the following expansion stands always possible.

$$\begin{aligned}
\gamma &= x^*_{q_1} \cdot \ldots \cdot x^*_{q_k} \\
&= (X_{q_1}^{a_{11}} \vee \ldots \vee X_{q_1}^{a_{1s_1}}) \cdot \ldots \cdot (X_{q_k}^{a_{k1}} \vee \ldots \vee X_{q_k}^{a_{ks_k}}) \\
&= \gamma_1 \vee \ldots \vee \gamma_l
\end{aligned}$$

where each $x^*_{q_i}$ denotes a conjunction of some of literals for a variable x_{q_i} ($q_i \in \{1, \ldots, n\}$), each $X_{q_i}^{a_{st}}$ denotes an atom for the variable x_{q_i} and each γ_l implies a product term constructed by only atoms.

If a variable x_i dose not exist in a product term γ, then the relation $\gamma = \gamma(X_i^0 \vee X_i^{11} \vee X_i^1) = \gamma X_i^0 \vee \gamma X_i^{11} \vee \gamma X_i^1$ holds, since $X_i^0 \vee X_i^{11} \vee X_i^1 = 1$ stands always true. Therefore, a product term in which a variable x_i dose not appear can be expanded into the disjunction of product terms appearing all variables. Hereafter, we call a product term, which is finally obtained by the expansion denoted above, a *minterm*, that is, a minterm is a product term appearing all variables and constructing of only atoms.

Example 4 *Let γ be a product term on variables x, y, z such as $\gamma = \neg x \sim y \neg y$. Then, γ is expanded into a disjunction of minterms as follows.*

$$\begin{aligned}
\gamma &= \neg x \sim y \neg y \\
&= X^1 Y^{1\alpha}(Z^0 \vee Z^{11} \vee Z^1) \\
&= X^1 Y^{1\alpha} Z^0 \vee X^1 Y^{1\alpha} Z^{11} \vee X^1 Y^{1\alpha} Z^1
\end{aligned}$$

A logic formula representing an α-KS logic function F can be expanded into the disjunctive form $F = \gamma_1 \vee \ldots \vee \gamma_s$, where γ_i ($i = 1, \ldots, s$) is a minter, from the above discussions and from the distributive laws, the absorption laws, De Morgan's laws and other laws as discussed in Section 2. Each minterm γ_i can be classified into one of the following four types.

type 1: A minterm γ consisting of only $X^{1\alpha}$, X^0, X^{11}, X^1 and $X^{\alpha 1}$ for any variable x.

type 2': A minterm γ consisting of only atoms of type 1 and X^{10}, X^{01} for any variable x, and appearing at least one of Y^{10} or Y^{01} for some variable y.

type 3': A minterm γ consisting of only atoms of type 2 and X^{00} for any variable x, and appearing at least one Y^{00} for some variable y.

type 4': A minterm γ appearing at least one of $X^{0\alpha}$ or $X^{\alpha 0}$ for some variable x.

Let γ be a minterm of type 2' appearing X^0 or X^1 for some variable x. Then, $X^0 Y^{ij} = X^{1\alpha} Y^{ij}$ ($i, j = 0$ or 1, and $i \neq j$) holds. Because let $\boldsymbol{a}$ be an element of V_7^n, then $X^0(\boldsymbol{a}) \neq 0$ if and only if $X^{1\alpha}(\boldsymbol{a}) \neq 0$. Therefore, $(X^0 Y^{ij})(\boldsymbol{a}) \neq 0$ if and only if $(X^{1\alpha})(\boldsymbol{a}) \neq 0$, and in such the element $\boldsymbol{a}$ we always obtain $Y^{ij}(\boldsymbol{a}) \leq X^0(\boldsymbol{a})$ and $Y^{ij}(\boldsymbol{a}) \leq X^{1\alpha}(\boldsymbol{a})$, that is, $(X^0 Y^{ij})(\boldsymbol{a}) = (X^{1\alpha} Y^{ij})(\boldsymbol{a}) = Y^{ij}(\boldsymbol{a})$. Accordingly $X^0 Y^{ij} = X^{1\alpha} Y^{ij}$ stands always true. Therefore, a minterm α of type 2' is equivalent to the minterm which is obtained by replacing any atom X^0 and X^1 appareling in α to $X^{1\alpha}$ and $X^{\alpha 1}$, respectively.

Let γ be a minterm of type 3' appearing X^{11} for some variable x. Then the relation $Y^{00} X^{11} = Y^{00}(X^{01} \vee X^{10}) = Y^{00} X^{01} \vee Y^{00} X^{10}$ holds. Because $(X^{01} \vee X^{10})(\boldsymbol{a}) \neq 0$ if and only if $X^{11}(\boldsymbol{a}) \neq 0$.

Therefore, $(Y^{00}X^{01} \vee Y^{00}X^{10})(\boldsymbol{a}) \neq$ if and only if $(Y^{00}X^{11})(\boldsymbol{a}) \neq 0$, and in such an element $\boldsymbol{a}$ Y^{00} is always smaller than any X^{01}, X^{10} and X^{11}, that is, $(Y^{00}X^{01} \vee Y^{00}X^{10})(\boldsymbol{a}) = (Y^{00}X^{11})(\boldsymbol{a}) = Y^{00}(\boldsymbol{a})$. Accordingly $Y^{00}X^{11} = Y^{00}(X^{01} \vee X^{10}) = Y^{00}X^{01} \vee Y^{00}X^{10}$ stands always true. Therefore, any type 3' minterm γ appearing X^{11} for some variables x can be expanded into the disjunction of minterms in which X^{11} never appear for any variable x. Next, X^0 or X^1 appears in a type 3' minterm for some variable x. Then $Y^{00}X^0 = Y^{00}X^{1\alpha}$ and $Y^{00}X^1 = Y^{00}X^{\alpha 1}$ holds in the similar manner with type 2' minterms. Therefore, any type 3' minterm appearing X^0 or X^1 for some variable x is equivalent to the minterm obtained by replacing any X^0 and X^1 to $X^{1\alpha}$ and $X^{\alpha 1}$, respectively.

If there is X^{11}, X^{01}, X^{10}, X^0, or X^1 in a minterm γ of type 4', since there is $Y^{\alpha 0}$ or $Y^{0\alpha}$ in γ we can show in the similar manner that $Y^{ij}X^{11} = Y^{ij}X^{01} = Y^{ij}X^{10} = Y^{ij}X^{00}$, $Y^{ij}X^0 = Y^{ij}X^{1\alpha}$, and $Y^{ij}X^1 = Y^{ij}X^{\alpha 1}$ (i, $j = 0$ or α, and $i \neq j$)

From the above, any α-KS logic function can be expanded into the disjunction of the following four types of minterms.

type 1:

type 2: A minterm consisting of only X^{11}, $X^{1\alpha}$, $X^{\alpha 1}$, X^{10} and X^{01} for any variable x, and appearing at least one of Y^{10} or Y^{01} for some variable y,

type 3: A minterm consisting of only X^{00}, $X^{1\alpha}$, $X^{\alpha 1}$, X^{10} and X^{01} for any variable x, and appearing at least Y^{00} for some variable y,

type 4: A minterm consisting of only X^{00}, $X^{1\alpha}$, $X^{\alpha 1}$, $X^{0\alpha}$ and $X^{\alpha 0}$ for any variable x, and appearing at least one of $Y^{0\alpha}$ or $Y^{\alpha 0}$ for some variable y.

Example 5 *$X^{1\alpha}Y^0Z^{11}$, $X^{1\alpha}Y^{01}Z^{11}$, $X^{1\alpha}Y^{01}Z^{00}$ and $X^{1\alpha}Y^{0\alpha}Z^{00}$ are minterms of type 1, type 2, type 3 and type 4, respectively.*

7.3.2 Properties of Type 1 ~ Type 4 Minterms

Definition 9 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of $\{0,\ \alpha,\ 1/2,\ \alpha',\ 1\}^n$. Then, the element $\boldsymbol{a}$ corresponds to a minterm $\gamma = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ of type 1 if the following relation holds form every $i = 1, \ldots, n$.*

$$x_i^{a_i} = \begin{cases} X_i^{1\alpha} & \text{if} \quad a_i = 0, \\ X_i^{0} & \text{if} \quad a_i = \alpha, \\ X_i^{11} & \text{if} \quad a_i = 1/2, \\ X_i^{1} & \text{if} \quad a_i = \alpha', \\ X_i^{\alpha 1} & \text{if} \quad a_i = 1 \end{cases}$$

It is evident that $\gamma(\boldsymbol{a}) = 1$ and $\gamma(\boldsymbol{b}) \in \{0,\ \alpha',\ 1\}$ for all elements $\boldsymbol{b} \in V_7^n$.

Definition 10 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of $\{0,\ \beta,\ 1/2,\ \beta',\ 1\}^n - \{0, 1/2, 1\}^n$. Then, the element $\boldsymbol{a}$ corresponds to a minterm $\gamma = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ of type 2 if the following relation holds for every $i = 1, \ldots n$.*

$$x_i^{a_i} = \begin{cases} X_i^{1\alpha} & \text{if} \quad a_i = 0, \\ X_i^{10} & \text{if} \quad a_i = \beta, \\ X_i^{11} & \text{if} \quad a_i = 1/2, \\ X_i^{01} & \text{if} \quad a_i = \beta', \\ X_i^{\alpha 1} & \text{if} \quad a_i = 1 \end{cases}$$

It is evident that $\gamma(\boldsymbol{a}) = \beta'$ and $\gamma(\boldsymbol{b}) \in \{0, \beta, 1/2, \beta'\}$ for all elements $\boldsymbol{b} \in V_7^n$.

Definition 11 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of $\{\alpha,\ \beta,\ 1/2,\ \beta',\ \alpha'\}^n - \{\alpha,\ \beta,\ \beta',\ \alpha'\}^n$. Then, the element $\boldsymbol{a}$ corresponds to a minterm $\gamma = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ of type 3 if the following relation holds for every $i = 1, \ldots n$.*

$$x_i^{a_i} = \begin{cases} X_i^{1\alpha} & \text{if} \quad a_i = \alpha, \\ X_i^{10} & \text{if} \quad a_i = \beta, \\ X_i^{00} & \text{if} \quad a_i = 1/2, \\ X_i^{01} & \text{if} \quad a_i = \beta', \\ X_i^{\alpha 1} & \text{if} \quad a_i = \alpha' \end{cases}$$

It is evident that $\gamma(\boldsymbol{a}) = 1/2$ and $\gamma(\boldsymbol{b}) \in \{0, \beta, 1/2\}$ for all elements $\boldsymbol{b} \in V_7^n$.

Definition 12 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of $\{0,\ \alpha,\ 1/2,\ \alpha',\ 1\}^n - \{0, 1/2, 1\}^n$. Then, the element a corresponds to a minterm $\gamma = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$ of type 4 if the following relation holds for every $i = 1, \ldots n$.*

$$x_i^{a_i} = \begin{cases} X_i^{1\alpha} & \text{if} \quad a_i = 0, \\ X_i^{0\alpha} & \text{if} \quad a_i = \alpha, \\ X_i^{00} & \text{if} \quad a_i = 1/2, \\ X_i^{\alpha 0} & \text{if} \quad a_i = \alpha', \\ X_i^{\alpha 1} & \text{if} \quad a_i = 1 \end{cases}$$

It is evident that $\gamma(\boldsymbol{a}) = \alpha$ and $\gamma(\boldsymbol{b}) \in \{0, \alpha\}$ for all elements $\boldsymbol{b} \in V_7^n$.

We will define a notation with respect to the partial order relation $\leq_\alpha$ as follows. Let a and b be elements of V. If a and b are comparable to each other, then we can find the infimum of a and b concerning with the relation $\leq_\alpha$, otherwise does not. We will write an infimum of a and b as $a\Delta_\alpha b$, and if it does not exist, then we will write as $a\Delta_\alpha b = \emptyset$. This can be extended among V^n as follows. For two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ of V^n, we will define $\boldsymbol{a}\Delta_\alpha \boldsymbol{b}$ as $(a_1\Delta_\alpha b_1, \ldots, a_n\Delta_\alpha b_n)$, and if $a_i\Delta_\alpha b_i = \emptyset$ for some i $(i = 1, \ldots, n)$, then we will define it as $\boldsymbol{a}\Delta_\alpha \boldsymbol{b} = \emptyset$.

Lemma 1 *Let $\boldsymbol{a}$ be an element of $\{0, \alpha, 1/2, \alpha', 1\}^n$, and γ the minterm of type 1 corresponding to $\boldsymbol{a}$. If $\boldsymbol{b}$ is an element of V_7^n, then*

(1) $\gamma(\boldsymbol{b}) = 1$ if and only if $\boldsymbol{b} \preceq_\alpha \boldsymbol{a}$,

(2) $\gamma(\boldsymbol{b}) = \alpha'$ if and only if $\boldsymbol{b} \leq_\alpha \mathbf{c}$ and $\boldsymbol{b} \not\preceq_\alpha \boldsymbol{a}$,

(3) $\gamma(\boldsymbol{b}) = 0$ if and only if $\boldsymbol{b} \not\leq_\alpha \mathbf{c}$,

where $\boldsymbol{a} \leq_\alpha \mathbf{c}$ and $\mathbf{c} \in \{0, 1/2, 1\}^n$.

Proof: Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$. $\gamma(\boldsymbol{b}) = \alpha'$ if and only if it has to be held for any $i = 1, \ldots n$ that $b_i \preceq_\alpha a_i$ when $a_i = \alpha$, $1/2$ or α', and $a_i = b_i$ when $a_i = 0$ or 1. Therefore, $\gamma(\boldsymbol{b}) = \alpha'$ if and only if $\boldsymbol{b} \preceq_\alpha \boldsymbol{a}$. Next, $\gamma(\boldsymbol{b}) = 0$ if and only if it has to be held for some $i = 1, \ldots, n$ that $b_i \neq 0$ and α when $a_i = 0$ or α, $b_i \neq 1$ and α' when $a_i = 1$ or α', or $b_i \neq \beta$, $1/2$ and β' when $a_i = 1/2$. Therefore, $\gamma(\boldsymbol{b}) = 0$ if and only if $\boldsymbol{b} \not\leq_\alpha \mathbf{c}$. (2) is derived directly from (1) and (3). This completes the proof of the lemma. ∎

Lemma 2 *Let $\boldsymbol{a}$ be an element of $\{0,\beta,1/2,\beta',1\}^n - \{0,1/2,1\}^n$, and γ the minterm of type 2 corresponding to $\boldsymbol{a}$. If $\boldsymbol{b}$ is an element of V_7^n, then*

(1) $\gamma(\boldsymbol{b}) = \beta'$ if and only if $\boldsymbol{b} \leq_\alpha \boldsymbol{a}$,

(2) $\gamma(\boldsymbol{b}) = 1/2$ if and only if $\boldsymbol{b} \leq_\alpha \mathbf{c}$, $\boldsymbol{b} \not\leq_\alpha \boldsymbol{a}$ and $\boldsymbol{a}\Delta_\alpha\boldsymbol{b} \neq \emptyset$,

(3) $\gamma(\boldsymbol{b}) = \beta$ if and only if $\boldsymbol{b} \leq_\alpha \mathbf{c}$ and $\boldsymbol{a}\Delta_\alpha\boldsymbol{b} = \emptyset$,

(4) $\gamma(\boldsymbol{b}) = 0$ if and only if $\boldsymbol{b} \not\leq_\alpha \mathbf{c}$,

where $\boldsymbol{a} \leq_\alpha \mathbf{c}$ and $\mathbf{c} \in \{0,1/2,1\}^n$.

Lemma 3 *Let $\boldsymbol{a}$ be an element of $\{\alpha, \beta, 1/2, \beta', \alpha'\}^n - \{\alpha, \beta, \beta', \alpha'\}^n$, and γ the minterm of type 3 corresponding to $\boldsymbol{a}$. If $\boldsymbol{b}$ is an element of V_7^n, then*

(1) $\gamma(\boldsymbol{b}) = 1/2$ if and only if $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$,

(2) $\gamma(\boldsymbol{b}) = \beta$ if and only if $\boldsymbol{b} \leq_\alpha \mathbf{c}$ and $\boldsymbol{a} \not\leq_\alpha \boldsymbol{b}$,

(3) $\gamma(\boldsymbol{b}) = 0$ if and only if $\boldsymbol{b} \not\leq_\alpha \mathbf{c}$,

where $\boldsymbol{a} \leq_\alpha \mathbf{c}$, $\mathbf{c} \in \{0,1/2,1\}^n$.

Lemma 4 *Let $\boldsymbol{a}$ be an element of $\{0,\alpha,1/2,\alpha',1\}^n - \{0,1/2,1\}^n$, and γ the minterm of type 4 corresponding to $\boldsymbol{a}$. If $\boldsymbol{b}$ is an element of V_7^n, then*

(1) $\gamma(\boldsymbol{b}) = \alpha$ if and only if $\boldsymbol{b} \leq_\alpha \boldsymbol{a}$,

(2) $\gamma(\boldsymbol{b}) = 0$ if and only if $\boldsymbol{b} \not\leq_\alpha \boldsymbol{a}$.

The proofs of Lemma 2, 3 and 4 are similar to that of Lemma 1.

Let γ_1, γ_2, γ_3, and γ_4 be minterms of type 1, 2, 3, and 4, respectively. Then, it is evident that $\gamma_i \vee \gamma_j \neq \gamma_j$ from the definition of each minterm, where $i < j$ (i, j=1, 2, 3, 4).

Lemma 5 *Let γ and γ' be any minterm of type 1 $\sim$ type 4, and $\boldsymbol{a}$ and $\boldsymbol{b}$ elements of V_7^n corresponding to γ and γ', respectively. Then, $\gamma \vee \gamma' = \gamma$, that is, $\gamma' \sqsubseteq \gamma$ if and only if*

(1) $\boldsymbol{a} \leq_{\mathrm{B}} \boldsymbol{b}$, when both γ and γ' are type 1.

(2) $\boldsymbol{b} \leq_{\mathrm{K}} \boldsymbol{a}$, when both γ and γ' are type 2.

(3) $\boldsymbol{a} \leq_{\mathrm{K}} \boldsymbol{b}$, when both γ and γ' are type 3.

(4) $\boldsymbol{b} \leq_{\mathrm{B}} \boldsymbol{a}$, when both γ and γ' are type 4.

(5) $\boldsymbol{a}\Delta_\alpha\boldsymbol{b} \neq \emptyset$, when γ and γ' are type 2 and type 3, respectively.

(6) $\boldsymbol{b} \leq_\alpha \mathbf{c}$, when γ is a type 1 and γ' is either one of type 2, 3, or 4.

(7) $\boldsymbol{b} \leq_\alpha \mathbf{c}$, when γ is either one of type 2 or 3, and γ' is a type 4.

where $\boldsymbol{a} \leq_\alpha \mathbf{c}$ and $\mathbf{c} \in \{0,1/2,1\}^n$.

Proof: Let us suppose γ and γ' be both type 1, and then $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n) \in \{0,$ α, $1/2$, α', $1\}$. First assume $\boldsymbol{b} \leq_{\mathrm{B}} \boldsymbol{a}$, that is, $b_i \leq_{\mathrm{B}} a_i$ for any i ($i = 1, \ldots, n$). This implies that $a_i = b_i$ when $a_i = \alpha$, $1/2$ and α', and $b_i \leq_{\mathrm{B}} a_i$ when $a_i = 0$ or 1. Therefore, $\gamma \vee \gamma' = \gamma$. Next, assume $\gamma \vee \gamma' = \gamma$, that is, $\gamma(\boldsymbol{a}') \geq \gamma'(\boldsymbol{a}')$ for any element $\boldsymbol{a}'$ of V_7^n. This implies $\gamma(\boldsymbol{b}) = 1$ since $\gamma'(\boldsymbol{b}) = 1$. By Lemma 1.1 $\boldsymbol{b} \preceq_\alpha \boldsymbol{a}$, and this implies that $a_i = b_i$ when $a_i = 0$, 1 or 1/2 and $a_i \leq_{\mathrm{B}} b_i$ when $a_i = \alpha$ or α' for any $i = 1, \ldots, n$. Therefore we have $\boldsymbol{a} \leq_{\mathrm{B}} \boldsymbol{b}$ From the above, we have $\gamma \vee \gamma' = \gamma$ if and only if $\boldsymbol{b} \leq_{\mathrm{B}} \boldsymbol{a}$ when γ and γ' are both type 1. The remaining cases of the lemma are proved in the similar manner. ∎

7.3.3 Definition and Proof of Uniqueness for Canonical Disjunctive Forms

In this section, we give a definition of canonical disjunctive form, and then show this form is uniquely determined for a given α-KS logic function.

Definition 13 *If an α-KS logic function F is represented by a logic formula $F = \gamma_1 \vee \ldots \vee \gamma_s$, then it is said that F is in the canonical disjunctive form where γ_i $(i = 1, \ldots, s)$ is a minterm of either type 1, type 2, type 3, or type 4 and $\gamma_i \vee \gamma_j \neq \gamma_i$ for any i, j $(i \neq j)$.*

Example 6 *The canonical disjunctive form of a 2-variable α-KS logic function $F = x\neg \sim y \vee \sim y\neg \sim y$ is obtained as follows.*

$$\begin{aligned}
F &= x\neg \sim y \vee \sim y \neg \sim y \\
&= (X^{0\alpha} \vee X^{01} \vee X^{\alpha 1}) \vee Y^1 \vee Y^{\alpha 0} \\
&= X^{0\alpha}Y^1 \vee X^{01}Y^1 \vee X^{\alpha 1}Y^1 \vee Y^{\alpha 0}(X^0 \vee X^{11} \vee X^1) \\
&= X^{0\alpha}Y^1 \vee X^{01}Y^1 \vee X^{\alpha 1}Y^1 \vee X^0 Y^{\alpha 0} \vee X^{11}Y^{\alpha 0} \vee X^1 Y^{\alpha 0} \\
&= X^{0\alpha}Y^{\alpha 1} \vee X^{01}Y^{\alpha 1} \vee X^{\alpha 1}Y^1 \vee X^{1\alpha}Y^{\alpha 0} \vee X^{00}Y^{\alpha 0} \vee X^{\alpha 1}Y^{\alpha 0}
\end{aligned}$$

Here, elements $(\alpha, 1)$, $(\beta, 1)$, $(\alpha', 1)$, $(0, \alpha')$, $(1/2, \alpha')$ and $(1, \alpha')$ are corresponding to minterms $X^{0\alpha}Y^{\alpha 1}$, $X^{01}Y^{\alpha 1}$, $X^{\alpha 1}Y^1$, $X^{1\alpha}Y^{\alpha 0}$, $X^{00}Y^{\alpha 0}$, and $X^{\alpha 1}Y^{\alpha 0}$, respectively. Then, $X^{00}Y^{\alpha 0}$, and $X^{\alpha 1}Y^{\alpha 0}$, are omitted by $X^{01}Y^{\alpha 1}$ and $X^{\alpha 1}Y^1$, respectively, from Lemma 5. Therefore, the canonical disjunctive form of F is

$$F = X^{0\alpha}Y^{\alpha 1} \vee X^{01}Y^{\alpha 1} \vee X^{\alpha 1}Y^1 \vee X^{1\alpha}Y^{\alpha 0}.$$

Lemma 6 *Let F be a canonical disjunctive form of an α-KS logic function, and γ a minterm of F. Then $F(\boldsymbol{a}) = \gamma(\boldsymbol{a})$ for the element $\boldsymbol{a}$ corresponding to γ.*

Proof: In the proof, the symbol γ' implies one of type 1, 2 or 3 minterm, and the symbol $\boldsymbol{b}$ implies the corresponding element to γ'. First suppose γ is a type 1, then since $\gamma(\boldsymbol{a}) = 1$ we have $F(\boldsymbol{a}) = 1$. Second suppose γ is a type 2, then $F(\boldsymbol{a}) = \beta'$. Because, if $F(\boldsymbol{a}) > \beta'$, then there is a type 1 minterm γ' in F such that $\gamma'(\boldsymbol{a}) = 1$ or α', and therefore by Lemma 1.3 $\boldsymbol{a} \leq_\alpha \boldsymbol{c}$ where $\boldsymbol{b} \leq_\alpha \boldsymbol{c}$ and $\boldsymbol{c} \in \{0,\ 1/2,\ 1\}^n$. By Lemma 5.6 we have $\gamma \sqsubseteq \gamma'$ and this contradicts that F is the canonical disjunctive form. Third suppose γ is a type 3, then $F(\boldsymbol{a}) = 1/2$. Because, if $F(\boldsymbol{a}) > 1/2$, then there is a type 1 or 2 minterm γ' in F such that $\gamma'(\boldsymbol{a}) > 1/2$. When γ' is a type 1, then by Lemma 1.3 and Lemma 5.6 we have $\gamma \sqsubseteq \gamma'$ in the similar manner. When γ' is a type 2, then $\gamma'(\boldsymbol{a}) = \beta'$ since $\gamma'(\boldsymbol{a}') \leq \beta'$ for any element $\boldsymbol{a}'$ of V_7^n and $\gamma'(\boldsymbol{a}) > 1/2$, and therefore by Lemma 2.1 $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$. Accordingly by Lemma 5.5 we have $\gamma \sqsubseteq \gamma'$. Thus, both two cases contradict that F is the canonical disjunctive form. Finally suppose γ is a type 4, then $F(\boldsymbol{a}) = \alpha$. Because, if $F(\boldsymbol{a}) > \alpha$, then in F there is a minterm γ', which is some of type 1, 2 or 3, such that $\gamma'(\boldsymbol{a}) \neq 0$. Therefore, by Lemma 1.3, 2.4 and 3.3 $\boldsymbol{a} \leq_\alpha \boldsymbol{c}$ where $\boldsymbol{b} \leq_\alpha \boldsymbol{c}$ and $\boldsymbol{c} \in \{0,\ 1/2,\ 1\}^n$. Therefore, $\gamma \sqsubseteq \gamma'$ by Lemma 5.6 and 5.7, and these contradict that F is the canonical disjunctive form. This completes the proof of the lemma. ∎

Theorem 4 *Any α-KS logic function can be represented by a canonical disjunctive form, and it is determined uniquely for a given a-KS logic function (ignoring the order of the minterms).*

Proof: Let us suppose $F_1 = \gamma_1 \vee \ldots \vee \gamma_s$ and $F_2 = \gamma'_1 \vee \ldots \vee \gamma'_t$ are two different canonical disjunctive forms of an α-KS logic function F (It is evident from the above discussions that there is at least one canonical disjunctive form of F). Now, we can suppose that a minterm γ exists in F_1, but F_2 without loss of generality. Note that in the proof, the each symbol γ' and γ'' implies some of type 1, 2, 3 or 4 minterm, and the symbols $\boldsymbol{a}$, $\boldsymbol{a}'$ and $\boldsymbol{a}''$ imply the corresponding elements to γ, γ' and γ'', respectively.

First, assume γ is a type 1. Then $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a}) = 1$ by Lemma 6, and therefore there is a type 1 minterm γ' in F_2 such that $\gamma'(\boldsymbol{a}) = 1$. By Lemma 1.1 this implies $\boldsymbol{a} \preceq_\alpha \boldsymbol{a}'$ ($\boldsymbol{a} \neq \boldsymbol{a}'$ form the assumption). In the similar manner, there is a type 1 minterm γ'' in F_1 such that $\gamma''(\boldsymbol{a}') = 1$, and accordingly we have $\boldsymbol{a}' \preceq_\alpha \boldsymbol{a}''$ by Lemma 1.1. Therefore, $\boldsymbol{a} \preceq \boldsymbol{a}''$ implies $\boldsymbol{a}'' \leq_B \boldsymbol{a}$ since both $\boldsymbol{a}$ and $\boldsymbol{a}''$ are elements of $\{0, \alpha, 1/2, \alpha', 1\}^n$, and by Lemma 5.1 $\gamma \sqsubseteq \gamma''$ holds. This contradicts that F_1 is the canonical disjunctive form.

Second assume γ is a type 2. Then $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a}) = \beta'$ by Lemma 6, and therefore there is a type 2 minterm γ' in F_2 such that $\gamma'(\boldsymbol{a}) = \beta'$. By Lemma 2.1 this implies $\boldsymbol{a} \leq_\alpha \boldsymbol{a}'$ ($\boldsymbol{a} \neq \boldsymbol{a}'$ from the assumption), and since both γ and γ' are both type 2 the relation $\boldsymbol{a} \leq_\alpha \boldsymbol{a}'$ implies $\boldsymbol{a} \leq_K \boldsymbol{a}'$. In the similar manner, there is a type 2 minterm γ'' in F_1 such that $\gamma''(\boldsymbol{a}') = \beta'$, and by Lemma 2.1 and both γ' and γ'' are type 2, we have $\boldsymbol{a}' \leq_K \boldsymbol{a}''$. Therefore by Lemma 5.2 $\gamma \sqsubseteq \gamma''$ holds, and this contradicts that F_1 is the canonical disjunctive form.

Third assume γ is a type 3. Then $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a}) = 1/2$ by Lemma 6, and therefore there is a type 2 or type 3 minterm γ' in F_2 such that $\gamma'(\boldsymbol{a}) = 1/2$. If γ' is a type 2, then by Lemma 2.2 $\boldsymbol{a}\Delta_\alpha\boldsymbol{a}' \neq \emptyset$ ($\boldsymbol{a} \neq \boldsymbol{a}'$ from the assumption). Since $F_1(\boldsymbol{a}') = F_2(\boldsymbol{a}') = \beta'$ there is a type 2 minterm γ'' in F_1 such that $\gamma''(\boldsymbol{a}') = \beta'$, and therefore by Lemma 2.1 $\boldsymbol{a}' \leq_\alpha \boldsymbol{a}''$. Accordingly we have $\gamma \sqsubseteq \gamma''$ by $\boldsymbol{a}\Delta_\alpha\boldsymbol{a}'' \neq \emptyset$ and Lemma 5.5, and this contradicts that F_1 is the canonical disjunctive form. If γ' is a type 3, then by Lemma 3.1 $\boldsymbol{a}' \leq_\alpha \boldsymbol{a}$ ($\boldsymbol{a} \neq \boldsymbol{a}'$ form the assumption), and this implies $\boldsymbol{a}' \leq_K \boldsymbol{a}$ since both γ and γ' are type 3. Since $F_1(\boldsymbol{a}') = F_2(\boldsymbol{a}') = 1/2$ there is a type 2 or type 3 minterm γ'' in F_1 such that $\gamma''(\boldsymbol{a}') = 1/2$. In the similar manner, $\boldsymbol{a}'\Delta_\alpha\boldsymbol{a}'' \neq \emptyset$ and this leads us to $\boldsymbol{a}\Delta_\alpha\boldsymbol{a}'' \neq \emptyset$ when γ'' is a type 2, and moreover when γ'' is a type 3 we have $\boldsymbol{a}'' \leq_K \boldsymbol{a}'$, that is, $\boldsymbol{a}'' \leq_K \boldsymbol{a}$. Therefore, by Lemma 5.5 and 5.3 we have $\gamma \sqsubseteq \gamma''$ whether γ'' is type 2 or type 3, and this contradicts that F_1 is the canonical disjunctive form.

Finally assume γ is a type 4. Then $F_1(\boldsymbol{a}) = F_2(\boldsymbol{a}) = \alpha$ by Lemma 6, and therefore there is a type 4 minterm γ' in F_2 such that $\gamma'(\boldsymbol{a}) = \alpha$. By Lemma 4.1 this implies $\boldsymbol{a} \leq_\alpha \boldsymbol{a}'$ ($\boldsymbol{a} \neq \boldsymbol{a}'$ form the assumption), and since both γ and γ' are type 4 the relation $\boldsymbol{a} \leq_\alpha \boldsymbol{a}'$ implies $\boldsymbol{a} \leq_B \boldsymbol{a}'$. In the similar manner, there is a type 4 minterm γ'' in F_1 such that $\gamma''(\boldsymbol{a}') = \alpha$, and by Lemma 4.1 and by both γ' and γ'' are type 4 $\boldsymbol{a}' \leq_B \boldsymbol{a}''$. Therefore, we have $\boldsymbol{a} \leq_B \boldsymbol{a}''$, that is, by Lemma 5.4 $\gamma \sqsubseteq \gamma''$. This contradicts that F_1 is the canonical disjunctive form.

From the above, any minterm appearing in F_1 also appear in F_2, and in the similar manner we can also show that any minterm appearing in F_2 also appear in F_1. This completes the proof of the theorem. ■

7.4 A Characterization of α-KS Logic Functions

By Theorem 3 there is a one-to-one and onto mapping between the set of α-KS logic functions and the set of 7-valued α-KS logic functions, which are defined as an α-KS logic function whose domain is restricted to the set V_7^n, that is, a 7-valued α-KS logic function is a function $F : V_7^n \to V_7$, called 7-valued function below, represented by a logic formula. Obviously, 7-valued α-KS logic functions are not functionally complete, that is, it is impossible that every 7-valued function can not obtained by means of a logic formula. Of course, α-KS logic functions are not functionally complete by Theorem 1. Therefore, in the section, we discuss a necessary and sufficient condition for a 7-valued function to be a 7-valued α-KS logic function, and then show a necessary and sufficient condition for a function $F : V^n \to V$, called an infinite-valued function below, to be an α-KS logic function.

7.4.1 Necessary and Sufficient Condition for 7-Valued α-KS Logic Functions

The following set of four conditions is a necessary and sufficient condition for a 7-valued function to be a 7-valued α-KS logic function.

(*a*) $\boldsymbol{a} \in \{0, \alpha, 1/2, \alpha', 1\}^n$ implies $F(\boldsymbol{a}) \in \{0, \alpha, 1/2, \alpha', 1\}$

(*b*) $\boldsymbol{a} \in \{0, \beta, 1/2, \beta', 1\}^n$ implies $F(\boldsymbol{a}) \in \{0, \beta, 1/2, \beta', 1\}$

(*c*) $\boldsymbol{a} \in \{0, \alpha, \beta, \beta', \alpha', 1\}^n$ implies $F(\boldsymbol{a}) \in \{0, \alpha, \beta, \beta', \alpha', 1\}$

(*d*) $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$ implies $F(\boldsymbol{a}) \leq_\alpha F(\boldsymbol{b})$

where $\boldsymbol{a}, \boldsymbol{b} \in V_7^n$.

Before showing that the set of Conditions $(a) \sim (d)$ is a necessary and sufficient condition for 7-valued α-KS logic functions, we will clarify some properties of 7-valued functions satisfying the conditions $(a) \sim (d)$.

(I): 7-valued Functions Satisfying Conditions $(a) \sim (d)$

Let F be a 7-valued function satisfying Conditions $(a) \sim (d)$. Then, we consider specific seven subsets

$$F^{-1}(i) = \{\boldsymbol{a} \in V_7^n \mid F(\boldsymbol{a}) = i\},$$

where $i = 0,\ \alpha,\ \beta,\ 1/2,\ \beta',\ \alpha'$ or 1. It is clear that $F^{-1}(i) \cap F^{-1}(j) = \emptyset$ ($i \neq j$ and $i,\ j \in V_7$) and $\bigcup_{i \in V_7} F^{-1}(i) = V_7^n$. Suppose $\boldsymbol{a}$ be an element of $F^{-1}(1)$ (or $F^{-1}(1/2)$). Then any element $\boldsymbol{a}'$ such that $\boldsymbol{a} \leq_\alpha \boldsymbol{a}'$ is also element of $F^{-1}(1)$ (or $F^{-1}(1/2)$) from Condition (d), Moreover, if $\boldsymbol{a}$ is an element of $F^{-1}(\beta')$ (or $F^{-1}(\alpha)$), then any element $\boldsymbol{a}'$ such that $\boldsymbol{a}' \leq_\alpha \boldsymbol{a}$ is also element of $F^{-1}(\beta')$ (or $F^{-1}(\alpha)$) in the same manner. Therefore, the subsets $F^{-1}(1)$, $F^{-1}(\beta')$, $F^{-1}(1/2)$ and $F^{-1}(\alpha)$ each form partial order finite sets concerning with the relation $\leq_\alpha$. For any given F, the sets of maximal elements of $F^{-1}(\beta')$ and $F^{-1}(\alpha)$, each denoted by $\partial F^{-1}(\beta')$ and $\partial F^{-1}(\alpha)$, and the set of minimal elements of $F^{-1}(1)$, $F^{-1}(1/2)$, each denoted by $\partial F^{-1}(1)$, $\partial F^{-1}(1/2)$, are uniquely determined, respectively.

Lemma 7 *Let F be a 7-valued function satisfying Conditions $(a) \sim (d)$. Then, the following (1), (2), (3) and (4)hold.*

(1) $\partial F^{-1}(1) \subseteq \{0,\ \alpha,\ \beta,\ \beta',\ \alpha',\ 1\}^n$,

(2) $\partial F^{-1}(\beta') \subseteq \{0,\ \beta,\ 1/2,\ \beta',\ 1\}^n - \{0, 1/2, 1\}^n$,

(3) $\partial F^{-1}(1/2) \subseteq \{\alpha,\ \beta,\ 1/2,\ \beta',\ \alpha'\}^n - \{\alpha,\ \beta,\ \beta',\ \alpha'\}^n$,

(4) $\partial F^{-1}(\alpha) \subseteq \{0,\ \alpha,\ 1/2,\ \alpha',\ 1\}^n - \{0, 1/2, 1\}^n$.

Proof: First suppose $\boldsymbol{a} = (a_1, \ldots, a_n)$ is an element of $\partial F^{-1}(1)$, and assume $a_i = 1/2$ for some $i = 1, \ldots, n$. Then by Condition (d) the element $\boldsymbol{a}' = (a_1, \ldots, a_{i-1}, \beta$ (or β'), $a_{i+1}, \ldots, a_n)$ is an element of $F^{-1}(1)$, and this contradicts to $\boldsymbol{a} \in \partial F^{-1}(1)$. Therefore, $\partial F^{-1} \subseteq \{0,\ \alpha,\ \beta,\ \beta',\ \alpha',\ 1\}^n$ and it has been shown the lemma (1).

Next let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of $\partial F^{-1}(\beta')$, and suppose $a_i = \alpha$ for some $i = 1, \ldots, n$. Then by Condition (d) the element $\boldsymbol{a}' = (a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n)$ is an element of $F^{-1}(\beta')$, and this contradicts to $\boldsymbol{a} \in \partial F^{-1}(\beta')$. We can also derive same contradiction even if $i = \alpha'$. Therefore at least we obtain $\partial F^{-1}(\beta') \subseteq \{0,\ \beta,\ 1/2,\ \beta',\ 1\}^n$. By Condition (a) an element $\boldsymbol{a}$ of $\partial F^{-1}(\beta')$ never satisfies $\boldsymbol{a} \in \{0,\ 1/2,\ 1\}^n$. Therefore $\partial F^{-1}(\beta') \subseteq \{0, \beta, 1/2, \beta', 1\}^n - \{0, 1/2, 1\}^n$, and it has been shown the lemma (2). In the remaining cases we can prove in the similar manner. ∎

Let define $\Delta F^{-1}(1)$ as the set

$$\Delta F^{-1}(1) = \left\{\boldsymbol{b} \in \{0,\ \alpha,\ 1/2,\ \alpha',\ 1\}^n \mid \boldsymbol{a} \in \partial F^{-1}(1) \text{ and } \boldsymbol{a} \leq_{\mathrm{K}} \boldsymbol{b}\right\},$$

then we can show the following lemma.

Lemma 8 *Let F be a 7-valued function satisfying Condition $(a) \sim (d)$. Then $\partial F^{-1}(1) \neq \emptyset$ if and only if $\Delta F^{-1}(1) \neq \emptyset$.*
(The proof is omitted)

Lemma 9 *Let F be a 7-valued function satisfying Conditions $(a) \sim (d)$. If $F(\boldsymbol{a}) = \alpha'$, then there is an element $\boldsymbol{b}$ of $\Delta F^{-1}(1)$ such that $\boldsymbol{a} \leq_\alpha \boldsymbol{c}$ where $\boldsymbol{b} \leq_\alpha \boldsymbol{c}$ and $\boldsymbol{c} \in \{0, 1/2, 1\}^n$.*

Proof: By $F(\boldsymbol{a}) = \alpha'$ we obtain an element $\boldsymbol{c}$ of $\{0, 1/2, 1\}^n$ such that $\boldsymbol{a} \leq_\alpha \boldsymbol{c}$, an by Condition (a) and (d) it has to be $F(\boldsymbol{c}) = 1$. Therefore, we can find an element $\boldsymbol{b}'$ of $\partial F^{-1}(1)$ such that $\boldsymbol{b}' \leq_\alpha \boldsymbol{c}$, that is, there is an element $\boldsymbol{b}$ of $\Delta F^{-1}(1)$ such that $\boldsymbol{b}' \leq_{\mathrm{K}} \boldsymbol{b}$. This implies $\boldsymbol{b} \leq_\alpha \boldsymbol{c}$, and we have been shown the lemma. ∎

(II): Proofs

In this section, we will show that the set of Conditions $(a) \sim (d)$ is a necessary and sufficient condition for a 7-valued function to be a 7-valued α-KS logic function.

Theorem 5 *If F is a 7-valued α-KS logic function, then F is a 7-valued function satisfying Conditions $(a) \sim (d)$.*

Proof: It is evident from the definition of each operation $(\cdot, \vee, \sim, \neg)$ and from Theorem 3. ∎

Theorem 6 *If F is a 7-valued function satisfying Conditions $(a) \sim (d)$, then F is a 7-valued α-KS logic function.*

Proof: For any given 7-valued function F satisfying Conditions $(a) \sim (d)$, we will be able to construct a 7-valued α-KS logic function representing F. Let F_i (i=1, 2, 3, 4) be a logic formula constructed by the disjunction of all minterms of type i (i=1, 2, 3, 4) corresponding to all elements of $\Delta F^{-1}(1)$, $\partial F^{-1}(\beta')$, $\partial F^{-1}(1/2)$ and $\partial F^{-1}(\alpha)$, respectively (We can always construct each type of minterms of F_i (i=1, 2, 3, 4) from Definition 9, 10, 11, 12 and Lemma 7). Then, we can show that $F_f = F_1 \vee F_2 \vee F_3 \vee F_4$ and F are equivalent 7-valued functions as follows.

(1) Suppose $F(\boldsymbol{a}) = 1$. Then, there is an element $\boldsymbol{a}' \in \partial F^{-1}(1)$ such that $\boldsymbol{a}' \leq_\alpha \boldsymbol{a}$, that is, we have an element $\boldsymbol{b} \in \Delta F^{-1}(1)$ such that $\boldsymbol{a}' \leq_{\mathrm{K}} \boldsymbol{b}$. The relations $\boldsymbol{a}' \leq_\alpha \boldsymbol{a}$ and $\boldsymbol{a}' \leq_{\mathrm{K}} \boldsymbol{b}$ imply $\boldsymbol{a} \preceq_\alpha \boldsymbol{b}$ since $\boldsymbol{a}' \in \{0, \alpha, \beta, \beta', \alpha', 1\}^n$ and $\boldsymbol{b} \in \{0, \alpha, 1/2, \alpha', 1\}^n$ by Lemma 7.1 and the definition of $\Delta F^{-1}(1)$. Therefore, by Lemma 1.1 we obtain $\gamma(\boldsymbol{a}) = 1$ where γ is the type 1 minterm corresponding to $\boldsymbol{b}$. Accordingly $F_f(\boldsymbol{a}) = 1$. Conversely suppose $F_f(\boldsymbol{a}) = 1$. Then, there is a type 1 minterm γ in F_1 such that $\gamma(\boldsymbol{a}) = 1$, and therefore for the corresponding element $\boldsymbol{b}$ to γ we obtain $\boldsymbol{a} \preceq_\alpha \boldsymbol{b}$ by Lemma 1.1. On the other hand, there is an element $\boldsymbol{a}'$ of $\partial F^{-1}(1)$ such that $\boldsymbol{a}' \leq_{\mathrm{K}} \boldsymbol{b}$ because $\boldsymbol{b}$ is an element of $\Delta F^{-1}(1)$. Since $\boldsymbol{a} \in \{0, \alpha, \beta, \beta', \alpha', 1\}^n$ and $\boldsymbol{b} \in \{0, \alpha, 1/2, \alpha', 1\}^n$ the relations $\boldsymbol{a} \preceq_\alpha \boldsymbol{b}$ and $\boldsymbol{a}' \leq_{\mathrm{K}} \boldsymbol{b}$ imply $\boldsymbol{a}' \leq_\alpha \boldsymbol{a}$, and therefore $F(\boldsymbol{a}) = 1$.

(2) Suppose $F(\boldsymbol{a}) = \alpha'$ (in this case, $F_f(\boldsymbol{a}) \leq \alpha'$ from the above discussions). Then, there is an element $\boldsymbol{b} \in \Delta F^{-1}(1)$ such that $\boldsymbol{a} \leq_\alpha \boldsymbol{c}$ from Lemma 9, where $\boldsymbol{b} \leq_\alpha \boldsymbol{c}$ and $\boldsymbol{c} \in \{0, 1/2, 1\}^n$. Therefore, $\gamma(\boldsymbol{a}) = 1$ or α' for the minterm γ corresponding to $\boldsymbol{b}$ from Lemma 1.3. $\gamma(\boldsymbol{a}) = 1$ dose not hold from the above discussions. Thus, we have $F_f(\boldsymbol{a}) = \alpha'$. Conversely, suppose $F_f(\boldsymbol{a}) = \alpha'$ ($F_f(\boldsymbol{a}) \leq \alpha'$ from the above discussions). Then, there is a minterm γ in F_1 such that $\gamma(\boldsymbol{a}) \neq 0$. Therefore, for the element $\boldsymbol{b}$ corresponding to γ we have $\boldsymbol{a} \leq_\alpha \boldsymbol{c}$ where $\boldsymbol{b} \leq_\alpha \boldsymbol{c}$ and $\boldsymbol{c} \in \{0, 1/2, 1\}^n$. Thus, $F(\boldsymbol{a}) = 1$ or α'. $F(\boldsymbol{a}) = 1$ dose not hold from the above discussions. Therefore, we have $F(\boldsymbol{a}) = \alpha'$.

(3) Suppose $F(\boldsymbol{a}) = \beta'$ ($F_f(\boldsymbol{a}) \leq \beta'$ from the above discussions). Then, there is an element $\boldsymbol{b} \in \partial F^{-1}(\beta')$ such that $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$. Therefore, $\gamma(\boldsymbol{a}) = \beta'$ for the minterm γ corresponding to $\boldsymbol{b}$ from Lemma 2.1. Thus, we have $F_f(\boldsymbol{a}) = \beta'$. Conversely, suppose $F_f(\boldsymbol{a}) = \beta'$ ($F(\boldsymbol{a}) \leq \beta'$ from the above discussions). Then, there is a minterm γ in F_2 such that $\gamma(\boldsymbol{a}) = \beta'$. Therefore, $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$ for the element $\boldsymbol{b}$ corresponding to γ from Lemma 2.1. Thus, we have $F(\boldsymbol{a}) = \beta'$.

(4) Suppose $F(\boldsymbol{a}) = 1/2$ ($F_f(\boldsymbol{a}) \leq 1/2$ from the above discussions). Then, there is an element $\boldsymbol{b} \in \partial F^{-1}(1/2)$ such that $\boldsymbol{b} \leq_\alpha \boldsymbol{a}$. Therefore, $\gamma(\boldsymbol{a}) = 1/2$ for the minterm γ corresponding to $\boldsymbol{b}$ from Lemma 3.1. Thus, we have $F_f(\boldsymbol{a}) = 1/2$. Conversely, suppose $F_f(\boldsymbol{a}) = 1/2$ ($F(\boldsymbol{a}) \leq 1/2$ from the above discussions). This implies at least one of the following cases.

(I) There is a minterm γ in F_2 such that $\gamma(\boldsymbol{a}) = 1/2$, or

(II) There is a minterm γ in F_3 such that $\gamma(\boldsymbol{a}) = 1/2$.

First suppose (I) holds, then $\boldsymbol{a}\Delta_\alpha\boldsymbol{b} \neq \emptyset$ from Lemma 2.2. Let $\boldsymbol{c} = \boldsymbol{a}\Delta_\alpha\boldsymbol{b}$, that is, $\boldsymbol{c} \leq_\alpha \boldsymbol{a}$ and $\boldsymbol{c} \leq_\alpha \boldsymbol{b}$. Then, $F(\boldsymbol{c}) = \beta'$ since $F(\boldsymbol{b}) = \beta'$. Therefore, $F(\boldsymbol{a}) = 1/2$ or β', but $F(\boldsymbol{a}) = \beta'$ dose not hold from the above discussions. Thus, we have $F(\boldsymbol{a}) = 1/2$. Next, suppose (II) holds then $\boldsymbol{b} \leq_\alpha \boldsymbol{a}$ for the element $\boldsymbol{b}$ corresponding to γ from Lemma 3.1. Therefore, we have $F(\boldsymbol{a}) = 1/2$.

(5) Suppose $F(\boldsymbol{a}) = \beta$ ($F_f(\boldsymbol{a}) \leq \beta$ from the above discussions). Then, there is an element $\boldsymbol{b}$ of $\partial F^{-1}(1/2)$ such that $\boldsymbol{b} \leq_\alpha \boldsymbol{c}$ where $\boldsymbol{a} \leq_\alpha \boldsymbol{c}$ and $\boldsymbol{c} \in \{0, 1/2, 1\}^n$. Because, for the element $\boldsymbol{c}$ such that $\boldsymbol{a} \leq_\alpha \boldsymbol{c}$ and $\boldsymbol{c} \in \{0, 1/2, 1\}^n$, we have $\boldsymbol{c} \in F^{-1}(1/2)$ and therefore there is a minimal element $\boldsymbol{b} \in \partial F^{-1}(1/2)$ such that $\boldsymbol{b} \leq_\alpha \boldsymbol{c}$. Accordingly there is a minterm γ in F_3, which is the corresponding minterm to $\boldsymbol{b}$, such that $\gamma(\boldsymbol{a}) = \beta$ or $1/2$ by Lemma 3.3, and $\gamma(\boldsymbol{a}) = 1/2$, however, does not hold from the above discussions. Thus, $F_f(\boldsymbol{a}) = \beta$. Conversely, suppose $F_f(\boldsymbol{a}) = \beta$ ($F(\boldsymbol{a}) \leq \beta$ from the above discussions). Then, there is a minterm γ in F_2 or F_3 such that $\gamma(\boldsymbol{a}) = \beta$, and therefore by Lemma 2.4 or Lemma 3.3 we have $\boldsymbol{a} \leq_\alpha \boldsymbol{c}$ where $\boldsymbol{b} \leq_\alpha \boldsymbol{c}$ and $\boldsymbol{c} \in \{0, 1/2, 1\}^n$ for the corresponding element $\boldsymbol{b}$ to γ. Since $\boldsymbol{c} \in F^{-1}(1/2)$ we obtain $F(\boldsymbol{a}) = \beta$, $1/2$ or β', and $F(\boldsymbol{a}) = 1/2$ or β', however, does not hold from the above discussions. Therefore $F(\boldsymbol{a}) = \beta$.

(6) Suppose $F(\boldsymbol{a}) = \alpha$ ($F_f(\boldsymbol{a}) \leq \alpha$ from the above discussions). Then, there is an element $\boldsymbol{b} \in \partial F^{-1}(\alpha)$ such that $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$. Therefore, $\gamma(\boldsymbol{a}) = \alpha$ by Lemma 4.1 for the corresponding minterm γ to $\boldsymbol{b}$, and this implies $F_f(\boldsymbol{a}) = \alpha$. Conversely, suppose $F_f(\boldsymbol{a}) = \alpha$ ($F(\boldsymbol{a}) \leq \alpha$ from the above discussions). Then, there is a minterm γ in F_4 such that $\gamma(\boldsymbol{a}) = \alpha$, and be Lemma 4.1 $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$ for the corresponding element $\boldsymbol{b}$ to γ. Therefore $F(\boldsymbol{a}) = \alpha$.

From the above we have that $F(\boldsymbol{a}) = F_f(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_7^n, and this completes the proof of the theorem. ■

Each one of $(a) \sim (d)$ can not derive from the remaining conditions, since each the following function F_i ($i = a, b, c$ or d) of Table 7.3 is an example satisfying three conditions except for i.

Table 7.3: Examples Showing Independence among Conditions $(a) \sim (d)$

x	0	α	β	$1/2$	β'	α'	1
$F_a(x)$	β	β	β	β	β	β	β
$F_b(x)$	α	α	α	α	α	α	α
$F_c(x)$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
$F_d(x)$	0	1	0	0	0	0	0

7.4.2 Necessary and Sufficient Condition for α-KS Logic Functions

The following set of five conditions is a necessary and sufficient condition for an infinite-valued function to be an α-KS logic function. However, these five conditions are not independent to each other. Therefore, in the section, first we show the proofs for the necessary and sufficient condition, and next discuss the relationship among these five conditions.

(A) $\boldsymbol{a} \in \{0, \alpha, 1/2, \alpha', 1\}^n$ implies $F(\boldsymbol{a}) \in \{0, \alpha, 1/2, \alpha', 1\}$

(B) $\boldsymbol{a} \in \{0, \beta, 1/2, \beta', 1\}^n$ implies $F(\boldsymbol{a}) \in \{0, \beta, 1/2, \beta', 1\}$

(C) $\boldsymbol{a} \in \{0, \alpha, \beta, \beta', \alpha', 1\}^n$ implies $F(\boldsymbol{a}) \in \{0, \alpha, \beta, \beta', \alpha', 1\}$

(D) $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$ implies $F(\boldsymbol{a}) \leq_\alpha F(\boldsymbol{b})$

(E) $\overline{F(\boldsymbol{a})}^{\varepsilon\delta} = F(\overline{\boldsymbol{a}}^{\varepsilon\delta})$ for any ε and δ such that $0 < \varepsilon \leq \alpha < \delta \leq 1/2$

where $\boldsymbol{a} \in V^n$.

(I): Proofs

Lemma 10 *Let F be an infinite-valued function. If F satisfies Condition (E), then F also satisfies that $F(\boldsymbol{a})$ is an element of V_7 whenever $\boldsymbol{a} \in V_7^n$.*

Proof: Let assume that $\boldsymbol{a}$ is an element of V_7^n such that $F(\boldsymbol{a}) \notin V_7$. Then $\boldsymbol{a} = \overline{\boldsymbol{a}}^{\varepsilon\delta}$ for $\varepsilon = \alpha$ and $\delta = 1/2$. Therefore, we obtain $F(\boldsymbol{a}) = F(\overline{\boldsymbol{a}}^{\varepsilon\delta}) = \overline{F(\boldsymbol{a})}^{\varepsilon\delta} \in V_7$ for $\varepsilon = \alpha$ and $\delta = 1/2$ from Condition (E). This contradicts to the assumption $F(\boldsymbol{a}) \notin V_7^n$, and we complete the proof of the lemma. ■

Theorem 7 *If F is an α-KS logic function, then F is an infinite-valued function satisfying Condition $(A) \sim (E)$.*

Proof: It is evident from the definition of each operations ($\cdot$, $\vee$, $\sim$, $\neg$) and Theorem 1 and Theorem 3. ■

Theorem 8 *If F is an infinite-valued function satisfying Condition (A) $\sim$ (E), then F is an α-KS logic function.*

Proof: By Lemma 10 and Condition (E) $F(\boldsymbol{a}) \in V_7$ whenever $\boldsymbol{a} \in V_7^n$. Therefore, there is a logic formula F_α such that $F(\boldsymbol{a}) = F_\alpha(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_7^n by Theorem 6. Then, we can prove that $F(\boldsymbol{a}) = F_\alpha(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V^n below. Let suppose $F(\boldsymbol{b}) \neq F_\alpha(\boldsymbol{b})$ for some element $\boldsymbol{b}$ of $V^n - V_7^n$. This implies at least one of

$$(1)\ F(\boldsymbol{b}) > F_\alpha(\boldsymbol{b}) \text{ or } \quad (2)\ F(\boldsymbol{b}) < F_\alpha(\boldsymbol{b})$$

In any case (1) or (2), we can always find ε and δ ($0 < \varepsilon \leq \alpha < \delta \leq 1/2$) such that $\overline{F(\boldsymbol{b})}^{\varepsilon\delta} \neq \overline{F_\alpha(\boldsymbol{b})}^{\varepsilon\delta}$ and by Theorem 2 and Condition (E) $\overline{F(\boldsymbol{b})}^{\varepsilon\delta} = F(\overline{\boldsymbol{b}}^{\varepsilon\delta}) \neq F_\alpha(\overline{\boldsymbol{b}}^{\varepsilon\delta}) = \overline{F_\alpha(\boldsymbol{b})}^{\varepsilon\delta}$. This contradicts that $F(\boldsymbol{a}) = F_\alpha(\boldsymbol{a})$ for any element $\boldsymbol{a}$ of V_7^n since $\overline{\boldsymbol{b}}^{\varepsilon\delta} \in V_7^n$. This completes the proof of the theorem. ■

(II): Relationship among Conditions $(A) \sim (E)$

Condition $(A) \sim (D)$ are not independent to each other, since (A) and (D) are derived from (E). In this section, we will show the above relations.

Theorem 9 *Let F be an infinite-valued function. If F satisfies Condition (E), then F also satisfies Condition (A).*

Proof: By Condition (E) and Lemma 10, $F(\boldsymbol{a}) \in V_7$ holds for any element $\boldsymbol{a} \in V_7^n$. Suppose that $\boldsymbol{b}$ is an element of $\{0, \alpha, 1/2, \alpha', 1\}^n$ such that $F(\boldsymbol{b}) \notin \{0, \alpha, 1/2, \alpha', 1\}$. This implies that $F(\boldsymbol{b}) \in V_7 - \{0, \alpha, 1/2, \alpha', 1\} = \{\beta, \beta'\}$ and $\boldsymbol{b} = \overline{\boldsymbol{b}}^{\varepsilon\delta}$ for any ε and δ such that $0 < \varepsilon \leq \alpha < \delta \leq 1/2$. Therefore, $F(\boldsymbol{b}) = F(\overline{\boldsymbol{b}}^{\varepsilon\delta}) = \overline{F(\boldsymbol{b})}^{\varepsilon\delta}$ holds for any ε and δ ($0 < \varepsilon \leq \alpha < \delta \leq 1/2$). On the other hand, we obtain $\overline{F(\boldsymbol{b})}^{\varepsilon\delta} = 1/2$ for $\delta = \beta$ and $0 < \varepsilon \leq \alpha$, and this completes the proof of the theorem. ■

Lemma 11 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V_7^n such that $\boldsymbol{a} \preceq_\alpha \boldsymbol{b}$. Then, there exist an element $\boldsymbol{t} = (t_1, \ldots, t_n)$ of V^n and constants ε_1, ε_2, δ_1 and δ_2 such that $\boldsymbol{a} = \overline{\boldsymbol{t}}^{\varepsilon_1\delta_1}$ and $\boldsymbol{b} = \overline{\boldsymbol{t}}^{\varepsilon_2\delta_2}$ where $0 < \varepsilon_1 < \varepsilon_2 \leq \alpha < \delta_2 \leq \delta_1 \leq 1/2$.*

Proof: By $\boldsymbol{a} \preceq_\alpha \boldsymbol{b}$, $a_i \preceq_\alpha b_i$ holds for any $i = 1, \ldots, n$. This implies one of the following relations holds for any i.

$$(1)\ a_i = b_i, \qquad (2)\ a_i = \alpha \text{ and } b_i = 0,$$
$$(3)\ a_i = \beta \text{ or } \beta' \text{ and } b_i = 1/2, \text{ or } \quad (4)\ a_i = \alpha' \text{ and } b_i = 1.$$

(1) is a trivial case. When (2) holds, in order to exist t_i such that $\overline{t_i}^{\varepsilon_1\delta_1} = \alpha$ and $\overline{t_i}^{\varepsilon_2\delta_2} = 0$, the relations $\varepsilon_1 \leq t_i \leq \alpha$ and $0 \leq t_i < \varepsilon_2$ have to be held, and therefore, we have $t_i^{\varepsilon_1\delta_1} = \alpha$ and $t_i^{\varepsilon_2\delta_2} = 0$ for any t_i such that $\varepsilon_1 \leq t_i < \varepsilon_2$. When (3) holds, in order to exist t_i such that $\overline{t_i}^{\varepsilon_1\delta_1} = \beta$ or β' and $\overline{t_i}^{\varepsilon_2\delta_2} = 1/2$, the relation $\alpha < t_i < \delta_1$ when $\overline{t_i}^{\varepsilon_1\delta_1} = \beta$ or $1 - \delta_1 < t_i < 1 - \alpha$ when $\overline{t_i}^{\varepsilon_1\delta_1} = \beta'$ and $\delta_2 < t_i < 1 - \delta_2$ have to be held, and therefore, we obtain $\overline{t_i}^{\varepsilon_1\delta_1} = \beta$ or β' and $\overline{t_i}^{\varepsilon_2\delta_2} = 1/2$ for any t_i such that $\delta_2 \leq t_i < \delta_1$. When (4) holds, then we can also have $\overline{t_i}^{\varepsilon_1\delta_1} = \alpha'$ and $\overline{t_i}^{\varepsilon_2\delta_2} = 1$ for any t_i such that $\varepsilon_1 \leq t_i < \varepsilon_2$ in the similar manner. This completes the proof of the lemma. ■

Lemma 12 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be elements of V^n. $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$ if and only if $\overline{\boldsymbol{a}}^{\varepsilon\delta} \leq_\alpha \overline{\boldsymbol{b}}^{\varepsilon\delta}$ for any ε and δ such that $0 < \varepsilon \leq \alpha < \delta \leq 1/2$.*

Proof: First suppose $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$. This implies $a_i \leq_\alpha b_i$ for any $i = 1, \ldots, n$. It is evident that $\overline{\boldsymbol{b}}^{\varepsilon\delta} \in V_7^n$ for any ε and δ such that $0 < \varepsilon \leq \alpha < \delta \leq 1/2$, that is, $\overline{b_i}^{\varepsilon\delta} \in V_7$ for any i. If $\overline{b_i}^{\varepsilon\delta} = 0$, then this implies $0 \leq b_i < \varepsilon$, and therefore, by $a_i \leq_\alpha b_i$ we have $0 \leq b_i \leq a_i \leq \alpha$. Accordingly, $\overline{a_i}^{\varepsilon\delta} = 0$ or α holds, that is, we obtain $\overline{a_i}^{\varepsilon\delta} \leq_\alpha \overline{b_i}^{\varepsilon\delta}$. If $\overline{b_i}^{\varepsilon\delta} = \alpha$, then this implies $\varepsilon \leq b_i \leq \alpha$, and therefore, by $a_i \leq_\alpha b_i$ we have $\varepsilon \leq b_i \leq a_i \leq \alpha$. Accordingly, $\overline{a_i}^{\varepsilon\delta} = \alpha$ holds, that is, we obtain $\overline{a_i}^{\varepsilon\delta} \leq_\alpha \overline{b_i}^{\varepsilon\delta}$. If $\overline{b_i}^{\varepsilon\delta} = \beta$, then this implies $\alpha < b_i < \delta$, and therefore, by $a_i \leq_\alpha b_i$ we have $\alpha < b_i < a_i \leq 1/2$. Accordingly $\overline{a_i}^{\varepsilon\delta} = \beta$ or $1/2$ holds, that is, we obtain $\overline{a_i}^{\varepsilon\delta} \leq_\alpha \overline{b_i}^{\varepsilon\delta}$. For the remaining cases, we can also prove $a_i^{\varepsilon\delta} \leq_{\mathrm{B}} b_i^{\varepsilon\delta}$ in the similar manner. Therefore, we have been shown the former half of the lemma.

Conversely suppose $\overline{\boldsymbol{a}}^{\varepsilon\delta} \leq_\alpha \overline{\boldsymbol{b}}^{\varepsilon\delta}$ for any ε and δ such that $0 < \varepsilon \leq \alpha < \delta \leq 1/2$. $\delta = 1/2$. We can assume without loss of generality that $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$ never holds. This implies that there exists at least one $i = 1, \ldots, n$ such that $a_i \leq_\alpha b_i$ never holds, that is, one of the following relations has to be held.

(1) a_i and b_i are not comparable to each other, or

(2) $b_i \leq_\alpha a_i$ and $a_i \neq b_i$.

If (1) holds, then $\overline{a_i}^{\varepsilon\delta}$ and $\overline{b_i}^{\varepsilon\delta}$ are not comparable to each other for any ε such that $0 < \varepsilon \leq \alpha$ and $\delta = 1/2$, and this contradicts to the assumption. Next, suppose (2) holds. If $0 \leq a_i < b_i \leq \alpha$ or $1 - \alpha \leq b_i < a_i \leq 1$, then we never have $\overline{a_i}^{\varepsilon'\delta} \leq_\alpha \overline{b_i}^{\varepsilon'\delta}$ for any δ such that $\alpha < \delta \leq 1/2$ and $\varepsilon' = \min(\varepsilon'', 1 - \varepsilon'')$ where $\varepsilon'' = (a_i + b_i)/2$. If $\alpha < b_i < a_i \leq 1/2$ or $1 - \alpha < b_i < a_i \leq 1/2$, then we also never have $\overline{a_i}^{\varepsilon\delta'} \leq_\alpha \overline{b_i}^{\varepsilon\delta'}$ for any ε such that $0 < \varepsilon \leq \alpha$ and $\delta' = \min(\delta'', 1 - \delta'')$ where $\delta'' = (a_i + b_i)/2$. Therefore, we have been shown the latter half of the lemma, and this completes the proof of the lemma. ∎

Lemma 13 *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be an element of V^n and ε_1, ε_2, δ_1 and δ_2 be constants such that $0 < \varepsilon_1 \leq \alpha < \delta_1 \leq 1/2$ and $0 < \varepsilon_2 \leq \alpha < \delta_2 \leq 1/2$, respectively. Then $\varepsilon_2 \leq_\alpha \varepsilon_1$ and $\delta_2 \leq_\alpha \delta_1$ imply $\overline{\boldsymbol{a}}^{\varepsilon_1\delta_1} \leq_\alpha \overline{\boldsymbol{a}}^{\varepsilon_2\delta_2}$.*

Proof: By $\varepsilon_2 \leq_\alpha \varepsilon_1$, we have $0 < \varepsilon_1 \leq \varepsilon_2 \leq \alpha$ or $1 - \alpha \leq \varepsilon_2 \leq \varepsilon_1 < 1$, and moreover by $\delta_2 \leq_\alpha \delta_1$ we also have $\alpha < \delta_2 \leq \delta_1 \leq 1/2$ or $1 - \alpha < \delta_2 \leq \delta_1 \leq 1/2$. Therefore, these conditions satisfy $\overline{a_i}^{\varepsilon_1\delta_1} \leq_\alpha \overline{a_i}^{\varepsilon_2\delta_2}$ for any $a_i \in V$. This completes the proof of the lemma. ∎

Theorem 10 *Let F be an infinite-valued function. If F satisfies Condition (E), then F also satisfies Condition (D).*

Proof: Suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ elements of V^n such that $\boldsymbol{a} \leq_\alpha \boldsymbol{b}$. Then, $\overline{\boldsymbol{a}}^{\varepsilon\delta} \leq_\alpha \overline{\boldsymbol{b}}^{\varepsilon\delta}$ holds for any ε and δ $(0 < \varepsilon \leq \alpha < \delta \leq 1/2)$ from Lemma 12. Therefore, there exist an element $\boldsymbol{t}$ of V^n and ε_1, ε_2, δ_1 and δ_2 $(0 < \varepsilon_1 < \varepsilon_2 \leq \alpha < \delta_2 \leq \delta_1 \leq 1/2)$ such that $\overline{\boldsymbol{a}}^{\varepsilon\delta} = \overline{\boldsymbol{t}}^{\varepsilon_1\delta_1}$ and $\overline{\boldsymbol{b}}^{\varepsilon\delta} = \overline{\boldsymbol{t}}^{\varepsilon_2\delta_2}$ from Lemma 11. Then, we obtain the following relations from Condition (E).

$$\begin{aligned} \overline{F(\boldsymbol{a})}^{\varepsilon\delta} = F(\overline{\boldsymbol{a}}^{\varepsilon\delta}) &= F(\overline{\boldsymbol{t}}^{\varepsilon_1\delta_1}) = \overline{F(\boldsymbol{t})}^{\varepsilon_1\delta_1}, \\ \overline{F(\boldsymbol{b})}^{\varepsilon\delta} = F(\overline{\boldsymbol{b}}^{\varepsilon\delta}) &= F(\overline{\boldsymbol{t}}^{\varepsilon_2\delta_2}) = \overline{F(\boldsymbol{t})}^{\varepsilon_2\delta_2}, \end{aligned}$$

where $\varepsilon_2 \leq_\alpha \varepsilon_1$ and $\delta_2 \leq_\alpha \delta_1$ since $0 < \varepsilon_1 < \varepsilon_2 \leq \alpha$ and $\alpha < \delta_2 \leq \delta_1 \leq 1/2$. Therefore, $\overline{F(t)}^{\varepsilon_1\delta_1} \leq_\alpha \overline{F(t)}^{\varepsilon_2\delta_2}$ from Lemma 13, that is, we obtain $\overline{F(t)}^{\varepsilon_1\delta_1} = \overline{F(a)}^{\varepsilon\delta} \leq_\alpha \overline{F(b)}^{\varepsilon\delta} = \overline{F(t)}^{\varepsilon_2\delta_2}$ for any ε and δ $(0 < \varepsilon \leq \alpha < \delta \leq 1/2)$. Thus, $F(a) \leq_\alpha F(b)$ holds from Lemma 12. This completes the proof of the theorem. ■

Each one of (B), (C) and (E) can not derive from the remaining two conditions. Because, it is evident that F_B of Figure 7.4 satisfies Condition (C) but not (B), and we can show it also satisfies (E) as follows. $\overline{a}^{\varepsilon\delta} \in V_7$ for any $a \in V$ and any ε and δ $(0 < \varepsilon \leq \alpha < \delta \leq 1/2)$, and obviously $F_B(\overline{a}^{\varepsilon\delta}) = \alpha$. We also obtain $\overline{F_B(a)}^{\varepsilon\delta}$ for any $a \in V$ and any ε and δ $(0 < \varepsilon \leq \alpha < \delta \leq 1/2)$ since $F_B(a) = \alpha$ for any element $a \in V$. Therefore, we necessary can not derive (B) from (C) and (E).

Next, F_C of Figure 7.5 is an example satisfying (B) and (E) but not (C). Because, it is evident it satisfies (B) but not (C). In the similar manner, we obtain $F_C(\overline{a}^{\varepsilon\delta}) = 1/2$ for any $a \in V$ and any ε and δ $(0 < \varepsilon \leq \alpha < \delta \leq 1/2)$. Also $\overline{F_C(a)}^{\varepsilon\delta} = 1/2$ for any $a \in V$ and any ε and δ $(0 < \varepsilon \leq \alpha < \delta \leq 1/2)$ since $F_C(a) = 1/2$ for any element $a \in V$.

Finally, F_E of Figure 7.6 is an example satisfying (B) and (C) but not (E). Because, it is evident it satisfies (B) and (C). On the other hand, when $\delta = \beta$, $F_E(\overline{\beta}^{\varepsilon\delta}) = F_E(1/2) = 1/2$ and whereas $\overline{F_E(\beta)}^{\varepsilon\delta} = \overline{0}^{\varepsilon\delta} = 0$. Therefore, F_E does not satisfy (E).

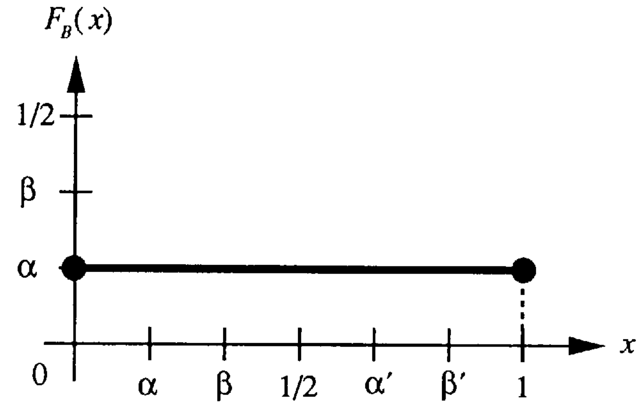

Figure 7.4: An Example Satisfying Only Conditions (C) and (E)

7.5 Conclusions

In the chapter, we discussed some properties of α-KS logic functions.

As compared with the discussions of previous chapters, the following two points are lacks of discussions in the chapter: minimization of α-KS logic functions and the number of n-variable α-KS logic functions. Therefore, these are still open problems. However, our postulation is that the number of n-variable α-KS logic functions is represented in the term of monotone Boolean functions and B-ternary logic functions.

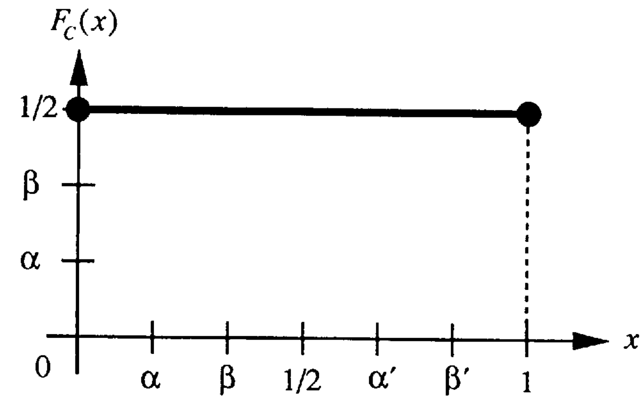

Figure 7.5: An Example Satisfying Only Conditions (B) and (E)

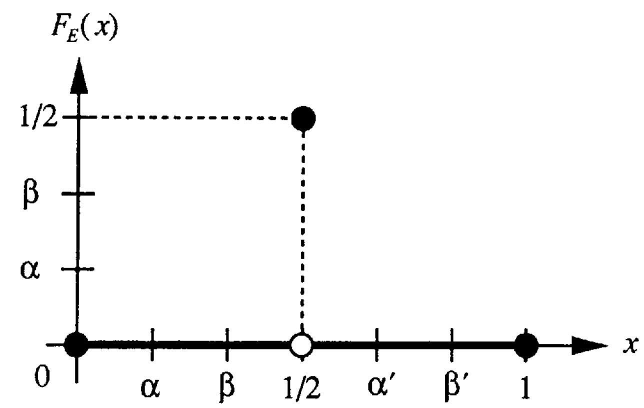

Figure 7.6: An Example Satisfying Only Conditions (B) and (C)

Chapter 8

Conclusions

This dissertation shows some properties of meaningful special classes of multiple-valued logic functions, especially, some mathematical aspects.

Chapter 5 described Kleenean functions which are an effective means to treat ambiguity. Kleenean functions are multiple-valued logic functions defined by extending the condition of Representation of a Regular Ternary Logic Formula, which is one of the conditions of regular ternary logic functions. We have more two different kinds conditions in representing regular ternary logic functions, and therefore, extending into multiple-valued of the remaining two conditions, that is, Regularity and Monotonicity of Ambiguity, are future problems.

In Chapter 5, we described some properties of Stone logic functions, which are defined as fuzzy logic functions not having the unary operation $\sim$ by $\neg$. In a series of multiple-valued logic functions taking the operations *min*, *max* and $(1-)$, for instance fuzzy logic function and Kleenean functions and so on, they each forms an algebraic system called a Kleene algebra. However, the set of Stone logic functions forms a Stone algebra which is different from a Kleene algebra. Therefore, Chapter 5 treats a new class of multiple-valued logic function besides the models of Kleene algebras.

Chapter 6 treated Kleene-Stone logic functions. Kleene-Stone logic functions are defined as fuzzy logic functions with the unary operation $\neg$ employed in Stone logic functions. As mentioned in the introduction of Chapter 6, Kleene-Stone logic functions have a connection with modal logic closely. The investigations of the relationship between Kleene-Stone logic functions and modal logic are interesting. We discussed minimization for Kleene-Stone logic functions. The minimal forms are defined in the terms of the number of literals, and we denoted an algorithm deriving a minimal form of a given Kleene-Stone logic functions and the algorithm is described in the terms of prime implicants. However, all minimal forms of a given Kleene-Stone logic function generally can not be represented by the terms of prime implicant only, and therefore, it can not derive all of the minimal forms of a given Kleene-Stone logic function. Finding a new algorithm which enables us to take all of the minimal form of any given Kleene-Stone logic functions is an open problem.

Chapter 7 discussed α-KS logic functions. They are defined as multiple-valued logic functions obtained by extending the operation $\neg$ of Kleene-Stone logic functions. The investigations for α-KS logic functions can easily lead us to the studies of fuzzy logic functions with α-*cut* operator $cut_\alpha : [0,1] \to \{0,1\}$ defined below.

$$cut_\alpha(x) = \begin{cases} 1 & \text{if } x \geq \alpha \\ 0 & \text{if } x < \alpha \end{cases}$$

where $\alpha \in [0,1]$. α-*cut* operator is a typical one of fuzzy set theory, and therefore, the investigations of fuzzy logic functions with α-*cut* operator will clear the theory of fuzzy set with α-*cut* operator.

Finally, the author hopes his investigations will be helpful for the development of multiple-valued logic and fuzzy set theory.

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