Almost Gorenstein 環

明治大学大学院 理工学研究科

2015年度

博士学位請求論文

Almost Gorenstein rings

（Almost Gorenstein 環）

学位請求者 基礎理工学専攻数学系

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December 9, 2015

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Acknowledgement

First of all, it is great pleasure that I heartily thank my academic adviser, Professor Shiro Goto who is a warm-hearted and generous teacher. He always passes on his abundant knowledge to students. I am deeply grateful to him for having the opportunity to benefit from not only his acute mathematical insight and valuable expertise but also precious lessons of social life. I have a firm belief that my life has changed by studying mathematics under him.

I also would like to thank Professors Ryo Takahashi and Ken-ichi Yoshida for their invaluable suggestions and profound expertise. In particular, I would like to express my deep gratitude to Professor Bernd Ulrich at Purdue University for his hospitality and interest to my research subjects. Besides them, I have benefited from comments from a lot of professors, co-workers, and friends, including Naoyuki Matsuoka, Tran Thi Phuong, Hoang Le Truong, Tran Nguyen An, Nguyen Thi Dung, Olgur Celikbas, Ela Celikbas, and Asuki Koura.

Moreover, this dissertation was partially supported by the Japan Society for the Promotion of Science Research Fellowships for Young Scientists during the two years of my doctoral courses.

Finally I wish to appreciate my family for their active support and reliance.

Preface

In the end of the 19th century, commutative ring theory was originally established by D. Hilbert throughout the study of invariant algebras. He then proved that every ideal in the polynomial ring over a field is finitely generated, which is nowadays known as *Hilbert's Basis Theorem*. After the breakthrough of his work, E. Noether played a central role of the developments of the theory of commutative algebra. At the middle of the 20th century, the notion of homological method was innovated into commutative ring theory by many researchers, say M. Auslander, D. A. Buchsbaum, D. Rees, D. G. Northcott, J.-P. Serre and others. Among them J.-P. Serre finally produced an innovative result which insists that any localization of a regular local ring is again regular. Since then, and up to the present day, commutative ring theory has been developed dramatically by investigating the theory of Cohen-Macaulay rings and modules.

Cohen-Macaulay rings are named after the results of F. S. Macaulay and I. S. Cohen. In 1916, F. S. Macaulay showed that the polynomial ring over a field satisfies *the unmixedness theorem*. Remember that an ideal I of a Noetherian ring R is called *unmixed* if I has no embedded associated prime divisors, more precisely, the associated prime ideals of R/I are exactly the minimal prime ideals of I. We say that a Noetherian ring R satisfies the unmixedness theorem, if every ideal I of R generated by $\mathrm{ht}_R I$ elements is unmixed. I. S. Cohen who was one of the students of O. Zariski proved in his Ph.D. thesis that the unmixedness theorem holds for every regular local ring. After their achievements, Cohen-Macaulay ring is defined to be a ring satisfies the unmixedness theorem. It is known that a Noetherian local ring is Cohen-Macaulay if its Krull dimension equals to the depth.

The origin of Gorenstein rings traces back to the article of D. Gorenstein [16] in 1952, which dealt with the plane curves. After that, A. Grothendieck introduced in 1957 the concept of Gorenstein local ring to be a Cohen-Macaulay local ring which is isomorphic to its canonical module. Hence Gorenstein rings are a special class of Cohen-Macaulay rings. Being inspired by this definition, H. Bass discovered in 1963 the deep relationship between the Gorenstein property and the finiteness of self-injective dimension and proved that the above two conditions are equivalent to each other (see [9]). Thereafter we confirm that Gorenstein rings are defined to be the rings which possess locally finite self-injective dimension. Gorenstein rings enjoy a beautiful symmetry. For instance, the numerical semigroup ring is Gorenstein if and only if the corresponding semigroup is symmetric. This fact is given by E. Kunz ([52]) in 1970, which was the starting point of the study of numerical semigroups and semigroup rings. Another example is the behavior of the h-vector $(h_0, h_1, \ldots, h_s)$ of a Cohen-Macaulay homogeneous domain R. In 1978, R. P. Stanley showed that R is Gorenstein if and only if $h_i = h_{s-i}$ for every $i = 0, 1, \ldots, \lfloor s/2 \rfloor$ ([68]).

There are known numerous examples of Cohen-Macaulay rings and among the progress of the theory of Cohen-Macaulay rings, we often encounter non-Gorenstein Cohen-Macaulay rings in the field of not only commutative algebra, but also algebraic geometry, representation theory, invariant theory, and combinatorics. On all such occasions, we have a natural query of why there are so many Cohen-Macaulay rings which are not Gorenstein. As we mentioned above, Gorenstein rings are defined by locally finite self-injective dimension. However there is a huge gap between the two conditions of finiteness and infiniteness of self-injective dimension. Based on this observation, the aim of this dissertation is to find a new class of Cohen-Macaulay rings, which may not be Gorenstein, but sufficiently good next to the Gorenstein rings.

One of the candidates for such a class is *almost Gorenstein rings*, which was originally studied by V. Barucci and R. Fröberg ([8]) in the case where the local rings are analytically unramified and of dimension one. After that, S. Goto, N. Matsuoka and T. T. Phuong ([26]) extended in 2013 the notion of almost Gorenstein property over one-dimensional Cohen-Macaulay local rings which are not necessarily analytically unramified. We are now in a position to ask the following question.

Problem A. Find a possible definition of almost Gorenstein rings of higher dimension.

To explain the aim and motivation of Problem A more precisely, let us review on the definition of almost Gorenstein rings of dimension one in the sense of Goto, Matsuoka, and Phuong ([26]).

For the moment, let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $\dim R = 1$. Let K_R stand for the canonical module of R. Then an ideal I of R is called *canonical*, if $I \neq R$ and $I \cong \mathrm{K}_R$ as an R-module. Notice that this definition implicitly assume the existence of the canonical module. By the result [43, Satz 6.21] of J. Herzog and E. Kunz, R possesses a canonical ideal if and only if the total ring of fractions $\mathrm{Q}(\widehat{R})$ of $\widehat{R}$ is Gorenstein, where we denote by $\widehat{R}$ the $\widehat{\mathfrak{m}}$-adic completion of R. Hence the ring R contains a canonical ideal I if it is analytically unramified. Since I is faithful and $\dim R = 1$, I is an $\mathfrak{m}$-primary ideal of R. Therefore there exist integers $\mathrm{e}_0(I) > 0$ and $\mathrm{e}_1(I)$ such that

$$\ell_R(R/I^{n+1}) = \mathrm{e}_0(I)\binom{n+1}{1} - \mathrm{e}_1(I)$$

for all integers $n \gg 0$. The integers $\mathrm{e}_i(I)$'s are called *the Hilbert coefficients of R with respect to I*. These integers describe the complexity of given local rings, and there are a huge number of preceding researches about them, e.g., [18, 19, 26, 28, 29]. In particular, the integer $\mathrm{e}_0(I) > 0$ is called *the multiplicity of R with respect to I* and has been explored very intensively.

Let $\mathrm{r}(R)$ stand for the Cohen-Macaulay type of R ([43, Definition 1.20]). Then the almost Gorenstein ring is defined as follows.

Definition B ([26])**.** *We say that R is an almost Gorenstein local ring, if R possesses a canonical ideal I of R such that* $\mathrm{e}_1(I) \leq \mathrm{r}(R)$.

Remember that if R is Gorenstein, then any parameter ideal Q of R is canonical and hence $\mathrm{e}_1(Q) < \mathrm{r}(R) = 1$, which implies that every Gorenstein local ring is an almost Gorenstien ring.

We now assume that I contains a parameter ideal $Q = (a)$ as a reduction, so that $I^{r+1} = QI^r$ for some integer $r \geq 0$. This assumption is automatically satisfied, if the residue class field $R/\mathfrak{m}$ of R is infinite. We set

$$K = \frac{I}{a} = \left\{\frac{x}{a} \mid x \in I\right\} \subseteq \mathrm{Q}(R).$$

Notice that K is a fractional ideal of R such that

$$R \subseteq K \subseteq \overline{R} \ \text{ and } \ K \cong \mathrm{K}_R$$

where $\overline{R}$ denotes the integral closure of R in $\mathrm{Q}(R)$. Then the result [26, Theorem 3.11] says that R is an almost Gorenstein ring if and only if $\mathfrak{m}K \subseteq R$, or equivalently

$\mathfrak{m}I = \mathfrak{m}Q$. The latter condition is the original definition of almost Gorenstein ring in the sense of [8]. Therefore if R is analytically unramified, that is $\widehat{R}$ is reduced, then the these two definitions of almost Gorenstein ring coincides, provided the residue class field $R/\mathfrak{m}$ of R is infinite.

In Chapter 1 of this thesis we introduce the notion of almost Gorenstein local ring of arbitrary dimension. In what follows, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 0$. Suppose that R possesses the canonical module K_R of R. Then my proposal for the definition of almost Gorenstein local ring is the following.

Definition C (Definition 1.1.1)**.** *We say that R is an almost Gorenstein local ring, if there exists an exact sequence*

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$. Here $\mu_R(C)$ (resp. $\mathrm{e}^0_{\mathfrak{m}}(C)$) denotes the number of elements in a minimal system of generators for C (resp. the multiplicity of C with respect to $\mathfrak{m}$).

Notice that every Gorenstein ring is by definition almost Gorenstein, and the converse holds if the ring R is Artinian. Thus Definition C requires that if R is an almost Gorenstein local ring, then R might be non-Gorenstein but the ring R can be embedded into its canonical module K_R so that the difference K_R/R should have good properties.

We look at an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules. Here we do not need to assume that R is almost Gorenstein. If $C \neq (0)$, then C is a Cohen-Macaulay R-module of dimension $d-1$. Suppose that the ring R possesses the infinite residue class field $R/\mathfrak{m}$. Set $\overline{R} = R/[(0):_R C]$ and let $\overline{\mathfrak{m}}$ denote the maximal ideal of $\overline{R}$. Choose elements $f_1, f_2, \ldots, f_{d-1} \in \mathfrak{m}$ such that $(f_1, f_2, \ldots, f_{d-1})\overline{R}$ forms a minimal reduction of $\overline{\mathfrak{m}}$. Then we have

$$\mathrm{e}^0_{\mathfrak{m}}(C) = \mathrm{e}^0_{\overline{\mathfrak{m}}}(C) = \ell_R(C/(f_1, f_2, \ldots, f_{d-1})C) \geq \ell_R(C/\mathfrak{m}C) = \mu_R(C).$$

Therefore $\mathrm{e}^0_{\mathfrak{m}}(C) \geq \mu_R(C)$ and we say that C is *an Ulrich R-module* if $\mathrm{e}^0_{\mathfrak{m}}(C) = \mu_R(C)$, since C is *a maximally generated maximal Cohen-Macaulay $\overline{R}$-module* in the sense of B. Ulrich ([10]). Thus C is an Ulrich R-module if and only if $\mathfrak{m}C = (f_1, f_2, \ldots, f_{d-1})C$.

Therefore if $\dim R = 1$, then the Ulrich property for C is equivalent to saying that C is a vector space over $R/\mathfrak{m}$.

One can construct many examples of almost Gorenstein rings of higher dimension. The significant examples of almost Gorenstein rings are one-dimensional Cohen-Macaulay local rings of finite Cohen-Macaulay representation type and two-dimensional rational singularity. Therefore, by using Auslander's result, every two-dimensional finite Cohen-Macaulay representation type is almost Gorenstein. Furthermore, for all the known examples of finite Cohen-Macaulay representation type are almost Gorenstein. Thus, it might be true that for any finite Cohen-Macaulay representation type is almost Gorenstein for arbitrary dimension, which we leave as an open question.

Let me explain how this thesis is organized. In Chapter 1 we shall give basic properties of almost Gorenstein local rings, including the so-called non-zerodivisor characterization. We obtain a lot of generalization of the results given by Goto, Matsuoka, and Phuong ([26]); for example, we have a characterization of almost Gorenstein rings in terms of canonical ideals, which extends the result [26, Theorem 3.11] to higher-dimensional local rings. The graded version is also posed and explored.

In Chapter 2 and 3 we focus our attention on the almost Gorenstein property for Rees algebras. The study of Cohen-Macaulay and Gorenstein properties for the Rees algebras are traced back to the research by S. Goto and Y. Shimoda ([33]) in 1979 and we nowadays have a satisfactorily developed theory about the Cohen-Macaulay property for the Rees algebras. Among Cohen-Macaulay Rees algebras, Gorenstein Rees algebras are rather rare. Nevertheless some of Cohen-Macaulay Rees algebras are still good and might be almost Gorenstein. This expectation naturally inspires the following question.

Problem D. Find the condition of when the Rees algebra $\mathcal{R}(I)$ of a given ideal I is almost Gorenstein.

In Chapter 2 we shall give the characterization for the Rees algebras of ideals generated by subsystem of parameters and ideals so-called socles ideals to be almost Gorenstein rings. In Chapter 3 we shall prove that the Rees algebras of integrally closed ideals over two-dimensional regular local rings are almost Gorenstein.

The main purpose of Chapter 4 is to clarify the structure of Ulrich ideals of almost Gorenstein local rings. Ulrich ideals are one of the inventions of S. Goto and their basic properties are provided by the two joint papers [30, 31]. The motivation for this research of Chapter 4 comes from a result of Kei-ichi Watanabe, who showed that non-Gorenstein almost Gorenstein numerical semigroup rings do not contain Ulrich monomial ideals except the maximal ideal. His result suggests that there should be some restriction of the distribution of Ulrich ideals of an almost Gorenstein but non-Gorenstein local ring, namely let me ask the following question.

Problem E. Determine Ulrich ideals in a given almost Gorenstein ring.

In Chapter 4, the structure of the complex $\mathbf{R}\mathrm{Hom}_R(R/I, R)$ is explored for an Ulrich ideal I in a Cohen-Macaulay local ring R. As a direct consequence we have that inside of a one-dimensional almost Gorenstein but non-Gorenstein local ring, the only possible Ulrich ideal is the maximal ideal. We shall also study the problem of when Ulrich ideals of almost Gorenstein local rings have the same minimal number of generators.

The results in Chapter 1 and 4 are based on the joint works [36, 37] with Shiro Goto and Ryo Takahashi. The paper [36] (Chapter 1) was published in the Journal of Pure and Applied Algebra and the paper [37] (Chapter 4) has been accepted for publication in the Proceeding of the American Mathematical Society. The researches in Chapter 2 and 3 are submitted for possible publication in the papers [34, 35] jointly with Shiro Goto, Naoyuki Matsuoka, and Ken-ichi Yoshida.

December, 2015 NAOKI TANIGUCHI

Chapter 1

Almost Gorenstien rings
– towards a theory of higher dimension –

1.1 Introduction

For the last fifty years, commutative algebra has been concentrated mainly in the study of Cohen-Macaulay rings/modules and has performed huge achievements ([11]). While tracking the development, the authors often encounter non-Gorenstein Cohen-Macaulay rings in divers branches of (and related to) commutative algebra. On all such occasions, they have a query why there are so many Cohen-Macaulay rings which are not Gorenstein rings. Gorenstein rings are, of course, defined by local finiteness of self-injective dimension ([9]), enjoying beautiful symmetry. However as a view from the very spot, there is a substantial estrangement between two conditions of finiteness and infiniteness of self-injective dimension, and researches for the fifty years also show that Gorenstein rings turn over some part of their roles to canonical modules ([43]). It seems, nevertheless, still reasonable to ask for a new class of non-Gorenstein Cohen-Macaulay rings that could be called *almost Gorenstein* and are good next to Gorenstein rings. This observation has already motivated the research [26] of one-dimensional case. The second step should be to detect the notion of almost Gorenstein local/graded ring of higher dimension and develop the theory.

Almost Gorenstein local rings of dimension one were originally introduced in 1997 by Barucci and Fröberg [8] in the case where the local rings are analytically unramified. As was mentioned by [6] as for the proof of [8, Proposition 25], their framework was not sufficiently flexible for the analysis of one-dimensional analytically ramified local rings. This observation has inspired Goto, Matsuoka, and Phuong [26], where they posed a modified definition of one-dimensional almost Gorenstein local rings, which works well also in the case where the rings are analytically ramified. The present research aims to go beyond [26] towards a theory of higher dimensional cases, asking for possible extensions of results known by [6, 8, 7, 26].

To explain our aim and motivation more precisely, let us start on our definition.

Definition 1.1.1. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then R is said to be an almost Gorenstein local ring, if the following conditions are satisfied.

(1) R is a Cohen-Macaulay local ring, which possesses the canonical module K_R and

(2) there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$.

Here $\mu_R(C)$ (resp. $\mathrm{e}^0_{\mathfrak{m}}(C)$) denotes the number of elements in a minimal system of generators for C (resp. the multiplicity of C with respect to $\mathfrak{m}$).

With this definition every Gorenstein local ring is almost Gorenstein (take $C = (0)$) and the converse is also true, if $\dim R = 0$. In the exact sequence quoted in Definition 1.1.1 (2), if $C \neq (0)$, then C is a Cohen-Macaulay R-module with $\dim_R C = \dim R - 1$ and one has the equality $\mathfrak{m}C = (f_2, f_3, \ldots, f_d)C$ for some elements $f_2, f_3, \ldots, f_d \in \mathfrak{m}$ ($d = \dim R$), provided the residue class field $R/\mathfrak{m}$ of R is infinite. Hence C is a *maximally generated Cohen-Macaulay* R-module in the sense of [10], which is called in the present chapter an *Ulrich* R-module. Therefore, roughly speaking, our Definition 1.1.1 requires that if R is an almost Gorenstein local ring, then R might be a non-Gorenstein local ring but the ring R can be embedded into its canonical module K_R so that the difference K_R/R should be tame and well-behaved.

In the case where $\dim R = 1$, if R is an almost Gorenstein local ring, then $\mathfrak{m}C = (0)$ and R is an almost Gorenstein local ring exactly in the sense of [26, Definition 3.1]. The converse is also true, if $R/\mathfrak{m}$ is infinite. (When the field $R/\mathfrak{m}$ is too small, the converse is not true in general; see Remark 1.3.5 and [26, Remark 2.10].) With Definition 1.1.1, as we will later show, many results of [26] of dimension one are extendable over higher-dimensional local rings, which supports the appropriateness of our definition.

Let us now state our results, explaining how this chapter is organized. In Section 1.2 we give a brief survey on Ulrich modules, which we will need throughout this chapter. In Section 1.3 we explore basic properties of almost Gorenstein local rings, including the so-called non-zerodivisor characterization. In Section 1.4, we will give a characterization of almost Gorenstein local rings in terms of the existence of certain exact sequences of R-modules. Let M be an R-module. For a sequence $\boldsymbol{x} = x_1, x_2, \dots, x_n$ of elements in R, the Koszul complex of M associated to $\boldsymbol{x}$ is denoted by $\mathbb{K}_\bullet(\boldsymbol{x}, M)$. For each $z \in M$, we define a complex

$$\mathbb{U}(z, M) = (\cdots \to \underset{2}{0} \to \underset{1}{R} \xrightarrow{\varphi} \underset{0}{M} \to \underset{-1}{0} \to \cdots),$$

where the map φ is given by $a \mapsto az$. Let us say that an R-complex $C = (\cdots \to C_2 \to C_1 \to C_0 \to 0)$ is called an acyclic complex over M, if $\mathrm{H}_0(C) \cong M$ and $\mathrm{H}_i(C) = (0)$ for all $i > 0$. With this notation the main result of Section 1.4 is stated as follows.

Theorem 1.1.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = d \geq 1$ and the Cohen-Macaulay type r. Suppose that R admits the canonical module K_R and that the residue class field $R/\mathfrak{m}$ of R is infinite. Then the following conditions are equivalent.*

(1) *R is an almost Gorenstein local ring.*

(2) *There exist an R-sequence $\boldsymbol{x} = x_1, x_2, \dots, x_{d-1}$ and an element $y \in \mathrm{K}_R$ such that $\mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes_R \mathbb{U}(y, \mathrm{K}_R)$ is an acyclic complex over k^{r-1}.*

(3) *There exist an R-sequence $\boldsymbol{x}$ (not necessarily of length $d-1$) and an element $y \in \mathrm{K}_R$ such that $\mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes_R \mathbb{U}(y, \mathrm{K}_R)$ is an acyclic complex over an R-module annihilated by $\mathfrak{m}$.*

In Section 1.5 we give the following characterization of almost Gorenstein local rings in terms of canonical ideals. When $\dim R = 1$, this result corresponds to [26, Theorem 3.11].

Theorem 1.1.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R \geq 1$ and infinite residue class field. Let I ($\neq R$) be an ideal of R and assume that $I \cong \mathrm{K}_R$ as an R-module. Then the following conditions are equivalent.*

(1) *R is an almost Gorenstein local ring.*

(2) *R contains a parameter ideal $Q = (f_1, f_2, \ldots, f_d)$ such that $f_1 \in I$ and $\mathfrak{m}(I+Q) = \mathfrak{m}Q$.*

With the same notation as Theorem 1.1.3, if R is not a Gorenstein ring, we then have $\mathrm{e}_1(I+Q) = \mathrm{r}(R)$ (here $\mathrm{e}_1(I+Q)$ (resp. $\mathrm{r}(R)$) denotes the first Hilbert coefficient of the ideal $I+Q$ of R (resp. the Cohen-Macaulay type of R)). A structure theorem of the Sally module $\mathcal{S}_Q(I+Q)$ of $I+Q$ with respect to the reduction Q shall be described. These results reasonably extend the corresponding ones in [26, Theorem 3.16] to higher-dimensional local rings.

In Section 1.6 we study the question of when the idealization $A = R \ltimes X$ of a given R-module X is an almost Gorenstein local ring. Our goal is the following, which extends [26, Theorem 6.5] to higher-dimensional cases.

Theorem 1.1.4. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$, which possesses the canonical module K_R. Suppose that $R/\mathfrak{m}$ is infinite. Let $\mathfrak{p} \in \operatorname{Spec} R$ such that $R/\mathfrak{p}$ is a regular local ring of dimension $d-1$. Then the following conditions are equivalent.*

(1) *$A = R \ltimes \mathfrak{p}$ is an almost Gorenstein local ring.*

(2) *R is an almost Gorenstein local ring.*

In Section 1.7 we explore a special class of almost Gorenstein local rings, which we call semi-Gorenstein. A structure theorem of minimal free resolutions of semi-Gorenstein local rings shall be given. The semi-Gorenstein property is preserved under localization while the almost Gorenstein property is not, which we will show later; see Section 1.9.

In Section 1.8 we search for possible definitions of almost Gorenstein graded rings. Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay graded ring with $k = R_0$ a local ring. Assume that R possesses the graded canonical module K_R. This condition is equivalent to saying that k is a homomorphic image of a Gorenstein local ring ([40, 41]). Let $\mathfrak{M}$ denote the unique graded maximal ideal of R and let $a = \mathrm{a}(R)$ be the a-invariant of R. Hence $a = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^d_{\mathfrak{M}}(R)]_n \neq (0)\}$ ([40, Definition (3.1.4)]), where $\{[\mathrm{H}^d_{\mathfrak{M}}(R)]_n\}_{n \in \mathbb{Z}}$ denotes the homogeneous components of the d-th graded local cohomology module $\mathrm{H}^d_{\mathfrak{M}}(R)$ of R with respect to $\mathfrak{M}$. With this notation our definition of almost Gorenstein graded ring is stated as follows, which we discuss in Section 1.8.

Definition 1.1.5. We say that R is an almost Gorenstein graded ring, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules with $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$. Here $\mathrm{K}_R(-a)$ denotes the graded R-module whose underlying R-module is the same as that of K_R and whose grading is given by $[\mathrm{K}_R(-a)]_n = [\mathrm{K}_R]_{n-a}$ for all $n \in \mathbb{Z}$.

In Section 1.9 we study almost Gorensteinness in the graded rings associated to filtrations of ideals. We shall prove that the almost Gorenstein property of base local rings is inherited from that of the associated graded rings with a certain condition on the Cohen-Macaulay type. In general, local rings of an almost Gorenstein local ring are not necessarily almost Gorenstein, which we will show in this section; see Remark 1.9.3.

In Section 1.10 we explore Cohen-Macaulay homogeneous rings $R = k[R_1]$ over an infinite field $k = R_0$. We shall prove the following, which one can directly apply, for instance, to the Stanley-Reisner rings $R = k[\Delta]$ of simplicial complexes Δ over k.

Theorem 1.1.6. *Let $R = k[R_1]$ be a Cohen-Macaulay homogeneous ring over an infinite field k and assume that R is not a Gorenstein ring. Let $d = \dim R \geq 1$ and set $a = \mathrm{a}(R)$. Then the following conditions are equivalent.*

(1) *R is an almost Gorenstein graded level ring.*

(2) *The total ring $\mathrm{Q}(R)$ of fractions of R is a Gorenstein ring and $a = 1 - d$.*

In Section 1.11 we study the relation between the almost Gorensteinness of Cohen-Macaulay local rings $(R, \mathfrak{m})$ and their tangent cones $\mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. We shall prove, provided $R/\mathfrak{m}$ is infinite and $\mathrm{v}(R) = \mathrm{e}^0_{\mathfrak{m}}(R) + \dim R - 1$ (here $\mathrm{v}(R)$ denotes the embedding dimension of R), that R is an almost Gorenstein local ring if and only if $\mathrm{Q}(\mathrm{gr}_{\mathfrak{m}}(R))$ is a Gorenstein ring, which will eventually show that every two-dimensional rational singularity is an almost Gorenstein local ring (Corollary 1.11.5).

In the final section we shall prove that every one-dimensional Cohen-Macaulay complete local ring of finite Cohen-Macaulay representation type is an almost Gorenstein local ring, if it possesses a coefficient field of characteristic 0.

As is confirmed in Sections 1.8, 1.9, 1.10, our definition of almost Gorenstein graded rings works well to analyze divers graded rings. We, however, note here the following.

By definition, the ring $R_{\mathfrak{M}}$ is an almost Gorenstein local ring, if R is an almost Gorenstein graded ring with unique graded maximal ideal $\mathfrak{M}$, but as Example **??** shows, the converse is not true in general. In fact, for the example, one has $\mathrm{a}(R) = -2$ and there is no exact sequence $0 \to R \to \mathrm{K}_R(2) \to C \to 0$ of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, while there exists an exact sequence $0 \to R \to \mathrm{K}_R(3) \to D \to 0$ such that $\mu_R(D) = \mathrm{e}^0_{\mathfrak{M}}(D)$. The example seems to suggest the existence of alternative and more flexible definitions of almost Gorensteinness for graded rings. We would like to leave the quest to forthcoming researches.

In what follows, unless otherwise specified, let R denote a Noetherian local ring with maximal ideal $\mathfrak{m}$. For each finitely generated R-module M, let $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators of M (resp. the length of M). We denote by $\mathrm{e}^0_{\mathfrak{m}}(M)$ the multiplicity of M with respect to $\mathfrak{m}$.

1.2 Survey on Ulrich modules

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. The purpose of this section is to summarize some preliminaries on Ulrich modules, which we will use throughout this chapter. We begin with the following.

Definition 1.2.1. Let M ($\neq (0)$) be a finitely generated R-module. Then M is said to be an Ulrich R-module, if M is a Cohen-Macaulay R-module and $\mu_R(M) = \mathrm{e}^0_{\mathfrak{m}}(M)$.

Proposition 1.2.2. *Let M be a finitely generated R-module of dimension $s \geq 0$. Then the following assertions hold true.*

(1) *Suppose $s = 0$. Then M is an Ulrich R-module if and only if $\mathfrak{m}M = (0)$, that is M is a vector space over the field $R/\mathfrak{m}$.*

(2) *Suppose that M is a Cohen-Macaulay R-module. If $\mathfrak{m}M = (f_1, f_2, \dots, f_s)M$ for some $f_1, f_2, \dots, f_s \in \mathfrak{m}$, then M is an Ulrich R-module. The converse is also true, if $R/\mathfrak{m}$ is infinite. (We actually have $\mathfrak{m}M = (f_1, f_2, \dots, f_s)M$ for any elements $f_1, f_2, \dots, f_s \in \mathfrak{m}$ whose images in $R/[(0) :_R M]$ generate a minimal reduction of the maximal ideal of $R/[(0) :_R M]$.) When this is the case, the elements $f_1, f_2, \dots, f_s$ form a part of a minimal system of generators for $\mathfrak{m}$.*

(3) *Let* $\varphi : R \to S$ *be a flat local homomorphism of Noetherian local rings such that* $S/\mathfrak{m}S$ *is a regular local ring. Then* M *is an Ulrich* R*-module if and only if* $S \otimes_R M$ *is an Ulrich* S*-module.*

(4) *Let* M *be an Ulrich* R*-module with* $s = \dim_R M \geq 1$. *Let* $f \in \mathfrak{m}$ *and assume that* f *is superficial for* M *with respect to* $\mathfrak{m}$. *Then* M/fM *is an Ulrich* R*-module of dimension* $s-1$.

(5) *Let* $f \in \mathfrak{m}$ *and assume that* f *is* M*-regular. If* M/fM *is an Ulrich* R*-module, then* M *is an Ulrich* R*-module and* $f \notin \mathfrak{m}^2$.

Proof. (1) This follows from the facts that $\mu_R(M) = \ell_R(M/\mathfrak{m}M)$ and $\mathrm{e}^0_{\mathfrak{m}}(M) = \ell_R(M)$.

(2) Suppose that $\mathfrak{m}M = (f_1, f_2, \ldots, f_s)M$ for some $f_1, f_2, \ldots, f_s \in \mathfrak{m}$. Then $f_1, f_2, \ldots, f_s$ is a system of parameters of M and $\mathfrak{m}^{n+1}M = (f_1, f_2, \ldots, f_s)^{n+1}M$ for all $n \geq 0$. Hence $\ell_R(M/\mathfrak{m}^{n+1}M) = \ell_R(M/(f_1, f_2, \ldots, f_s)M) \cdot \binom{n+s}{s}$ and therefore $\mathrm{e}^0_{\mathfrak{m}}(M) = \ell_R(M/(f_1, f_2, \ldots, f_s)M) = \ell_R(M/\mathfrak{m}M)$, so that M is an Ulrich R-module. Let $\overline{R} = R/[(0) :_R M]$ and let $\overline{f_i}$ denote the image of f_i in $\overline{R}$. We then have $\overline{\mathfrak{m}}M = (\overline{f_1}, \overline{f_2}, \ldots, \overline{f_s})M$, where $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$. Hence $(\overline{f_1}, \overline{f_2}, \ldots, \overline{f_s})$ is a minimal reduction of $\overline{\mathfrak{m}}$, because M is a faithful $\overline{R}$-module, so that $f_1, f_2, \ldots, f_s$ form a part of a minimal system of generators for the maximal ideal $\mathfrak{m}$.

Conversely, suppose that $R/\mathfrak{m}$ is infinite and that M is an Ulrich R-module. Let us choose elements $f_1, f_2, \ldots, f_s \in \mathfrak{m}$ so that $(\overline{f_1}, \overline{f_2}, \ldots, \overline{f_s})$ is a minimal reduction of $\overline{\mathfrak{m}}$. Then $\mathrm{e}^0_{\mathfrak{m}}(M) = \mathrm{e}^0_{\overline{\mathfrak{m}}}(M) = \mathrm{e}^0_{(\overline{f_1}, \overline{f_2}, \ldots, \overline{f_s})}(M) = \ell_R(M/(f_1, f_2, \ldots, f_s)M)$. Hence $\mathfrak{m}M = (f_1, f_2, \ldots, f_s)M$ as $\ell_R(M/\mathfrak{m}M) = \mathrm{e}^0_{\mathfrak{m}}(M)$.

(3) Choose a regular system $g_1, g_2, \ldots, g_n \in \mathfrak{n}$ of parameters for the regular local ring $S/\mathfrak{m}S$ (here $n = \dim S/\mathfrak{m}S$) and set $\overline{S} = S/(g_1, g_2, \ldots, g_n)S$. Then the composite map $\psi : R \to S \to \overline{S}$ is flat ([43, Lemma 1.23]) and

$$(S \otimes_R M)/(g_1, g_2, \ldots, g_n)(S \otimes_R M) \cong \overline{S} \otimes_R M,$$

so that passing to the homomorphism ψ, we may assume $\mathfrak{m}S = \mathfrak{n}$. We then have

$$\mu_S(S \otimes_R M) = \mu_R(M), \quad \mathrm{e}^0_{\mathfrak{n}}(S \otimes_R M) = \mathrm{e}^0_{\mathfrak{m}}(M).$$

Hence M is an Ulrich R-module if and only if $S \otimes_R M$ is an Ulrich S-module.

(4) Since f is superficial for M with respect to $\mathfrak{m}$ and $s > 0$, f is M-regular and $\mathrm{e}^0_{\mathfrak{m}}(M/fM) = \mathrm{e}^0_{\mathfrak{m}}(M)$. Therefore M/fM is a Cohen-Macaulay R-module, and

consequently M/fM is an Ulrich R-module, because $\mu_R(M/fM) = \mu_R(M) = \mathrm{e}^0_{\mathfrak{m}}(M) = \mathrm{e}^0_{\mathfrak{m}}(M/fM)$.

(5) We put $R(X) = R[X]_{\mathfrak{m}R[X]}$ and $S(X) = S[X]_{\mathfrak{n}S[X]}$, where X is an indeterminate. Then, since $\mathfrak{m}R[X] = \mathfrak{n}S[X] \cap R[X]$, we get a flat local homomorphism $\psi : R(X) \to S(X)$, extending $\varphi : R \to S$. Because $\mu_{R(X)}(R(X) \otimes_R M) = \mu_R(M)$ and $\mathrm{e}^0_{\mathfrak{m}R(X)}(R(X) \otimes_R M) = \mathrm{e}^0_{\mathfrak{m}}(M)$, $R(X) \otimes_R M$ is an Ulrich $R(X)$-module. For the same reason, $S(X) \otimes_S (S \otimes_R M)$ is an Ulrich $S(X)$-module if and only if $S \otimes_R M$ is an Ulrich S-module. Therefore, since $S(X)/\mathfrak{m}S(X) = (S/\mathfrak{m}S)(X)$ is a regular local ring, passing to the homomorphism $\psi : R(X) \to S(X)$, without loss of generality we may assume that the residue class field $R/\mathfrak{m}$ of R is infinite. We now choose elements $f_2, f_3, \ldots, f_s \in \mathfrak{m}$ so that $\mathfrak{m}{\cdot}(M/fM) = (f_2, f_3, \ldots, f_s){\cdot}(M/fM)$. Then $\mathfrak{m}M = (f_1, f_2, \ldots, f_s)M$ with $f_1 = f$. Therefore by assertion (2), M is an Ulrich R-module and $f \notin \mathfrak{m}^2$. $\square$

1.3 Almost Gorenstein local rings

Let R be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R \geq 0$, possessing the canonical module K_R. Hence R is a homomorphic image of a Gorenstein ring ([58]). The purpose of this section is to define almost Gorenstein local rings and explore their basic properties.

We begin with the following.

Lemma 1.3.1. *Let $R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$ be an exact sequence of R-modules. Then the following assertions hold true.*

(1) *If $\dim_R C < d$, then φ is injective and the total ring $\mathrm{Q}(R)$ of fractions of R is a Gorenstein ring.*

(2) *Suppose that φ is injective. If $C \neq (0)$, then C is a Cohen-Macaulay R-module of dimension $d-1$.*

(3) *If φ is injective and $d = 0$, then φ is an isomorphism.*

Proof. (1) Let $L = \operatorname{Ker}\varphi$ and assume that $L \neq (0)$. Choose $\mathfrak{p} \in \operatorname{Ass}_R L$ and we have the exact sequence $0 \to L_{\mathfrak{p}} \to R_{\mathfrak{p}} \xrightarrow{\varphi_{\mathfrak{p}}} (\mathrm{K}_R)_{\mathfrak{p}} \to C_{\mathfrak{p}} \to 0$ of $R_{\mathfrak{p}}$-modules. Since $\mathfrak{p} \in \operatorname{Ass} R$ and $\dim_R C < d$, we get $C_{\mathfrak{p}} = (0)$, whence $\varphi_{\mathfrak{p}}$ is an epimorphism. Therefore, because $(\mathrm{K}_R)_{\mathfrak{p}} \cong \mathrm{K}_{R_{\mathfrak{p}}}$ ([43, Korollar 6.2]) and $\ell_{R_{\mathfrak{p}}}(\mathrm{K}_{R_{\mathfrak{p}}}) = \ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})$, $\varphi_{\mathfrak{p}}$ is necessarily an isomorphism and hence $L_\mathfrak{p} = (0)$, which is impossible. Thus $L = (0)$ and φ is injective. The second assertion is clear, because $R_\mathfrak{p} \cong (\mathrm{K}_R)_\mathfrak{p} \cong \mathrm{K}_{R_\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Ass} R$.

(2) Let $\mathfrak{p} \in \operatorname{Supp}_R C$ with $\dim R/\mathfrak{p} = \dim_R C$. If $\dim_R C = d$, then $\mathfrak{p} \in \operatorname{Ass} R$ and hence $\ell_{R_\mathfrak{p}}(R_\mathfrak{p}) = \ell_{R_\mathfrak{p}}(\mathrm{K}_{R_\mathfrak{p}}) = \ell_{R_\mathfrak{p}}((\mathrm{K}_R)_\mathfrak{p})$, so that $C_\mathfrak{p} = (0)$, because the homomorphism $\varphi_\mathfrak{p} : R_\mathfrak{p} \to (\mathrm{K}_R)_\mathfrak{p}$ is injective, which is impossible. Hence $\dim_R C < d$, while we get $\operatorname{depth}_R C \geq d-1$, applying the depth lemma to the exact sequence $0 \to R \to \mathrm{K}_R \to C \to 0$. Thus C is a Cohen-Macaulay R-module of dimension $d-1$.

(3) This is clear. □

Remark 1.3.2. Suppose that $d > 0$ and that $\mathrm{Q}(R)$ is a Gorenstein ring. Then R contains an ideal I ($\neq R$) such that $I \cong \mathrm{K}_R$ as an R-module. When this is the case, R/I is a Gorenstein local ring of dimension $d-1$ ([43, Satz 6.21]).

We are now ready to define almost Gorenstein local rings.

Definition 1.3.3. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring which possesses the canonical module K_R. Then R is said to be an almost Gorenstein local ring, if there is an exact sequence $0 \to R \to \mathrm{K}_R \to C \to 0$ of R-modules such that $\mu_R(C) = \mathrm{e}^0_\mathfrak{m}(C)$.

In Definition 1.3.3, if $C \neq (0)$, then C is an Ulrich R-module of dimension $d-1$ (Definition 1.2.1 and Lemma 1.3.1 (2)). Note that every Gorenstein local ring R is almost Gorenstein (take $C = (0)$) and that R is a Gorenstein local ring, if R is an almost Gorenstein local ring of dimension 0 (Lemma 1.3.1 (3)).

Almost Gorenstein local rings were defined in 1997 by Barucci and Fröberg [8] in the case where R is analytically unramified and $\dim R = 1$. Goto, Matsuoka and Phuong [26] extended the notion to the case where R is not necessarily analytically unramified but still of dimension one. Our definition 1.3.3 is a higher-dimensional proposal. In fact we have the following.

Proposition 1.3.4. *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring. If R is an almost Gorenstein local ring in the sense of Definition 1.3.3, then R is an almost Gorenstein local ring in the sense of [26, Definition 3.1]. The converse also holds, when $R/\mathfrak{m}$ is infinite.*

Proof. Firstly, assume that R is an almost Gorenstein local ring in the sense of Definition 1.3.3 and choose an exact sequence $0 \to R \xrightarrow{\varphi} I \to C \to 0$ of R-modules so that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$, where I ($\neq R$) is an ideal of R such that $I \cong \mathrm{K}_R$ as an R-module (Lemma 1.3.1 (1) and Remark 1.3.2). Then, because $\mathfrak{m}C = (0)$ by Proposition 1.2.2 (1), we get $\mathfrak{m}I \subseteq (f)$, where $f = \varphi(1)$. We set $Q = (f)$. Then, since $\mathfrak{m}Q \subseteq \mathfrak{m}I \subseteq Q$, we have either $\mathfrak{m}Q = \mathfrak{m}I$ or $\mathfrak{m}I = Q$. If $\mathfrak{m}I = \mathfrak{m}Q$, then Q is a reduction of I, so that R is an almost Gorenstein local ring in the sense of [26, Definition 3.1] (see [26, Theorem 3.11] also). If $\mathfrak{m}I = Q$, then the maximal ideal $\mathfrak{m}$ of R is invertible, so that R is a discrete valuation ring. Hence in any case, R is an almost Gorenstein local ring in the sense of [26].

Conversely, assume that R is an almost Gorenstein local ring in the sense of [26] and that $R/\mathfrak{m}$ is infinite. Let us choose an R-submodule K of $\mathrm{Q}(R)$ such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ as an R-module, where $\overline{R}$ denotes the integral closure of R in $\mathrm{Q}(R)$. Then by [26, Theorem 3.11], we get $\mathfrak{m}K \subseteq R$, and therefore R is an almost Gorenstein local ring in the sense of Definition 1.3.3 (use Proposition 1.2.2 (1)). $\square$

Remark 1.3.5. When the field $R/\mathfrak{m}$ is finite, R is not necessarily an almost Gorenstein local ring in the sense of Definition 1.3.3, even though R is an almost Gorenstein local ring in the sense of [26]. The ring

$$R = k[[X, Y, Z]]/[(X, Y) \cap (Y, Z) \cap (Z, X)]$$

is a typical example, where $k[[X, Y, Z]]$ is the formal power series ring over $k = \mathbb{Z}/(2)$ ([26, Remark 2.10]). This example also shows that R is not necessarily an almost Gorenstein local ring in the sense of Definition 1.3.3, even if it becomes an almost Gorenstein local ring in the sense of Definition 1.3.3, after enlarging the residue class field $R/\mathfrak{m}$ of R.

We note the following.

Proposition 1.3.6. *Suppose that R is not a Gorenstein ring and consider the following two conditions.*

(1) *R is an almost Gorenstein local ring.*

(2) *There exist an exact sequence $0 \to R \to \mathrm{K}_R \to C \to 0$ of R-modules, a non-zerodivisor $f \in (0) :_R C$, and a parameter ideal $\mathfrak{q}$ ($\subsetneq R$) for $R/(f)$ such that $\mathfrak{m}C = \mathfrak{q}C$.*

Then the implication (2) $\Rightarrow$ (1) *holds. If* $R/\mathfrak{m}$ *is infinite, the reverse implication* (1) $\Rightarrow$ (2) *is also true.*

Proof. (2) $\Rightarrow$ (1) We have $C \neq (0)$, so that by Lemma 1.3.1 (2) C is a Cohen-Macaulay R-module of dimension $d-1$. Hence Proposition 1.2.2 (2) shows C is an Ulrich R-module, because $\mathfrak{m}C = \mathfrak{q}C$.

(1) $\Rightarrow$ (2) We take an exact sequence $0 \to R \to \mathrm{K}_R \to C \to 0$ of R-modules such that C is an Ulrich R-module of dimension $d-1$. Hence $[(0) :_R C]$ contains a non-zerodivisor f of R. We choose elements $\{f_i\}_{2 \le i \le d}$ of $\mathfrak{m}$ so that their images in $R/(f)$ generate a minimal reduction of the maximal ideal of $R/(f)$. We then have $\mathfrak{m}C = \mathfrak{q}C$ by Proposition 1.2.2 (2), because the images of $\{f_i\}_{2 \le i \le d}$ in $R/[(0) :_R C]$ also generate a minimal reduction of the maximal ideal of $R/[(0) :_R C]$. $\square$

Let us now explore basic properties of almost Gorenstein local rings. We begin with the non-zerodivisor characterization.

Theorem 1.3.7. *Let* $f \in \mathfrak{m}$ *and assume that* f *is* R*-regular.*

(1) *If* $R/(f)$ *is an almost Gorenstein local ring, then* R *is an almost Gorenstein local ring. If* R *is moreover not a Gorenstein ring, then* $f \notin \mathfrak{m}^2$.

(2) *Conversely, suppose that* R *is an almost Gorenstein local ring which is not a Gorenstein ring. Consider the exact sequence*

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R*-modules such that* $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$*. If* f *is superficial for* C *with respect to* $\mathfrak{m}$ *and* $d \ge 2$*, then* $R/(f)$ *is an almost Gorenstein local ring.*

Proof. We set $\overline{R} = R/(f)$. Remember that $\mathrm{K}_R/f\mathrm{K}_R = \mathrm{K}_{\overline{R}}$ ([43, Korollar 6.3]), because f is R-regular.

(1) We choose an exact sequence $0 \to \overline{R} \xrightarrow{\psi} \mathrm{K}_{\overline{R}} \to D \to 0$ of $\overline{R}$-modules so that D is an Ulrich $\overline{R}$-module of dimension $d-2$. Let $\xi \in \mathrm{K}_R$ such that $\psi(1) = \overline{\xi}$, where $\overline{\xi}$ denotes the image of ξ in $\mathrm{K}_{\overline{R}} = \mathrm{K}_R/f\mathrm{K}_R$. We now consider the exact sequence

$$R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules with $\varphi(1) = \xi$. Then, because $\psi = \overline{R} \otimes_R \varphi$, we get $D = C/fC$, whence $\dim_R C < d$, because $\dim_R D = d-2$. Consequently, by Lemma 1.3.1 (1) the homomorphism φ is injective, and hence by Lemma 1.3.1 (2), C is a Cohen-Macaulay R-module of dimension $d-1$. Therefore, f is C-regular, so that by Proposition 1.2.2 (5), C is an Ulrich R-module and $f \notin \mathfrak{m}^2$. Hence R is almost Gorenstein.

(2) The element f is C-regular, because f is superficial for C with respect to $\mathfrak{m}$ and $\dim_R C = d-1 > 0$. Therefore the exact sequence $0 \to R \to \mathrm{K}_R \to C \to 0$ gives rise to the exact sequence of $\overline{R}$-modules

$$0 \to \overline{R} \to \mathrm{K}_{\overline{R}} \to C/fC \to 0,$$

where C/fC is an Ulrich $\overline{R}$-module by Proposition 1.2.2 (4). Hence $\overline{R}$ is almost Gorenstein. □

The following is a direct consequence of Theorem 1.3.7 (1).

Corollary 1.3.8. *Suppose that $d > 0$. If $R/(f)$ is an almost Gorenstein local ring for every non-zerodivisor $f \in \mathfrak{m}$, then R is a Gorenstein local ring.*

We are now interested in the question of how the almost Gorenstein property is inherited under flat local homomorphisms. Let us begin with the following. Notice that the converse of the first assertion of Theorem 1.3.9 is not true in general, unless $R/\mathfrak{m}$ is infinite (Remark 1.3.5).

Theorem 1.3.9. *Let $(S, \mathfrak{n})$ be a Noetherian local ring and let $\varphi : R \to S$ be a flat local homomorphism such that $S/\mathfrak{m}S$ is a regular local ring. Then S is an almost Gorenstein local ring, if R is an almost Gorenstein local ring. The converse also holds, when $R/\mathfrak{m}$ is infinite.*

Proof. Suppose that R is an almost Gorenstein local ring and consider an exact sequence $0 \to R \to \mathrm{K}_R \to C \to 0$ of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$. If $C = (0)$, then R is a Gorenstein local ring, so that S is a Gorenstein local ring. Suppose $C \neq (0)$. Then $S \otimes_R C$ is an Ulrich S-module by Proposition 1.2.2 (2), since C is an Ulrich R-module. Besides, $\mathrm{K}_S \cong S \otimes_R \mathrm{K}_R$ as an S-module ([43, Satz 6.14]), since $S/\mathfrak{m}S$ is a Gorenstein local ring. Thus S is almost Gorenstein, thanks to the exact sequence of S-modules

$$0 \to S \to \mathrm{K}_S \to S \otimes_R C \to 0.$$

Suppose now that $R/\mathfrak{m}$ is infinite and S is an almost Gorenstein local ring. Let $n = \dim S/\mathfrak{m}S$. We have to show that R is an almost Gorenstein local ring. Assume the contrary and choose a counterexample S so that $\dim S = n + d$ is as small as possible. Then S is not a Gorenstein ring, so that $\dim S = n + d > 0$. Choose an exact sequence

$$0 \to S \to \mathrm{K}_S \to D \to 0$$

of S-modules with $\mu_S(D) = \mathrm{e}^0_{\mathfrak{n}}(D)$. Suppose $n > 0$. If $d > 0$, then we take an element $g \in \mathfrak{n}$ so that g is superficial for D with respect to $\mathfrak{n}$ and $\overline{g}$ is a part of a regular system of parameters of $S/\mathfrak{m}S$, where $\overline{g}$ denotes the image of g in $S/\mathfrak{m}S$. Then g is S-regular and the composite homomorphism $R \xrightarrow{\varphi} S \to S/gS$ is flat. Therefore the minimality of $n + d$ forces R to be an almost Gorenstein local ring, because S/gS is an almost Gorenstein local ring by Theorem 1.3.7 (2). Thus $d = 0$ and $\mathfrak{p} = \mathfrak{m}S$ is a minimal prime ideal of S. Hence the induced flat local homomorphism $R \xrightarrow{\varphi} S \to S_{\mathfrak{p}}$ shows that R is a Gorenstein ring, because $S_{\mathfrak{p}}$ is a Gorenstein ring (Lemma 1.3.1 (1)). Consequently $n = 0$ and $\mathfrak{n} = \mathfrak{m}S$.

Suppose now that $d \geq 2$. Then because $\mathfrak{n} = \mathfrak{m}S$, we may choose an element $f \in \mathfrak{m}$ so that f is R-regular and $\varphi(f)$ is superficial for D with respect to $\mathfrak{n}$. Then by Theorem 1.3.7 (2) S/fS is an almost Gorenstein local ring, while the homomorphism $R/fR \to S/fS$ is flat. Consequently, R/fR is an almost Gorenstein local ring, so that by Theorem 1.3.7 (1) R is an almost Gorenstein local ring.

Thus $d = 1$ and $\mathfrak{n} = \mathfrak{m}S$, so that R is an almost Gorenstein local ring by [26, Proposition 3.3], which is the required contradiction. □

Let $\mathrm{r}(R) = \ell_R(\mathrm{Ext}^d_R(R/\mathfrak{m}, R))$ denote the Cohen-Macaulay type of R.

Corollary 1.3.10. *Let R be an almost Gorenstein local ring and choose an exact sequence $0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$ of R-modules so that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$. If $\varphi(1) \in \mathfrak{m}\mathrm{K}_R$, then R is a regular local ring. Therefore, $\mu_R(C) = \mathrm{r}(R) - 1$, if R is not a regular local ring.*

Proof. Enlarging the residue class field of R, by Theorem 1.3.9 we may assume that $R/\mathfrak{m}$ is infinite. Suppose $\varphi(1) \in \mathfrak{m}\mathrm{K}_R$. Then $C \neq (0)$ and therefore $d > 0$ (Lemma 1.3.1 (3)). Assume $d = 1$. Then by Lemma 1.3.1 (1) $\mathrm{Q}(R)$ is a Gorenstein ring. Therefore by Remark 1.3.2 we get an exact sequence $0 \to R \xrightarrow{\psi} I \to C \to 0$ of R-modules with

$\psi(1) \in \mathfrak{m}I$, where I ($\subsetneq R$) is an ideal of R such that $I \cong \mathrm{K}_R$ as an R-module. Let $a = \psi(1)$. Then $\mathfrak{m}I = (a)$, because $\mathfrak{m}C = (0)$ and $a \in \mathfrak{m}I$. Hence R is a discrete valuation ring, because the maximal ideal $\mathfrak{m}$ of R is invertible.

Let $d > 1$ and assume that our assertion holds true for $d-1$. Let $f \in \mathfrak{m}$ be a non-zerodivisor of R such that f is superficial for C with respect to $\mathfrak{m}$. We set $\overline{R} = R/(f)$ and $\overline{C} = C/fC$. Then by Theorem 1.3.7 (2) (and its proof) $\overline{R}$ is an almost Gorenstein local ring with the exact sequence $0 \to \overline{R} \xrightarrow{\overline{\varphi}} \mathrm{K}_{\overline{R}} \to \overline{C} \to 0$ of $\overline{R}$-modules, where $\overline{\varphi} = \overline{R} \otimes_R \varphi$ and $\mathrm{K}_{\overline{R}} = \mathrm{K}_R/f\mathrm{K}_R$. Therefore, because $\overline{\varphi}(1) \in \mathfrak{m}\mathrm{K}_{\overline{R}}$, the hypothesis of induction on d shows $\overline{R}$ is regular and hence so is R.

The second assertion follows from the fact that $\mu_R(C) = \mu_R(\mathrm{K}_R) - 1 = \mathrm{r}(R) - 1$ ([43, Korollar 6.11]), because $\varphi(1) \notin \mathfrak{m}\mathrm{K}_R$. $\square$

The following is an direct consequence of Theorem 1.3.9. See [43, Satz 6.14] for the equality $\mathrm{r}(S) = \mathrm{r}(R)$.

Corollary 1.3.11. *Suppose that R is an almost Gorenstein local ring. Then for every $n \geq 1$ the formal power series ring $S = R[[X_1, X_2, \ldots, X_n]]$ is also an almost Gorenstein local ring with $\mathrm{r}(S) = \mathrm{r}(R)$.*

Proposition 1.3.12. *Let $(S, \mathfrak{n})$ be a Noetherian local ring and let $\varphi : R \to S$ be a flat local homomorphism such that $S/\mathfrak{m}S$ is a Gorenstein ring. Assume the following three conditions are satisfied.*

(1) *The field $R/\mathfrak{m}$ is infinite.*

(2) *R and S are almost Gorenstein local rings.*

(3) *S is not a Gorenstein ring.*

If $\dim R = 1$, then $S/\mathfrak{m}S$ is a regular local ring.

Proof. When $\dim S/\mathfrak{m}S > 0$, we pass to the flat local homomorphism $R \to S/gS$, choosing $g \in \mathfrak{n}$ so that g is $S/\mathfrak{m}S$-regular and S/gS is an almost Gorenstein local ring. Therefore we may assume that $\dim S/\mathfrak{m}S = 0$. Choose an ideal $I \subsetneq R$ of R so that $I \cong \mathrm{K}_R$ as an R-module. Therefore $IS \cong \mathrm{K}_S$ as an S-module, since $S/\mathfrak{m}S$ is a Gorenstein local ring. Let $\mathrm{e}_1(I)$ (resp. $\mathrm{e}_1(IS)$) be the first Hilbert coefficient of R (resp.

S) with respect to I (resp. IS). Then by [26, Theorem 3.16] and [43, Satz 6.14] we have

$$\mathrm{e}_1(I) = \mathrm{r}(R) = \mathrm{r}(S) = \mathrm{e}_1(IS) = \ell_S(S/\mathfrak{m}S){\cdot}\mathrm{e}_1(I),$$

because R and S are almost Gorenstein local rings and both of them are not Gorenstein rings. Thus $\ell_S(S/\mathfrak{m}S) = 1$, so that $S/\mathfrak{m}S$ is a field. $\square$

Unless $\dim R = 1$, Proposition 1.3.12 does not hold true in general, as we show in the following.

Example 1.3.13. Let T be an almost Gorenstein local ring with maximal ideal $\mathfrak{m}_0$, $\dim T = 1$, and $\mathrm{r}(T) = 2$. Let $R = T[[X]]$ be the formal power series ring and let $R[Y]$ be the polynomial ring. We set $S = R[Y]/(Y^2 - X)$. Then the following assertions hold true.

(1) R and S are two-dimensional almost Gorenstein local rings with $\mathrm{r}(R) = \mathrm{r}(S) = 2$.

(2) S is a finitely generated free R-module of rank two but $S/\mathfrak{m}S$ is not a field, where $\mathfrak{m} = \mathfrak{m}_0 R + XR$ denotes the maximal ideal of R.

Proof. We set $k = T/\mathfrak{m}_0$. Notice that S is a local ring with maximal ideal $\mathfrak{m}S + yS$, where y denotes the image of Y in S. The R-module S is free of rank two and $S/\mathfrak{m}S = (R/\mathfrak{m})[Y]/(Y^2) = k[Y]/(Y^2)$. The T-algebra S is flat with

$$S/\mathfrak{m}_0 S = (k[[X]])[Y]/(Y^2 - X),$$

which is a discrete valuation ring. By Corollary 1.3.11 R is an almost Gorenstein local ring with $\mathrm{r}(R) = 2$. We now consider the exact sequence

$$(\sharp) \quad 0 \to T \to \mathrm{K}_T \to T/\mathfrak{m}_0 \to 0$$

of T-modules. Then since $\mathrm{K}_S \cong S \otimes_T \mathrm{K}_T$, tensoring exact sequence $(\sharp)$ by S, we get the exact sequence

$$0 \to S \to \mathrm{K}_S \to S/\mathfrak{m}_0 S \to 0$$

of S-modules. Therefore S is an almost Gorenstein local ring by definition, since $S/\mathfrak{m}_0 S$ is a discrete valuation ring, while $\mathrm{r}(S) = 2$ by [43, Satz 6.14], since $S/\mathfrak{m}S$ is a Gorenstein ring. $\square$

Let us note a few basic examples of almost Gorenstein local rings.

Example 1.3.14. Let $U = k[[X_1, X_2, \ldots, X_n, Y_1, Y_2, , \ldots, Y_n]]$ $(n \geq 2)$ be the formal power series ring over a field k and put $R = U/\mathrm{I}_2(\mathbb{M})$, where $\mathrm{I}_2(\mathbb{M})$ denotes the ideal of U generated by 2×2 minors of the matrix $\mathbb{M} = \left(\begin{smallmatrix} X_1 & X_2 & \cdots & X_n \\ Y_1 & Y_2 & \cdots & Y_n \end{smallmatrix}\right)$. Then R is almost Gorenstein with $\dim R = n+1$ and $\mathrm{r}(R) = n-1$.

Proof. It is well-known that R is a Cohen-Macaulay normal local ring with $\dim R = n+1$ and $\mathrm{r}(R) = n-1$ ([12]). The sequence $\{X_i - Y_{i-1}\}_{1 \leq i \leq n}$ (here $Y_0 = Y_n$ for convention) forms a regular sequence in R and

$$R/(X_i - Y_{i-1} \mid 1 \leq i \leq n)R \cong k[[X_1, X_2, \ldots, X_n]]/\mathrm{I}_2(\mathbb{N}),$$

where $\mathbb{N} = \left(\begin{smallmatrix} X_1 & X_2 & \cdots & X_{n-1} & X_n \\ X_2 & X_3 & \cdots & X_n & X_1 \end{smallmatrix}\right)$. Let $S = k[[X_1, X_2, \ldots, X_n]]/\mathrm{I}_2(\mathbb{N})$. Then S is a Cohen-Macaulay local ring of dimension one, such that $\mathfrak{n}^2 = x_1\mathfrak{n}$ and $\mathrm{K}_S \cong (x_1, x_2, \ldots, x_{n-1})$, where $\mathfrak{n}$ is the maximal ideal of S and x_i is the image of X_i in S. Hence S is an almost Gorenstein local ring, because $\mathfrak{n}(x_1, x_2, \ldots, x_{n-1}) \subseteq (x_1)$. Thus R is an almost Gorenstein local ring by Theorem 1.3.7 (1). $\square$

Example 1.3.15. Let $S = k[[X, Y, Z]]$ be a formal power series ring over a field k and let $\mathbb{M} = \left(\begin{smallmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{smallmatrix}\right)$ be a matrix such that $f_{ij} \in kX + kY + kZ$ for each $1 \leq i \leq 2$ and $1 \leq j \leq 3$. Assume that $\mathrm{ht}_S \mathrm{I}_2(\mathbb{M}) = 2$ and set $R = S/\mathrm{I}_2(\mathbb{M})$. Then R is an almost Gorenstein local ring if and only if $R \not\cong S/(Y, Z)^2$.

Proof. Since $\mathrm{Q}(S/(Y, Z)^2)$ is not a Gorenstein ring, the *only if* part follows from Lemma 1.3.1 (1). Suppose that $R \not\cong S/(Y, Z)^2$. Then, thanks to [32, Classification Table 6.5], without loss of generality we may assume that our matrix $\mathbb{M}$ has the form $\mathbb{M} = \left(\begin{smallmatrix} g_1 & g_2 & g_3 \\ X & Y & Z \end{smallmatrix}\right)$, where $g_i \in kX + kY + kZ$ for every $1 \leq i \leq 3$. Let $d_1 = Yg_3 - Zg_2$, $d_2 = Zg_1 - Xg_3$, and $d_3 = Xg_2 - Yg_1$. Then the S-module $R = S/(d_1, d_2, d_3)$ has a minimal free resolution

$$0 \to S^2 \xrightarrow{{}^t\mathbb{M}} S^3 \xrightarrow{[d_1 d_2 d_3]} S \to R \to 0$$

and, taking the S-dual, we get the presentation $S^3 \xrightarrow{\mathbb{M}} S^2 \xrightarrow{\varepsilon} \mathrm{K}_R \to 0$ of the canonical module $\mathrm{K}_R = \mathrm{Ext}_R^2(R, S)$. Hence $\mathrm{K}_R/R\xi \cong S^2/L \cong S/(X, Y, Z)$, where $\xi = \varepsilon(\binom{0}{1})$, and L denotes the S-submodule of S^2 generated by $\binom{X}{g_1}$, $\binom{Y}{g_2}$, $\binom{Z}{g_3}$, and $\binom{0}{1}$. We therefore have an exact sequence $R \to \mathrm{K}_R \to R/\mathfrak{m} \to 0$ of R-modules, where $\mathfrak{m}$ is the maximal ideal of R. Hence R is almost Gorenstein by Lemma 1.3.1 (1). $\square$

Example 1.3.16. Let $a,\ell \in \mathbb{Z}$ such that $a \geq 4$, $\ell \geq 2$ and let

$$R = k[[t^a, t^{a\ell-1}, \{t^{a\ell+i}\}_{1 \leq i \leq a-3}\}]] \ \subseteq k[[t]]$$

be the semigroup ring of the numerical semigroup $H = \langle a, a\ell - 1, \{a\ell + i\}_{1 \leq i \leq a-3} \rangle$, where $k[[t]]$ denotes the formal power series ring over a field k. Then R is an almost Gorenstein local ring with $\mathrm{r}(R) = a - 2$. Therefore the formal power series rings $R[[X_1, X_2, \ldots, X_n]]$ $(n \geq 1)$ are also almost Gorenstein.

Proof. Let $I = (t^{2a\ell-a-1}, \{t^{3a\ell-2a-i-1}\}_{1 \leq i \leq a-3})$. Then $I \cong \mathrm{K}_R$ and $\mathfrak{m}I = \mathfrak{m}t^{2a\ell-a-1}$, where $\mathfrak{m}$ denotes the maximal ideal of R. Hence R is an almost Gorenstein local ring (see [20, Example 2.13] for details). $\square$

Remark 1.3.17. The local rings $R_{\mathfrak{p}}$ $(\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\})$ of an almost Gorenstein local ring $(R, \mathfrak{m})$ are not necessarily Gorenstein rings (Example 1.10.9). Also, the local rings $R_{\mathfrak{p}}$ of an almost Gorenstein local ring R are not necessarily almost Gorenstein (Example 1.9.13).

1.4 Characterization in terms of existence of certain exact sequences

In this section we investigate the almost Gorenstein property of local rings in terms of the existence of certain exact sequences.

Throughout this section, let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension d and Cohen-Macaulay type r, admitting the canonical module K_R. In what follows, all R-modules assumed to be finitely generated. For each sequence $\boldsymbol{x} = x_1, x_2, \ldots, x_n$ of elements in R and an R-module M, let $\mathbb{K}_\bullet(\boldsymbol{x}, M)$ be the Koszul complex of M associated to $\boldsymbol{x}$. Hence

$$\mathbb{K}_\bullet(\boldsymbol{x}, M) = \mathbb{K}_\bullet(x_1, R) \otimes_R \cdots \otimes_R \mathbb{K}_\bullet(x_n, R) \otimes_R M.$$

For each $z \in M$ let $R \xrightarrow{z} M$ stand for the homomorphism $a \mapsto az$. Denote by $\mathbb{U}_R(z, M)$ the complex

$$\mathbb{U}_R(z, M) = (\cdots \to \underset{2}{0} \to \underset{1}{R} \xrightarrow{z} \underset{0}{M} \to \underset{-1}{0} \to \cdots).$$

When there is no danger of confusion, we simply write $\mathbb{U}(z, M)$ as $\mathbb{U}_R(z, M)$. Let $\mathcal{D}(R)$ denote the derived category of R. Hence for two complexes X, Y of R-modules, one has $\mathrm{H}_i(X) \cong \mathrm{H}_i(Y)$ for all $i \in \mathbb{Z}$, if $X \cong Y$ in $\mathcal{D}(R)$.

Let us begin with the following.

Lemma 1.4.1. *Let $\boldsymbol{x} = x_1, x_2, \ldots, x_n$ be an R-sequence and let $y \in \mathrm{K}_R$. Then*

$$R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R \mathbb{U}_R(y, \mathrm{K}_R) \cong \mathbb{U}_{R/(\boldsymbol{x})}(\overline{y}, \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R)$$

in $\mathcal{D}(R)$, where $\overline{y}$ denotes the image of y in $\mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R$.

Proof. Since $\mathbb{U}_R(y, \mathrm{K}_R)$ is the mapping cone of the map $R \xrightarrow{y} \mathrm{K}_R$, we get an exact triangle $R \xrightarrow{y} \mathrm{K}_R \to \mathbb{U}_R(y, \mathrm{K}_R) \rightsquigarrow$ in $\mathcal{D}(R)$, which gives rise to, applying the triangle functor $R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R -$, an exact triangle

$$R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R R \xrightarrow{R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R y} R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R \mathrm{K}_R \to R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R \mathbb{U}_R(y, \mathrm{K}_R) \rightsquigarrow .$$

Notice that $R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R R \cong R/(\boldsymbol{x})$ and that $\mathrm{Tor}^R_i(R/(\boldsymbol{x}), \mathrm{K}_R) \cong \mathrm{H}_i(\mathbb{K}_\bullet(\boldsymbol{x}, \mathrm{K}_R)) = (0)$ for all $i > 0$, since $\boldsymbol{x}$ is also an K_R-sequence; hence $R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R \mathrm{K}_R \cong \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R$. Observe that $R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R \mathbb{U}_R(y, \mathrm{K}_R)$ is isomorphic to the mapping cone of the map $R/(\boldsymbol{x}) \xrightarrow{\overline{y}} \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R$, which is nothing but $\mathbb{U}_{R/(\boldsymbol{x})}(\overline{y}, \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R)$. □

We firstly give a characterization of Gorenstein local rings.

Proposition 1.4.2. *The following conditions are equivalent.*

(1) *R is a Gorenstein ring.*

(2) *There exist an R-sequence $\boldsymbol{x}$ and an element $y \in \mathrm{K}_R$ such that $\mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes_R \mathbb{U}(y, \mathrm{K}_R)$ is an exact sequence.*

Proof. (1) $\Rightarrow$ (2) Choose $y \in \mathrm{K}_R$ so that $\mathrm{K}_R = Ry$ and notice that $\mathbb{U}(y, \mathrm{K}_R) \cong \mathbb{K}_\bullet(1, R) \cong (0)$ in $\mathcal{D}(R)$. Therefore for each R-sequence $\boldsymbol{x}$ we get

$$\mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes_R \mathbb{U}(y, \mathrm{K}_R) \cong \mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes^{\mathbf{L}}_R \mathbb{U}(y, \mathrm{K}_R) \cong \mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes^{\mathbf{L}}_R (0) \cong (0)$$

in $\mathcal{D}(R)$, where the first isomorphism comes from the fact that $\mathbb{K}_\bullet(\boldsymbol{x}, R)$ is a complex of free R-modules. Thus the complex $\mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes_R \mathbb{U}(y, \mathrm{K}_R)$ is exact.

(2) ⇒ (1) By Lemma 1.4.1

$$(0) \cong \mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes_R \mathbb{U}(y, \mathrm{K}_R) \cong R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R \mathbb{U}_R(y, \mathrm{K}_R) \cong \mathbb{U}_{R/(\boldsymbol{x})}(\overline{y}, \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R)$$

in $\mathcal{D}(R)$. Hence the map $R/(\boldsymbol{x}) \xrightarrow{\overline{y}} \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R$ is an isomorphism, which shows $R/(\boldsymbol{x}) \cong \mathrm{K}_{R/(\boldsymbol{x})}$. Therefore $R/(\boldsymbol{x})$ is a Gorenstein ring and hence so is R. □

The following theorem 1.4.3 is the main result of this section, characterizing almost Gorenstein local rings. For an R-module M and a complex $C = (\cdots \to C_2 \to C_1 \to C_0 \to 0)$ of R-modules, we say that C is acyclic over M, if $\mathrm{H}_0(C) \cong M$ and $\mathrm{H}_i(C) = (0)$ for all $i > 0$.

Theorem 1.4.3. *Assume that $d = \dim R \geq 1$ and the field $k = R/\mathfrak{m}$ is infinite. Then the following conditions are equivalent.*

(1) *R is an almost Gorenstein local ring.*

(2) *There exist an R-sequence $\boldsymbol{x} = x_1, x_2, \dots, x_{d-1}$ and an element $y \in \mathrm{K}_R$ such that $\mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes_R \mathbb{U}(y, \mathrm{K}_R)$ is an acyclic complex over k^{r-1}.*

(3) *There exist an R-sequence $\boldsymbol{x}$ and an element $y \in \mathrm{K}_R$ such that $\mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes_R \mathbb{U}(y, \mathrm{K}_R)$ is an acyclic complex over an R-module M such that $\mathfrak{m}M = (0)$.*

Proof. (1) ⇒ (2) By Theorem 1.3.7 (2) applied repeatedly, we get an R-sequence $\boldsymbol{x} = x_1, x_2, \dots, x_{d-1}$ such that $R/(\boldsymbol{x})$ is an almost Gorenstein local ring of dimension one. Choose an exact sequence

$$0 \to R/(\boldsymbol{x}) \xrightarrow{\varphi} \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R \to k^{r-1} \to 0.$$

Setting $\overline{y} = \varphi(1)$ with $y \in \mathrm{K}_R$, we see that $\mathbb{U}(\overline{y}, \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R)$ is quasi-isomorphic to k^{r-1}. By Lemma 1.4.1

$$\mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes_R \mathbb{U}(y, \mathrm{K}_R) \cong R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R \mathbb{U}(y, \mathrm{K}_R) \cong \mathbb{U}(\overline{y}, \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R) \cong k^{r-1}$$

in $\mathcal{D}(R)$. Hence $\mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes_R \mathbb{U}(y, \mathrm{K}_R)$ is an acyclic complex over k^{r-1}.

(2) ⇒ (3) This is obvious.

(3) ⇒ (1) By Lemma 1.4.1

$$\mathbb{U}(\overline{y}, \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R) \cong R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R \mathbb{U}(y, \mathrm{K}_R) \cong \mathbb{K}_\bullet(\boldsymbol{x}, R) \otimes_R \mathbb{U}(y, \mathrm{K}_R) \cong k^n$$

for some $n \geq 0$. Hence there exists an exact triangle $R/(\boldsymbol{x}) \xrightarrow{\overline{y}} \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R \to k^n \rightsquigarrow$ in $\mathcal{D}(R)$, which gives the exact sequence $0 \to R/(\boldsymbol{x}) \xrightarrow{\overline{y}} \mathrm{K}_R/\boldsymbol{x}\mathrm{K}_R \to k^n \to 0$ of homology. Thus $R/(\boldsymbol{x})$ is an almost Gorenstein local ring of dimension one, whence by Theorem 1.3.7 (1) R is an almost Gorenstein local ring. □

Applying Theorem 1.4.3 to local rings of lower dimension, we readily get the following.

Corollary 1.4.4. *Assume that $k = R/\mathfrak{m}$ is infinite.*

(1) *Let $d = 1$. Then R is an almost Gorenstein local ring if and only if there exist an element $y \in \mathrm{K}_R$ and an exact sequence*

$$0 \to R \xrightarrow{y} \mathrm{K}_R \to k^{r-1} \to 0.$$

(2) *Let $d = 2$. Then R is an almost Gorenstein local ring if and only if there exist an R-regular element x, an element $y \in \mathrm{K}_R$, and an exact sequence*

$$0 \to R \xrightarrow{\left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right)} \mathrm{K}_R \oplus R \xrightarrow{(x,y)} \mathrm{K}_R \to k^{r-1} \to 0.$$

(3) *Let $d = 3$. Then R is an almost Gorenstein local ring if and only if there exist an R-sequence x_1, x_2, an element $y \in \mathrm{K}_R$, and an exact sequence*

$$0 \to R \xrightarrow{\left(\begin{smallmatrix} y \\ x_2 \\ -x_1 \end{smallmatrix}\right)} \mathrm{K}_R \oplus R^2 \xrightarrow{\left(\begin{smallmatrix} x_2 & -y & 0 \\ -x_1 & 0 & -y \\ 0 & x_1 & x_2 \end{smallmatrix}\right)} \mathrm{K}_R^2 \oplus R \xrightarrow{(x_1,x_2,y)} \mathrm{K}_R \to k^{r-1} \to 0.$$

For an R-module M let $\mathrm{pd}_R\, M$ and $\mathrm{Gdim}_R\, M$ denote the projective dimension and the G-dimension of M, respectively (we refer the reader to [13] for details of G-dimension).

Corollary 1.4.5. *Assume that R is an almost Gorenstein local ring of dimension $d \geq 1$. Then the following assertions hold true.*

(1) *The exact sequence*

$$0 \to R \to \mathrm{K}_R \oplus R^{d-1} \to \mathrm{K}_R^{d-1} \oplus R^{\binom{d-1}{2}} \to \cdots \to \mathrm{K}_R^{\binom{d-1}{2}} \oplus R^{d-1} \to \mathrm{K}_R^{d-1} \oplus R \to \mathrm{K}_R \to k^{r-1} \to 0$$

arising from Theorem 1.4.3 (2) is self-dual with respect to K_R, that is, after dualizing this exact sequence by K_R, one obtains the same exact sequence (up to isomorphisms).

(2) *Suppose that R is not a Gorenstein ring. Then R is G-regular in the sense of [71], that is* $\operatorname{Gdim}_R M = \operatorname{pd}_R M$ *for every finitely generated R-module M.*

Proof. (1) Let $X[n]$ denote, for a complex X of R-modules and $n \in \mathbb{Z}$, the complex X shifted by n (to the left). Then with the same notation as in Theorem 1.4.3 (2), $\mathbb{K} = \mathbb{K}_\bullet(\boldsymbol{x}, R)$ and $U = \mathbb{U}(y, \mathrm{K}_R)$. Therefore $\operatorname{Hom}_R(\mathbb{K}, R) \cong \mathbb{K}[1-d]$ and $\operatorname{Hom}_R(U, \mathrm{K}_R) \cong U[-1]$, which show

$$\begin{aligned}\operatorname{Hom}_R(\mathbb{K} \otimes_R U, \mathrm{K}_R) &\cong \operatorname{Hom}_R(\mathbb{K}, \operatorname{Hom}_R(U, \mathrm{K}_R)) \cong \operatorname{Hom}_R(\mathbb{K}, U[-1]) \\ &\cong \operatorname{Hom}_R(\mathbb{K}, R) \otimes_R U[-1] \cong \mathbb{K}[1-d] \otimes_R U[-1] \cong (\mathbb{K} \otimes_R U)[-d]\end{aligned}$$

(for the third isomorphism, remember that $\mathbb{K}$ is a bounded complex of free R-modules). Hence we get the assertion, because $\mathrm{H}_0(\mathbb{K} \otimes_R U) \cong k^{r-1}$ and $\mathrm{H}_i(\mathbb{K} \otimes_R U) = (0)$ for $i > 0$ by Theorem 1.4.3 (2).

(2) It suffices to show that every R-module M of finite G-dimension is of finite projective dimension. Let N be a high syzygy of M. Then since N is totally reflexive and maximal Cohen-Macaulay, we have

$$\operatorname{Ext}_R^i(N, R) = (0) = \operatorname{Ext}_R^i(N, \mathrm{K}_R)$$

for all $i > 0$. Apply the functor $\operatorname{Hom}_R(N, -)$ to the exact sequence in assertion (1) and we get

$$\operatorname{Ext}_R^i(N, k^{r-1}) = (0)$$

for $i \gg 0$. Since $r - 1 > 0$ (as R is not Gorenstein), N has finite projective dimension, and so does M. $\square$

Let us consider the Poincaré and Bass series over almost Gorenstein local rings. First of all let us fix some terminology. Let X (respectively, Y) be a homologically right (respectively, left) bounded complex of R-modules, possessing finitely generated homology modules. The *Poincaré series* of X and the *Bass series* of Y are defined as the following formal Laurent series with coefficients among non-negative integers:

$$\mathbb{P}_X(t) = \sum_{n \in \mathbb{Z}} \dim_k \operatorname{Tor}_n^R(X, k) {\cdot} t^n, \ \ \mathbb{I}^Y(t) = \sum_{n \in \mathbb{Z}} \dim_k \operatorname{Ext}_R^n(k, Y) {\cdot} t^n.$$

We then have the following, in which the Poincaré and Bass series of $C = \operatorname{Coker} \varphi$ are described in terms of the Bass series of R.

Theorem 1.4.6. *Let $(R, \mathfrak{m}, k)$ be an almost Gorenstein local ring of dimension $d \geq 1$ and assume that R is not a Gorenstein ring. Consider an exact sequence*

$$(\sharp) \quad 0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules such that C is an Ulrich R-module. We then have the following.

(1) $\mathbf{R}\mathrm{Hom}_R(C, \mathrm{K}_R) \cong C[-1]$ *in* $\mathcal{D}(R)$.

(2) $\mathbb{P}_C(t) = t^{1-d}\mathbb{I}^R(t) - 1$ *and* $\mathbb{I}^C(t) = \mathbb{I}^R(t) - t^{d-1}$.

Proof. (1) Since C is a Cohen-Macaulay R-module of dimension $d-1$, $\mathrm{Ext}_R^i(C, \mathrm{K}_R) = (0)$ for all $i \neq 1$ ([43, Satz 6.1]). To see $C \cong \mathrm{Ext}_R^1(C, \mathrm{K}_R)$, take the K_R-dual of exact sequence $(\sharp)$ and we get the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(\mathrm{K}_R, \mathrm{K}_R) & \xrightarrow{\mathrm{Hom}_R(\varphi, \mathrm{K}_R)} & \mathrm{Hom}_R(R, \mathrm{K}_R) & \longrightarrow & \mathrm{Ext}_R^1(C, \mathrm{K}_R) & \longrightarrow & 0 \\
 & & \cong\uparrow & & \cong\uparrow & & & & \\
0 & \longrightarrow & R & \xrightarrow{\varphi} & \mathrm{K}_R & \longrightarrow & C & \longrightarrow & 0,
\end{array}$$

where the vertical isomorphisms are canonical ones. Hence $C \cong \mathrm{Ext}_R^1(C, \mathrm{K}_R)$, so that $\mathrm{r}_R(C) = \mu_R(C) = r - 1$ by Corollary 1.3.10 and [43, Satz 6.10].

(2) By [13, (A.7.7)]

$$t\mathbb{I}^C(t) = \mathbb{I}^{C[-1]}(t) = \mathbb{I}^{\mathbf{R}\mathrm{Hom}(C, \mathrm{K}_R)}(t) = \mathbb{P}_C(t)\mathbb{I}_R^{\mathrm{K}}(t),$$

while

$$\mathbb{I}^C(t) = t^{d-1}\mathbb{P}_C(t),$$

as $\mathbb{I}_R^{\mathrm{K}}(t) = t^d$. Therefore, since $\mathrm{r}_R(C) = \mu_R(C) = r - 1$, applying $\mathrm{Hom}_R(k, -)$ to exact sequence $(\sharp)$ and writing the long exact sequence, we get

$$\mathrm{Ext}_R^i(k, C) \cong \begin{cases} (0) & (i \leq d-2), \\ k^{r-1} & (i = d-1), \\ \mathrm{Ext}_R^{i+1}(k, R) & (i \geq d). \end{cases}$$

Hence $\mathbb{I}^C(t) = \mathbb{I}^R(t) - t^{d-1}$. □

1.5 Characterization in terms of canonical ideals

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$, which possesses the canonical module K_R. The main result of this section is the following characterization of almost Gorenstein local rings in terms of canonical ideals, which is a natural generalization of [26, Theorem 3.11].

Theorem 1.5.1. *Suppose that* $\mathrm{Q}(R)$ *is a Gorenstein ring and take an ideal* I $(\neq R)$ *such that* $I \cong \mathrm{K}_R$ *as an* R*-module. Consider the following two conditions*:

(1) R *is an almost Gorenstein local ring.*

(2) R *contains a parameter ideal* $Q = (f_1, f_2, \ldots, f_d)$ *such that* $f_1 \in I$ *and* $\mathfrak{m}(I + Q) = \mathfrak{m}Q$.

Then the implication (2) $\Rightarrow$ (1) *holds. If* $R/\mathfrak{m}$ *is infinite, the implication* (1) $\Rightarrow$ (2) *is also true.*

Proof. (2) $\Rightarrow$ (1) Let $\mathfrak{q} = (f_2, f_3, \ldots, f_d)$. Then $\mathfrak{q}$ is a parameter ideal for the Cohen-Macaulay local ring R/I, because $I + Q = I + \mathfrak{q}$. We set $\overline{R} = R/\mathfrak{q}$, $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$, and $\overline{I} = I\overline{R}$. Notice that $\overline{I} \cong I/\mathfrak{q}I \cong \mathrm{K}_{\overline{R}}$, since $\mathfrak{q} \cap I = \mathfrak{q}I$. Let $\overline{f_1}$ be the image of f_1 in $\overline{R}$. Then since $\overline{\mathfrak{m}}{\cdot}\overline{I} = \overline{\mathfrak{m}}{\cdot}\overline{f_1}$, by [26, Theorem 3.11] $\overline{R}$ is an almost Gorenstein local ring, so that R is an almost Gorenstein local ring by Theorem 1.3.7.

(1) $\Rightarrow$ (2) Suppose that $R/\mathfrak{m}$ is infinite. We may assume that R is not a Gorenstein ring (because I is a principal ideal, if R is a Gorenstein ring). We consider the exact sequence

$$0 \to R \xrightarrow{\varphi} I \to C \to 0 \qquad (\sharp)$$

of R-modules such that C is an Ulrich R-module. Let $f_1 = \varphi(1) \in I$. Choose an R-regular sequence $f_2, f_3, \ldots, f_d \in \mathfrak{m}$ so that (1) $f_1, f_2, \ldots, f_d$ is a system of parameters of R, (2) $f_2, f_3, \ldots, f_d$ is a system of parameters for the ring R/I, and (3) $\mathfrak{m}C = (f_2, f_3, \ldots, f_d)C$ (this choice is, of course, possible; see Proposition 1.2.2 (2)). Let $\mathfrak{q} = (f_2, f_3, \ldots, f_d)$ and set $\overline{R} = R/\mathfrak{q}$, $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$, and $\overline{I} = I\overline{R}$. Then exact sequence $(\sharp)$ gives rise to the exact sequence

$$0 \to \overline{R} \xrightarrow{\overline{\varphi}} \overline{I} \to \overline{C} \to 0$$

of $\overline{R}$-modules where $\overline{C} = C/\mathfrak{q}C$, because $\overline{I} \cong I/\mathfrak{q}I$ and $f_2, f_3, \ldots, f_d$ form a C-regular sequence. Therefore, since $\overline{I} \cong \mathrm{K}_{\overline{R}}$ and $\overline{\mathfrak{m}}\overline{C} = (0)$, $\overline{R}$ is an almost Gorenstein local ring. We furthermore have that $\overline{\mathfrak{m}}{\cdot}\overline{I} = \overline{\mathfrak{m}}{\cdot}\overline{f_1}$ (here $\overline{f_1}$ denotes the image of f_1 in $\overline{R}$), because $\overline{R}$ is not a discrete valuation ring (see Corollary 1.3.10; remember that $\overline{f_1} = \overline{\varphi}(1)$). Consequently, since $\mathfrak{m}I \subseteq \mathfrak{m}f_1 + \mathfrak{q}$, we get

$$\mathfrak{m}I \subseteq (\mathfrak{m}f_1 + \mathfrak{q}) \cap I = \mathfrak{m}f_1 + (\mathfrak{q} \cap I) = \mathfrak{m}f_1 + \mathfrak{q}I \subseteq \mathfrak{m}Q,$$

where $Q = (f_1, f_2, \ldots, f_d)$. Hence $\mathfrak{m}(I + Q) = \mathfrak{m}Q$ as wanted. $\square$

Let R be an almost Gorenstein local ring of dimension $d \geq 2$. Let I be an ideal of R such that $I \cong \mathrm{K}_R$ as an R-module. Suppose that R is not a Gorenstein ring but contains a parameter ideal $Q = (f_1, f_2, \ldots, f_d)$ such that $f_1 \in I$ and $\mathfrak{m}(I + Q) = \mathfrak{m}Q$. Let $\mathfrak{q} = (f_2, f_3, \ldots, f_d)$. We set $\overline{R} = R/\mathfrak{q}$, $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$ and $\overline{I} = I\overline{R}$. Then $\overline{R}$ is an almost Gorenstein local ring with $\overline{I} = \mathrm{K}_{\overline{R}}$ and $(\overline{f})$ is a reduction of $\overline{I}$ with $\overline{\mathfrak{m}}\overline{I} = \overline{\mathfrak{m}}\overline{f}$, where $\overline{f}$ denotes the image of f in $\overline{R}$ (see Proof of Theorem 1.5.1).

We explore what kind of properties the ideal $J = I + Q$ enjoys. To do this, we fix the following notation, which we maintain throughout this section.

Notation 1.5.2. Let $\mathcal{T} = R[Qt] \subseteq \mathcal{R} = R[It] \subseteq R[t]$ where t is an indeterminate and set $\mathrm{gr}_J(R) = \mathcal{R}/J\mathcal{R}$ $(= \bigoplus_{n \geq 0} J^n/J^{n+1})$. We set

$$\mathcal{S} = \mathcal{S}_Q(J) = J\mathcal{R}/J\mathcal{T}$$

(the Sally module of J with respect to Q; [76]). Let

$$\mathcal{B} = \mathcal{T}/\mathfrak{m}\mathcal{T}$$

$(= (R/\mathfrak{m})[T_1, T_2, \ldots, T_d]$, the polynomial ring) and

$$\mathrm{red}_Q(J) = \min\{n \geq 0 \mid J^{n+1} = QJ^n\}.$$

We denote by $\{\mathrm{e}_i(J)\}_{0 \leq i \leq d}$ the Hilbert coefficients of R with respect to J.

Let us begin with the following. We set $f = f_1$.

Corollary 1.5.3. *The following assertions hold true.*

(1) $\mathrm{red}_Q(J) = 2$.

(2) $\mathcal{S}_Q(J) \cong \mathcal{B}(-1)$ *as a graded* $\mathcal{T}$*-module.*

(3) $\ell_R(R/J^{n+1}) = \ell_R(R/Q)\cdot\binom{n+d}{d} - \mathrm{r}(R)\cdot\binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}$ *for all* $n \geq 0$*. Hence* $\mathrm{e}_1(J) = \mathrm{r}(R)$, $\mathrm{e}_2(J) = 1$, *and* $\mathrm{e}_i(J) = 0$ *for* $3 \leq i \leq d$*.*

(4) *Let* $G = \mathrm{gr}_J(R)$*. Then the elements* $f_2t, f_3t, \ldots, f_dt$ $(\in \mathcal{T}_1)$ *form a regular sequence in* G *but* ft *is a zero-divisor in* G*. Hence* $\operatorname{depth} G = d-1$ *and the graded local cohomology module* $\mathrm{H}^d_{\mathfrak{M}}(G)$ *of* G *is not finitely generated, where* $\mathfrak{M} = \mathfrak{m}G + G_+$*.*

Proof. Let $K = Q :_R \mathfrak{m}$. Then $Q \subseteq J \subseteq K$. Notice that $\ell_R(J/Q) = \mu_R(J/Q) = \mu_R(\overline{I}/(\overline{f})) = \mathrm{r}(\overline{R}) - 1 = \mathrm{r}(R) - 1$, because $\overline{f} \notin \overline{\mathfrak{m}}\overline{I}$ and $\overline{I} = \mathrm{K}_{\overline{R}}$. Therefore $\ell_R(K/J) = 1$ since $\ell_R(J/Q) = \mathrm{r}(R)$, so that $K = J + (x)$ for some $x \in K$, while $K^2 = QK$ ([14]), as R is not a regular local ring. Consequently, $J^3 = QJ^2$ by [29, Proposition 2.6]. Thus $\mathrm{red}_Q(J) = 2$, since $\overline{I}^2 \neq \overline{f}\cdot\overline{I}$ ([26, Theorem 3.7]).

Let us show $\ell_R(J^2/QJ) = 1$. We have $\ell_R(\overline{I}^2/\overline{f}\cdot\overline{I}) = 1$ by [26, Theorem 3.16]. Choose $g \in I^2$ so that $I^2 \subseteq fI + (g) + \mathfrak{q}$. Then

$$I^2 = (fI + (g) + \mathfrak{q}) \cap I^2 \subseteq fI + (g) + \mathfrak{q}I,$$

since $\mathfrak{q} \cap I = \mathfrak{q}I$. Hence $J^2 = QJ + (g)$, because $J^2 = QJ + I^2$. Consequently $\ell_R(J^2/QJ) = 1$, since $\mathfrak{m}J^2 = \mathfrak{m}Q^2$ (remember that $\mathfrak{m}J = \mathfrak{m}Q$). Therefore, thanks to [63, 76], we have $\mathcal{S}_Q(J) \cong \mathcal{B}(-1)$ as a graded $\mathcal{T}$-module, $\mathrm{e}_1(J) = \mathrm{e}_0(J) - \ell_R(R/J) + 1$, and

$$\ell_R(R/J^{n+1}) = \mathrm{e}_0(J)\cdot\binom{n+d}{d} - \mathrm{e}_1(J)\cdot\binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}$$

for all $n \geq 0$. Hence

$$\mathrm{e}_1(J) = \mathrm{e}_0(J) - \ell_R(R/J) + 1 = \ell_R(R/Q) - \ell_R(R/J) + 1 = \ell_R(J/Q) + 1 = \mathrm{r}(R).$$

Thus assertions (1), (2), and (3) follow.

To see assertion (4), we claim the following, which shows the sequence $f_2t, f_3t, \ldots, f_dt$ is $\mathrm{gr}_J(R)$-regular.

Claim. $\mathfrak{q} \cap J^n = \mathfrak{q}J^{n-1}$ *for all* $n \in \mathbb{Z}$*.*

Proof of Claim. As $J^2 = QJ + (g) = fJ + (g) + \mathfrak{q}J$, we have

$$\mathfrak{q} \cap J^2 = \mathfrak{q}J + \mathfrak{q} \cap [fJ + (g)] \subseteq \mathfrak{q}J + (\mathfrak{q} \cap I) = \mathfrak{q}J.$$

Suppose that $n \geq 3$ and that our assertion holds true for $n-1$. Then

$$\mathfrak{q}\cap J^n = \mathfrak{q}\cap QJ^{n-1} = \mathfrak{q}J^{n-1}+(\mathfrak{q}\cap fJ^{n-1}) = \mathfrak{q}J^{n-1}+f\cdot(\mathfrak{q}\cap J^{n-1}) = \mathfrak{q}J^{n-1}+f\cdot\mathfrak{q}J^{n-2} = \mathfrak{q}J^{n-1}.$$

Hence $\mathfrak{q} \cap J^n = \mathfrak{q}J^{n-1}$ for all $n \in \mathbb{Z}$. $\square$

To show that ft is a zero-divisor in $\mathrm{gr}_J(R)$, remember that $g \notin QJ$, because $J^2 \neq QJ$. Since $J^2 \subseteq Q$, we may write $g = fy + h$ with $y \in R$ and $h \in \mathfrak{q}$. Then because $fy = g - h \in I^2 + \mathfrak{q}$, we see

$$fy \in (I^2 + \mathfrak{q}) \cap I = I^2 + \mathfrak{q}I \subseteq J^2,$$

while $y \notin J$. In fact, if $y \in J$, then

$$h = g - fy \in \mathfrak{q} \cap J^2 = \mathfrak{q}J \subseteq QJ,$$

so that $g = fy + h \in QJ$, which is impossible. Thus ft is a zero-divisor in $\mathrm{gr}_J(R)$. $\square$

Let $\rho : \mathrm{gr}_I(R) \xrightarrow{\varphi} \mathrm{gr}_J(R) \xrightarrow{\psi} \mathrm{gr}_{\overline{I}}(\overline{R})$ be the composite of canonical homomorphisms of associated graded rings and set $\mathcal{A} = \mathrm{Im}\,\varphi$. We then have $\mathrm{gr}_J(R) = \mathcal{A}[\xi_2, \xi_3, \dots, \xi_d]$, where $\xi_i = \overline{f_i t}$ denotes the image of $f_i t$ in $\mathrm{gr}_J(R)$. We are now interested in the question of when $\{\xi_i\}_{2 \leq i \leq d}$ are algebraically independent over $\mathcal{A}$. Our goal is Theorem 1.5.7 below.

We begin with the following, which readily follows from the fact that $\mathrm{Ker}\,\rho = \bigoplus_{n\geq 0}[I^{n+1} + (\mathfrak{q} \cap I^n)]/I^{n+1}$.

Lemma 1.5.4. $\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\rho$ *if and only if* $\mathfrak{q} \cap I^n \subseteq J^{n+1}$ *for all* $n \geq 2$.

Lemma 1.5.5. *Let* $n \geq 2$*. Then* $\mathfrak{q} \cap I^n = \mathfrak{q}I^n$ *if and only if* R/I^n *is a Cohen-Macaulay ring.*

Proof. If R/I^n is a Cohen-Macaulay ring, then $\mathfrak{q} \cap I^n = \mathfrak{q}I^n$, because $f_2, f_3, \dots, f_d$ form a regular sequence in R/I^n. Conversely, suppose that $\mathfrak{q} \cap I^n = \mathfrak{q}I^n$. Then the descending induction on i readily yields that

$$(f_2, f_2, f_3, \dots, f_i) \cap I^n = (f_2, f_3, \dots, f_i)I^n$$

for all $2 \leq i \leq d$, from which it follows that the sequence $f_2, f_3, \dots, f_d$ is R/I^n-regular. $\square$

Proposition 1.5.6. *The following assertions hold true.*

(1) *If R/I^3 is a Cohen-Macaulay ring, then $I^3 = fI^2$ and therefore the ideal I has analytic spread one and* $\mathrm{red}_{(f)}(I) = 2$.

(2) *If R/I^2 is a Cohen-Macaulay ring and $I^3 = fI^2$, then R/I^n is a Cohen-Macaulay ring for all $n \geq 1$*

Proof. (1) We have $\mathfrak{q} \cap I^3 = \mathfrak{q}I^3$ by Lemma 1.5.5, while $\overline{I}^3 = \overline{f}{\cdot}\overline{I}^2$ by Corollary 1.5.3 (1). Therefore $I^3 \subseteq (fI^2 + \mathfrak{q}) \cap I^3 = fI^2 + \mathfrak{q}I^3$, so that $I^3 = fI^2$ by Nakayama's lemma. Hence I is of analytic spread one and $\mathrm{red}_{(f)}(I) = 2$, because $\overline{I}^2 \neq \overline{f}{\cdot}\overline{I}$ ([26, Theorem 3.7]).

(2) We show that $\mathfrak{q} \cap I^n = \mathfrak{q}I^n$ for all $n \in \mathbb{Z}$. By Lemma 1.5.5 we may assume that $n \geq 3$ and that our assertion holds true for $n-1$. Then

$$\mathfrak{q} \cap I^n = \mathfrak{q} \cap fI^{n-1} = f(\mathfrak{q} \cap I^{n-1}) = f{\cdot}\mathfrak{q}I^{n-1} \subseteq \mathfrak{q}I^n.$$

Hence $\mathfrak{q} \cap I^n = \mathfrak{q}I^n$ for all $n \in \mathbb{Z}$, whence R/I^n is a Cohen-Macaulay ring by Lemma 1.5.5. □

We are now ready to prove the following.

Theorem 1.5.7. *Suppose that R/I^2 is a Cohen-Macaulay ring and $I^3 = fI^2$. Then $\mathcal{A}$ is a Buchsbaum ring and $\xi_2, \xi_3, \ldots, \xi_d$ are algebraically independent over $\mathcal{A}$, whence* $\mathrm{gr}_J(R) = \mathcal{A}[\xi_2, \xi_3, \ldots, \xi_d]$ *is the polynomial ring.*

Proof. We have $\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\rho$ by Lemma 1.5.4, 1.5.5, and Proposition 1.5.6, which shows that the composite homomorphism $\mathcal{A} \overset{\iota}{\hookrightarrow} \mathrm{gr}_J(R) \overset{\psi}{\longrightarrow} \mathrm{gr}_{\overline{I}}(\overline{R})$ is an isomorphism, where $\iota : \mathcal{A} \to \mathrm{gr}_J(R)$ denotes the embedding. Hence $\mathcal{A}$ is a Buchsbaum ring ([26, Theorem 3.16]). Let $k = \mathcal{A}_0$ and let $\mathcal{C} = k[X_2, X_3, \ldots, X_d]$ denote the polynomial ring. We regard $\mathcal{C}$ to be a $\mathbb{Z}$-graded ring so that $\mathcal{C}_0 = k$ and $\deg X_i = 1$. Let $B = \mathcal{A} \otimes_k \mathcal{C}$. Then B is a $\mathbb{Z}$-graded ring whose grading is given by $B_n = \sum_{i+j=n} \mathcal{A}_i \otimes_k \mathcal{C}_j$ for all $n \in \mathbb{Z}$. We put $Y_i = 1 \otimes X_i$ and consider the homomorphism Ψ: $B = \mathcal{A}[Y_2, Y_3, \ldots, Y_d] \to \mathrm{gr}_J(R)$ of $\mathcal{A}$-algebras defined by$\Psi(Y_i) = \xi_i$ for all $2 \leq i \leq d$. Let $\mathcal{K} = \mathrm{Ker}\,\Psi$. We then have the exact sequence $0 \to \mathcal{K} \to B \to \mathrm{gr}_J(R) \to 0$, which gives rise to the exact sequence

$$0 \to \mathcal{K}/(Y_2, Y_3, \ldots, Y_d)\mathcal{K} \to B/(Y_2, Y_3, \ldots, Y_d) \to \mathrm{gr}_J(R)/(\xi_2, \xi_3, \ldots, \xi_d) \to 0,$$

since the sequence $\xi_2, \xi_3, \ldots, \xi_d$ is $\mathrm{gr}_J(R)$-regular (Corollary 1.5.3 (4)). Because

$$B/(Y_2, Y_3, \ldots, Y_d) = \mathcal{A} \cong \mathrm{gr}_{\overline{I}}(\overline{R}) = \mathrm{gr}_J R/(\xi_2, \xi_3, \ldots, \xi_d),$$

we have $\mathcal{K}/(Y_2, Y_3, \ldots, Y_d)\mathcal{K} = (0)$ and hence $\mathcal{K} = (0)$ by graded Nakayama's lemma. Thus $\Psi\colon\ B \to \mathrm{gr}_J(R)$ is an isomorphism of $\mathcal{A}$-algebras. $\square$

The ring R/I^n is not necessarily a Cohen-Macaulay ring. Let us explore one example.

Example 1.5.8. Let $S = k[[s,t]]$ be the formal power series ring over a field k and set $R = k[[s^3, s^2t, st^2, t^3]]$ in S. Then R is a Cohen-Macaulay local ring of dimension 2. Setting $x = s^3$, $y = s^2t$, $z = st^2$, and $w = t^3$, we have $I = (y,z) \cong \mathrm{K}_R$ as an R-module and $\mathfrak{m}{\cdot}(I+Q) = \mathfrak{m}{\cdot}Q$, where $Q = (y, x-w)$. Hence R is an almost Gorenstein local ring with $\mathrm{r}(R) = 2$. Let $\mathfrak{q} = (x-w)$. Then $\mathfrak{q} \cap I^2 = \mathfrak{q}I^2$, because

$$(x-w) \cap I^2 \subseteq (x-w){\cdot}(s^2t^2S \cap R) \subseteq (x-w){\cdot}I^2.$$

However, $\mathfrak{q} \cap I^3 \neq \mathfrak{q}I^3$. In fact, if $\mathfrak{q} \cap I^3 = \mathfrak{q}I^3$, by Proposition 1.5.6 (1) I is of analytic spread one, which is however impossible, because $(R/\mathfrak{m}) \otimes_R \mathrm{gr}_I(R) \cong k[y,z]$. Hence R/I^2 is a Cohen-Macaulay ring but R/I^3 is not a Cohen-Macaulay ring (Lemma 1.5.5). We have $\mathrm{e}_1(J) = \mathrm{r}(R) = 2$ and $\operatorname{depth}\ \mathrm{gr}_J(R) = 1$, where $J = I + Q = (x-w, y, z)$.

Question 1.5.9. Let T be an almost Gorenstein but non-Gorenstein local ring of dimension 1 and let K be an ideal of T with $K \cong \mathrm{K}_T$ as a T-module. Let $R = T[[X_2, X_3, \ldots, X_d]]$ $(d \geq 2)$ be a formal power series ring and set $I = KR$. Then $I \cong \mathrm{K}_R$ as an R-module and R/I^n is Cohen-Macaulay for all $n \geq 1$. We suspect that this is the unique case for $\mathrm{gr}_J(R)$ to be the polynomial ring over $\mathcal{A}$.

1.6 Almost Gorenstein local rings obtained by idealization

Throughout this section let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, which possesses the canonical module K_R. For each R-module M let $M^\vee = \mathrm{Hom}_R(M, \mathrm{K}_R)$. We study the question of when the idealization $R \ltimes M^\vee$ is an almost Gorenstein local ring.

Let us begin with the following, which is based on [26, Proposition 6.1] and gives an extension of the result to higher dimensional local rings.

Theorem 1.6.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$, which possesses the canonical module K_R. Let I $(\neq R)$ be an ideal of R and assume that R/I is a Cohen-Macaulay ring of dimension $d-1$. We consider the following two conditions*:

(1) *$A = R \ltimes I^\vee$ is an almost Gorenstein local ring.*

(2) *R contains a parameter ideal $Q = (f_1, f_2, \ldots, f_d)$ such that $f_1 \in I$, $\mathfrak{m}(I+Q) = \mathfrak{m}Q$, and $(I+Q)^2 = Q(I+Q)$.*

Then one has the implication (2) $\Rightarrow$ (1). *If $R/\mathfrak{m}$ is infinite, the reverse implication* (1) $\Rightarrow$ (2) *is also true.*

Proof. (2) $\Rightarrow$ (1) Let $\mathfrak{q} = (f_2, f_3, \ldots, f_d)$ and set $\overline{R} = R/\mathfrak{q}$, $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$, and $\overline{I} = I\overline{R}$. Then $\overline{I} \cong I/\mathfrak{q}I$, since $f_2, f_3, \ldots, f_d$ is a regular sequence in R/I, while $\mathrm{Hom}_{\overline{R}}(\overline{I}, \mathrm{K}_{\overline{R}}) \cong I^\vee/\mathfrak{q}I^\vee$, because $f_2, f_3, \ldots, f_d$ is an R-sequence and I is a maximal Cohen-Macaulay R-module ([43, Lemma 6.5]). Therefore since $\overline{I}^2 = \overline{f_1}{\cdot}\overline{I}$ and $\overline{\mathfrak{m}}{\cdot}\overline{I} = \overline{\mathfrak{m}}{\cdot}\overline{f_1}$ (here $\overline{f_1}$ denotes the image of f_1 in $\overline{R}$), by [26, Proposition 6.1] the idealization $A/\mathfrak{q}A = \overline{R} \ltimes \mathrm{Hom}_{\overline{R}}(\overline{I}, \mathrm{K}_{\overline{R}})$ is an almost Gorenstein local ring. Hence $A = R \ltimes I^\vee$ is an almost Gorenstein local ring by Theorem 1.3.7, because $f_2, f_3, \ldots, f_d$ form a regular sequence in A.

(1) $\Rightarrow$ (2) Suppose that $R/\mathfrak{m}$ is infinite and that $A = R \ltimes I^\vee$ is an almost Gorenstein local ring. Choose an exact sequence

$$0 \to A \to \mathrm{K}_A \to C \to 0 \qquad (\sharp)$$

of A-modules such that $\mu_A(C) = \mathrm{e}^0_{\mathfrak{n}}(C)$, where $\mathfrak{n} = \mathfrak{m} \times I^\vee$ is the maximal ideal of A. If $C = (0)$, then A is a Gorenstein local ring. Hence $I^\vee \cong \mathrm{K}_R$ ([58]), whence $I \cong \mathrm{K}_R^\vee = R$ and assertion (2) is certainly true. Assume that $C \neq (0)$. Then C is an Ulrich A-module of dimension $d-1$. We put $\mathfrak{a} = (0) :_A C$ and consider R to be a subring of A via the homomorphism $R \to A$, $r \mapsto (r, 0)$. Then, since $B = A/\mathfrak{a}$ is a module-finite extension of $S = R/[\mathfrak{a} \cap R]$, $\dim S = \dim B = \dim_A C = d-1$. We set $\mathfrak{n}_B = \mathfrak{n}B$ and $\mathfrak{m}_S = \mathfrak{m}S$. Then $\mathfrak{m}_S B$ is a reduction of $\mathfrak{n}_B$, because $[(0) \times I^\vee]^2 = (0)$ in A. We choose a subsystem $f_2, f_3, \ldots, f_d$ of parameters of R so that $f_2, f_3, \ldots f_d$ is a system of parameters for R/I and $(f_2, f_3, \ldots, f_d)B$ is a reduction of $\mathfrak{n}_B$. Then $\mathfrak{n}C = (f_2, f_3, \ldots, f_d)C$ by Proposition 1.2.2 (2). Consequently, since $f_2, f_3, \ldots, f_d$ is a C-regular sequence, from exact sequence ($\sharp$) above we get the exact sequence

$$0 \to A/\mathfrak{q}A \to \mathrm{K}_A/\mathfrak{q}\mathrm{K}_A \to C/\mathfrak{q}C \to 0$$

of $A/\mathfrak{q}A$-modules, where $\mathfrak{q} = (f_2, f_3, \dots, f_d)$. Hence $A/\mathfrak{q}A$ is an almost Gorenstein local ring of dimension one, because $\mathrm{K}_A/\mathfrak{q}\mathrm{K}_A \cong \mathrm{K}_{A/\mathfrak{q}A}$ and $\mathfrak{n}(C/\mathfrak{q}C) = (0)$. Let $\overline{R} = R/\mathfrak{q}$, $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$, and $\overline{I} = I\overline{R}$. Then since $\mathfrak{q} \cap I = \mathfrak{q}I$, we get $\overline{I} = [I+\mathfrak{q}]/\mathfrak{q} \cong I/\mathfrak{q}I$ and therefore the ring

$$\overline{R} \ltimes \mathrm{Hom}_{\overline{R}}(\overline{I}, \mathrm{K}_{\overline{R}}) = (R/\mathfrak{q}) \ltimes (I^\vee/\mathfrak{q}I^\vee) = A/\mathfrak{q}A$$

is an almost Gorenstein local ring. Consequently, by [26, Proposition 6.1] we may choose $f_1 \in I$ so that $\overline{\mathfrak{m}}{\cdot}\overline{I} = \overline{\mathfrak{m}}{\cdot}\overline{f_1}$ and $\overline{I}^2 = \overline{f_1}{\cdot}\overline{I}$, where $\overline{f_1}$ denotes the image of f_1 in $\overline{R}$. Let $Q = (f_1, f_2, \dots, f_d)$. We will show that $\mathfrak{m}(I+Q) = \mathfrak{m}Q$ and $(I+Q)^2 = Q(I+Q)$. Firstly, since $\overline{\mathfrak{m}}{\cdot}\overline{I} = \overline{\mathfrak{m}}{\cdot}\overline{f_1}$, we get $\mathfrak{m}I \subseteq (\mathfrak{m}f_1 + \mathfrak{q}) \cap I = \mathfrak{m}f_1 + (\mathfrak{q} \cap I)$. Hence $\mathfrak{m}I \subseteq \mathfrak{m}Q$, because $\mathfrak{q} \cap I = \mathfrak{q}I$, so that $\mathfrak{m}(I+Q) = \mathfrak{m}Q$. Since $\overline{I}^2 = \overline{f_1}{\cdot}\overline{I}$, we similarly have

$$I^2 \subseteq (f_1 I + \mathfrak{q}) \cap I^2 \subseteq f_1 I + \mathfrak{q}I = QI,$$

whence $(I+Q)^2 = Q(I+Q)$. Notice that Q is a parameter ideal of R, because $\sqrt{Q} = \sqrt{I+Q} = \mathfrak{m}$, which proves Theorem 1.6.1. $\square$

Let us consider the case where R is a Gorenstein ring. The following result extends [26, Corollary 6.4] to local rings of higher-dimension.

Corollary 1.6.2. *Suppose that $(R, \mathfrak{m})$ is a Gorenstein local ring of dimension $d \geq 1$. Let M be a Cohen-Macaulay faithful R-module and consider the following two conditions*:

(1) *$A = R \ltimes M$ is an almost Gorenstein local ring.*

(2) *$M \cong R$ or $M \cong \mathfrak{p}$ as an R-module for some $\mathfrak{p} \in \operatorname{Spec} R$ such that $R/\mathfrak{p}$ is a regular local ring of dimension $d-1$.*

Then the implication (2) $\Rightarrow$ (1) *holds. If $R/\mathfrak{m}$ is infinite, the reverse implication* (1) $\Rightarrow$ (2) *is also true.*

Proof. (2) $\Rightarrow$ (1) We may assume $M \cong \mathfrak{p}$, where $\mathfrak{p} \in \operatorname{Spec} R$ such that $R/\mathfrak{p}$ is a regular local ring of dimension $d-1$. We choose a subsystem $f_2, f_3, \dots, f_d$ of parameters of R so that $\mathfrak{m} = \mathfrak{p} + (f_2, f_3, \dots, f_d)$ and set $\mathfrak{q} = (f_2, f_3, \dots, f_d)$. Then

$$\mathfrak{p}/\mathfrak{q}\mathfrak{p} \cong [\mathfrak{p}+\mathfrak{q}]/\mathfrak{q} = \mathfrak{m}/\mathfrak{q}$$

and hence $A/\mathfrak{q}A \cong R/\mathfrak{q} \ltimes \mathfrak{m}/\mathfrak{q}$. Therefore $A/\mathfrak{q}A$ is an almost Gorenstein local ring by [26, Corollary 6.4] and hence A is an almost Gorenstein local ring by Theorem 1.3.7.

(1) $\Rightarrow$ (2) Suppose that $R/\mathfrak{m}$ is infinite and let $\mathfrak{p} \in \operatorname{Ass} R$. Then $A_{\mathfrak{p}\times M} \cong R_\mathfrak{p} \ltimes M_\mathfrak{p}$ and $M_\mathfrak{p} \neq (0)$. Because $\mathrm{Q}(A)$ is a Gorenstein ring, we get $M_\mathfrak{p} \cong \mathrm{K}_{R_\mathfrak{p}} \cong R_\mathfrak{p}$ ([58]). Hence $\mathrm{Q}(R) \otimes_R M \cong \mathrm{Q}(R)$ and therefore we have an exact sequence

$$0 \to R \to M \to X \to 0 \qquad (\sharp_1)$$

of R-modules such that $\mathrm{Q}(R) \otimes_R X = (0)$. Notice that X is a Cohen-Macaulay R-module of dimension $d-1$, because $X \neq (0)$ and $\operatorname{depth}_R M = d$. Take the K_R-dual (in fact, $\mathrm{K}_R = R$) of exact sequence ($\sharp_1$) and we get the exact sequence

$$0 \to M^\vee \xrightarrow{\varphi} R \to \operatorname{Ext}^1_R(M, \mathrm{K}_R) \to 0$$

of R-modules. Let $I = \varphi(M^\vee)$. Then $M \cong I^\vee$ and R/I is a Cohen-Macaulay local ring of dimension $d-1$. Consequently, because $A = R \ltimes I^\vee$ is an almost Gorenstein local ring, by Theorem 1.6.1 R contains a parameter ideal $Q = (f_1, f_2, \ldots, f_d)$ such that $f_1 \in I$, $\mathfrak{m}(I+Q) = \mathfrak{m}Q$, and $(I+Q)^2 = Q(I+Q)$. We set $\mathfrak{q} = (f_2, f_3, \ldots, f_d)$. Then, since $Q \subseteq I + Q \subseteq Q :_R \mathfrak{m}$ and R is a Gorenstein local ring, we have either $Q = I + Q$ or $I + Q = Q :_R \mathfrak{m}$.

If $Q = I + Q$, then

$$I \subseteq Q \cap I = [(f_1) + \mathfrak{q}] \cap I = (f_1) + \mathfrak{q}I,$$

since $\mathfrak{q} \cap I = \mathfrak{q}I$, so that $I = (f_1) \cong R$. Therefore $M \cong I^\vee \cong R$, which is impossible. Hence $I + Q = Q :_R \mathfrak{m}$ and $I \nsubseteq Q$. Choose $x \in I \setminus Q$. We then have $I + Q = Q + (x)$ and therefore $I = [(Q + (x)] \cap I = (f_1, x) + \mathfrak{q}I$, so that $I = (f_1, x)$. Notice that $\mu_R(I) = 2$, because $I \ncong R$. Let $\mathfrak{p} = (f_1) :_R x$. Then $I/(f_1) \cong R/\mathfrak{p}$ and hence $\dim R/\mathfrak{p} < d$. On the other hand, thanks to the depth lemma applied to the exact sequence

$$0 \to R \xrightarrow{\varphi} I \to R/\mathfrak{p} \to 0 \qquad (\sharp_2)$$

of R-modules where $\varphi(1) = f_1$, we get $\operatorname{depth} R/\mathfrak{p} \geq d-1$. Hence $\dim R/\mathfrak{p} = d-1$. Set $\overline{R} = R/\mathfrak{q}$, $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$, and $\overline{I} = I\overline{R}$. Then $\overline{\mathfrak{m}}{\cdot}\overline{I} \subseteq (\overline{f_1})$, where $\overline{f_1}$ denotes the image of f_1 in $\overline{R}$. Since $\overline{I} \cong \overline{R} \otimes_R I$, we see

$$\overline{R} \otimes_R R/\mathfrak{p} \cong \overline{R} \otimes_R [I/(f_1)] \cong \overline{I}/(\overline{f_1})$$

and therefore $\mathfrak{m} = \mathfrak{p} + \mathfrak{q}$, because $\overline{\mathfrak{m}}{\cdot}(\overline{I}/(\overline{f_1})) = (0)$. Thus $R/\mathfrak{p}$ is a regular local ring. Now we take the K_R-dual of exact sequence ($\sharp_2$) and get the exact sequence

$$0 \to I^\vee \to R \to \mathrm{Ext}_R^1(R/\mathfrak{p}, \mathrm{K}_R) \to 0$$

of R-modules. Because $\mathrm{Ext}_R^1(R/\mathfrak{p}, \mathrm{K}_R) \cong R/\mathfrak{p}$, we then have $I^\vee \cong \mathfrak{p}$. Hence $M \cong I^\vee \cong \mathfrak{p}$ as R-modules, which proves the implication (1) ⇒ (2). □

When R contains a prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p}$ is a regular local ring of dimension $d-1$, we have the following characterization for $A = R \ltimes \mathfrak{p}$ to be an almost Gorenstein local ring, which is an extension of [26, Theorem 6.5].

Theorem 1.6.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$, which possesses the canonical module K_R. Suppose that $R/\mathfrak{m}$ is infinite. Let $\mathfrak{p} \in \operatorname{Spec} R$ and assume that $R/\mathfrak{p}$ is a regular local ring of dimension $d-1$. Then the following conditions are equivalent.*

(1) *$A = R \ltimes \mathfrak{p}$ is an almost Gorenstein local ring.*

(2) *R is an almost Gorenstein local ring.*

Proof. By [26, Theorem 6.5] we may assume that $d > 1$ and that our assertion holds true for $d-1$.

(1) ⇒ (2) Let $0 \to A \to \mathrm{K}_A \to Y \to 0$ be an exact sequence of A-modules such that $\mu_A(Y) = \mathrm{e}_{\mathfrak{n}}^0(Y)$, where $\mathfrak{n} = \mathfrak{m} \times \mathfrak{p}$ is the maximal ideal of A. Let us choose a parameter f of R so that f is superficial for Y with respect to $\mathfrak{n}$ and $R/[\mathfrak{p} + (f)]$ is a regular local ring of dimension $d-2$. Then A/fA is an almost Gorenstein local ring (see the proof of Theorem 1.3.7 (2)) and

$$A/fA = R/(f) \ltimes \mathfrak{p}/f\mathfrak{p} \cong R/(f) \ltimes [\mathfrak{p} + (f)]/(f),$$

which shows $R/(f)$ is an almost Gorenstein local ring. Thus R is almost Gorenstein by Theorem 1.3.7.

(2) ⇒ (1) We consider the exact sequence $0 \to R \to \mathrm{K}_R \to X \to 0$ of R-modules with $\mu_R(X) = \mathrm{e}_{\mathfrak{m}}^0(X)$ and choose a parameter f of R so that f is superficial for X with respect to $\mathfrak{m}$ and $R/[\mathfrak{p} + (f)]$ is a regular local ring of dimension $d-2$. Then because $A/fA \cong R/(f) \ltimes [\mathfrak{p} + (f)]/(f)$, the ring A/fA is an almost Gorenstein local ring. Hence by Theorem 1.3.7 A is an almost Gorenstein local ring. □

The following example extends [26, Example 6.10].

Example 1.6.4. Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 1$ and infinite residue class field. Let $\mathfrak{p} \in \operatorname{Spec} R$ and assume that $R/\mathfrak{p}$ is a regular local ring of dimension $d-1$. We set $A = R \ltimes \mathfrak{p}$. Then, thanks to Theorem 1.6.3, A is an almost Gorenstein local ring. Therefore because $\mathfrak{p} \times \mathfrak{p} \in \operatorname{Spec} A$ with $A/[\mathfrak{p} \times \mathfrak{p}] \cong R/\mathfrak{p}$, setting

$$R_n = \begin{cases} R & (n=0) \\ R_{n-1} \ltimes \mathfrak{p}_{n-1} & (n>0) \end{cases}, \qquad \mathfrak{p}_n = \begin{cases} \mathfrak{p} & (n=0) \\ \mathfrak{p}_{n-1} \ltimes \mathfrak{p}_{n-1} & (n>0) \end{cases},$$

we get an infinite family $\{R_n\}_{n \geq 0}$ of almost Gorenstein local rings. Notice that R_n is not a Gorenstein ring, if $n \geq 2$ ([26, Lemma 6.6]).

1.7 Generalized Gorenstein local rings

Throughout this section let $(R, \mathfrak{m})$ denote a Noetherian local ring of dimension $d \geq 0$. We explore a special class of almost Gorenstein local rings, which we call semi-Gorenstein.

We begin with the definition.

Definition 1.7.1. We say that R is a semi-Gorenstein local ring, if R is an almost Gorenstein local ring, that is R is a Cohen-Macaulay local ring having a canonical module K_R equipped with an exact sequence

$$(\sharp) \quad 0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$, where either $C = (0)$, or $C \neq (0)$ and there exist R-submodules $\{C_i\}_{1 \leq i \leq \ell}$ of C such that $C = \oplus_{i=1}^{\ell} C_i$ and $\mu_R(C_i) = 1$ for all $1 \leq i \leq \ell$.

Therefore, every Gorenstein local ring is a semi-Gorenstein local ring (take $C = (0)$) and every one-dimensional almost Gorenstein local ring is semi-Gorenstein, since $\mathfrak{m}C = (0)$. We notice that in exact sequence $(\sharp)$ of Definition 1.7.1, if $C \neq (0)$, then each C_i is a cyclic Ulrich R-module of dimension $d-1$, whence $C_i \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} R$ such that $R/\mathfrak{p}_i$ is a regular local ring of dimension $d-1$.

We note the following. This is no longer true for the almost Gorenstein property, as we will show in Section 1.9 (see Remark 1.9.12).

Proposition 1.7.2. *Let R be a semi-Gorenstein local ring. Then for every $\mathfrak{p} \in \operatorname{Spec} R$ the local ring $R_\mathfrak{p}$ is semi-Gorenstein.*

Proof. We may assume that R is not a Gorenstein ring. Choose an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules which satisfies the condition in Definition 1.7.1. Hence $C = \oplus_{i=1}^{\ell} R/\mathfrak{p}_i$, where for each $1 \le i \le \ell$, $\mathfrak{p}_i \in \operatorname{Spec} R$ and $R/\mathfrak{p}_i$ is a regular local ring of dimension $d-1$. Let $\mathfrak{p} \in \operatorname{Spec} R$. Then since $\mathrm{K}_{R_\mathfrak{p}} = (\mathrm{K}_R)_\mathfrak{p}$, we get an exact sequence

$$0 \to R_\mathfrak{p} \to \mathrm{K}_{R_\mathfrak{p}} \to C_\mathfrak{p} \to 0$$

of $R_\mathfrak{p}$-modules, where $C_\mathfrak{p} = \oplus_{\mathfrak{p}_i \subseteq \mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p}$ is a direct sum of finite cyclic Ulrich $R_\mathfrak{p}$-modules $R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p}$, so that by definition the local ring $R_\mathfrak{p}$ is semi-Gorenstein. $\square$

Let us define the following.

Definition 1.7.3 (cf. [7])**.** An almost Gorenstein local ring R is said to be pseudo-Gorenstein, if $\mathrm{r}(R) \le 2$.

Proposition 1.7.4. *Let R be a pseudo-Gorenstein local ring. Then R is semi-Gorenstein and for every $\mathfrak{p} \in \operatorname{Spec} R$ the local ring $R_\mathfrak{p}$ is pseudo-Gorenstein.*

Proof. We may assume that $\mathrm{r}(R) = 2$. Because R is not a regular local ring, in the exact sequence $0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$ of Definition 1.3.3 we get $\varphi(1) \notin \mathfrak{m}\mathrm{K}_R$ by Corollary 1.3.10, whence $\mu_R(C) = \mu_R(\mathrm{K}_R) - 1 = 1$. Therefore R is semi-Gorenstein, and the second assertion follows from Proposition 1.7.2. $\square$

We note one example.

Example 1.7.5. Let $k[[t]]$ be the formal poser series ring over a field k. For an integer $a \ge 4$ we set

$$R = k[[t^{a+i} \mid 0 \le i \le a-1 \text{ but } i \ne a-2]]$$

in $k[[t]]$. Then $\mathrm{K}_R = R + Rt^{a-1}$ and $\mathfrak{m}\mathrm{K}_R \subseteq R$. Hence R is a pseudo-Gorenstein local ring with $\mathrm{r}(R) = 2$.

Whether C is decomposed into a direct sum of cyclic R-modules depends on the choice of exact sequences $0 \to R \to \mathrm{K}_R \to C \to 0$ with $\mu_R(C) = \mathrm{e}_\mathfrak{m}^0(C)$, although R is semi-Gorenstein. Let us note one example.

Example 1.7.6. Let $S = k[X, Y]$ be the polynomial ring over a field k and consider the Veronesean subring $R = k[X^4, X^3Y, X^2Y^2, XY^3, Y^4]$ of S with order 4. Then $\mathrm{K}_R = (X^3Y, X^2Y^2, XY^3)$ is the graded canonical module of R. The exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R(1) \to R/(X^3Y, X^2Y^2, XY^3, Y^4) \oplus R/(X^4, X^3Y, X^2Y^2, XY^3) \to 0$$

of graded R-modules with $\varphi(1) = X^2Y^2$ shows that the local ring $R_{\mathfrak{M}}$ is semi-Gorenstein, where $\mathfrak{M} = R_+$. However in the exact sequence

$$0 \to R \xrightarrow{\psi} \mathrm{K}_R(1) \to D \to 0$$

with $\psi(1) = XY^3$, $D_{\mathfrak{M}}$ is an Ulrich $R_{\mathfrak{m}}$-module of dimension one, but $D_{\mathfrak{M}}$ is indecomposable. In fact, setting $A = R_{\mathfrak{M}}$ and $C = D_{\mathfrak{M}}$, suppose $C \cong A/\mathfrak{p}_1 \oplus A/\mathfrak{p}_2$ for some regular local rings $A/\mathfrak{p}_i$ ($\mathfrak{p}_i \in \operatorname{Spec} A$) of dimension one. Let $\mathfrak{a} = XY^3A :_A (X^3Y, X^2Y^2, XY^3)A$. Then $\mathfrak{a} = (0) :_A C = \mathfrak{p}_1 \cap \mathfrak{p}_2$, so that $\mathfrak{a}$ should be a radical ideal of A, which is impossible, because $X^6Y^2 \in \mathfrak{a}$ but $X^3Y \notin \mathfrak{a}$.

Let us examine the non-zerodivisor characterization.

Theorem 1.7.7. *Suppose that $R/\mathfrak{m}$ is infinite. If R is a semi-Gorenstein local ring of dimension $d \geq 2$, then $R/(f)$ is a semi-Gorenstein local ring for a general non-zerodivisor $f \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

Proof. We may assume R is not a Gorenstein ring. We look at exact sequence ($\sharp$) $0 \to R \to \mathrm{K}_R \to C \to 0$ of Definition 1.7.1, where $C = \oplus_{i=1}^{r-1} R/\mathfrak{p}_i$ ($r = \mathrm{r}(R)$) and each $R/\mathfrak{p}_i$ is a regular local ring of dimension $d - 1$. Then $R/(f)$ is a semi-Gorenstein local ring for every $f \in \mathfrak{m}$ such that $f \notin \bigcup_{i=1}^{\ell} [\mathfrak{m}^2 + \mathfrak{p}_i] \cup \bigcup_{\mathfrak{p} \in \operatorname{Ass} R} \mathfrak{p}$. $\square$

We now give a characterization of semi-Gorenstein local rings in terms of their minimal free resolutions, which is a broad generalization of [26, Corollary 4.2].

Theorem 1.7.8. *Let $(S, \mathfrak{n})$ be a regular local ring and $\mathfrak{a} \subsetneq S$ an ideal of S with $n = \mathrm{ht}_S\, \mathfrak{a}$. Let $R = S/\mathfrak{a}$. Then the following conditions are equivalent.*

(1) *R is a semi-Gorenstein local ring but not a Gorenstein ring.*

(2) *R is Cohen-Macaulay, $n \geq 2$, $r = \mathrm{r}(R) \geq 2$, and R has a minimal S-free resolution of the form:*

$$0 \to F_n = S^r \xrightarrow{\mathbb{M}} F_{n-1} = S^q \to F_{n-2} \to \cdots \to F_1 \to F_0 = S \to R \to 0$$

where

$$
{}^t\mathbb{M} = \begin{pmatrix} y_{21}y_{22}\cdots y_{2\ell} & y_{31}y_{32}\cdots y_{3\ell} & \cdots & y_{r1}y_{r2}\cdots y_{r\ell} & z_1z_2\cdots z_m \\ x_{21}x_{22}\cdots x_{2\ell} & 0 & 0 & 0 & 0 \\ 0 & x_{31}x_{32}\cdots x_{3\ell} & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & x_{r1}x_{r2}\cdots x_{r\ell} & 0, \end{pmatrix},
$$

$\ell = n+1$, $q \geq (r-1)\ell$, $m = q-(r-1)\ell$, and $x_{i1}, x_{i2}, \ldots, x_{i\ell}$ is a part of a regular system of parameters of S for every $2 \leq i \leq r$.

When this is the case, one has the equality

$$
\mathfrak{a} = (z_1, z_2, \ldots, z_m) + \sum_{i=2}^{r} \mathrm{I}_2\left(\begin{smallmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_{i1} & y_{i2} & \cdots & x_{i\ell} \end{smallmatrix}\right),
$$

where $\mathrm{I}_2(\mathbb{N})$ denotes the ideal of S generated by 2×2 minors of the submatrix $\mathbb{N} = \left(\begin{smallmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_{i1} & y_{i2} & \cdots & x_{i\ell} \end{smallmatrix}\right)$ of $\mathbb{M}$.

Proof. (1) $\Rightarrow$ (2) Choose an exact sequence

$$
0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0
$$

of R-modules so that $C = \oplus_{i=2}^{r} S/\mathfrak{p}_i$, where each $S/\mathfrak{p}_i$ ($\mathfrak{p}_i \in \operatorname{Spec} S$) is a regular local ring of dimension $d-1$. Let $\mathfrak{p}_i = (x_{ij} \mid 1 \leq j \leq \ell)$ with a part $\{x_{ij}\}_{1 \leq j \leq \ell}$ of a regular system of parameters of S, where $\ell = n+1$ ($= \mathrm{ht}_S\, \mathfrak{p}_i$). We set $f_1 = \varphi(1) \in \mathrm{K}_R$ and consider the S-isomorphism

$$
\mathrm{K}_R/Sf_1 \xrightarrow{\psi} S/\mathfrak{p}_2 \oplus S/\mathfrak{p}_3 \oplus \cdots \oplus S/\mathfrak{p}_r,
$$

choosing elements $\{f_i \in \mathrm{K}_R\}_{2 \leq i \leq r}$ so that

$$
\psi(\overline{f_i}) = (0, \ldots, 0, \overset{\overset{i}{\vee}}{1}, 0 \ldots, 0) \in S/\mathfrak{p}_2 \oplus S/\mathfrak{p}_3 \oplus \cdots \oplus S/\mathfrak{p}_r,
$$

where $\overline{f_i}$ denotes the image of f_i in K_R/Sf_1. Hence $\{f_i\}_{1 \leq i \leq r}$ is a minimal system of generators of the S-module K_R. Let $\{\mathbf{e}_i\}_{1 \leq i \leq r}$ denote the standard basis of S^r and let $\varepsilon : S^r \to \mathrm{K}_R$ be the homomorphism defined by $\varepsilon(\mathbf{e}_i) = f_i$ for each $1 \leq i \leq r$. We now look at the exact sequence

$$
0 \to L \to S^r \xrightarrow{\varepsilon} \mathrm{K}_R \to 0.
$$

Then because $x_{ij}\overline{f_i} = 0$ in K_R/Sf_1, we get $y_{ij}f_1 + x_{ij}f_i = 0$ in K_R for some $y_{ij} \in \mathfrak{n}$. Set $\mathbf{a}_{ij} = y_{ij}\mathbf{e}_1 + x_{ij}\mathbf{e}_i \in L$ for each $2 \le i \le r$ and $1 \le j \le \ell$. Then $\{\mathbf{a}_{ij}\}_{2 \le i \le r, 1 \le j \le \ell}$ is a part of a minimal basis of L, because $\{x_{ij}\}_{1 \le j \le \ell}$ is a part of a regular system of parameters of S (use the fact that $L \subseteq \mathfrak{n}S^r$). Hence $q \ge (r-1)\ell$.

Let $\mathbf{a} \in L$ and write $\mathbf{a} = \sum_{i=1}^r a_i\mathbf{e}_i$ with $a_i \in S$. Then $a_i \in \mathfrak{p}_i = (x_{ij} \mid 1 \le j \le \ell)$ for every $2 \le i \le r$, because $\sum_{i=2}^r a_i\overline{f_i} = 0$ in K_R/Sf_1. Therefore, writing $a_i = \sum_{j=1}^{\ell} c_{ij}x_{ij}$ with $c_{ij} \in S$, we get $\mathbf{a} - \sum_{i=2}^r c_{ij}\mathbf{a}_{ij} = c\mathbf{e}_1$ for some $c \in S$, which shows L is minimally generated by $\{\mathbf{a}_{ij}\}_{2 \le i \le r, 1 \le j \le \ell}$ together with some elements $\{z_k\mathbf{e}_1\}_{1 \le k \le m}$ ($z_k \in S$). Thus the S-module $\mathrm{K}_R = \mathrm{Ext}_S^n(R, S)$ possesses a minimal free resolution

$$\cdots \to S^q \xrightarrow{\mathbb{M}} S^r \xrightarrow{\varepsilon} \mathrm{K}_R \to 0$$

with $q = m + (r-1)\ell$, in which the matrix $\mathbb{M}$ has the required form. Since $R \cong \mathrm{Ext}_S^n(\mathrm{K}_R, S)$ ([43, Satz 6.1]), the minimal S-free resolution of R is obtained, by taking the S-dual, from the minimal free resolution of K_R, so that assertion (2) follows.

(2) $\Rightarrow$ (1) We look at the presentation

$$S^q \xrightarrow{{}^t\mathbb{M}} S^r \xrightarrow{\varepsilon} \mathrm{K}_R \to 0$$

of $\mathrm{K}_R = \mathrm{Ext}_S^n(R, S)$. Let $\{\mathbf{e}_i\}_{1 \le i \le r}$ be the standard basis of S^r and set $f_1 = \varepsilon(\mathbf{e}_1)$. Then

$$\mathrm{K}_R/Rf_1 \cong S^r/[\mathrm{Im}\,{}^t\mathbb{M} + S\mathbf{e}_1] \cong \bigoplus_{i=2}^{r} S/(x_{ij} \mid 1 \le j \le \ell),$$

where each $S/(x_{ij} \mid 1 \le j \le \ell)$ is a regular local ring of dimension $d-1$, so that R is a semi-Gorenstein local ring, because the sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to \bigoplus_{i=2}^{r} S/(x_{ij} \mid 1 \le j \le \ell) \to 0$$

is exact by Lemma 1.3.1 (1), where $\varphi(1) = f_1$. This completes the proof of equivalence (1) $\Leftrightarrow$ (2).

To prove the last equality in Theorem 1.7.8, we need a preliminary step. Let S be a Noetherian local ring. Let $\boldsymbol{x} = x_1, x_2, \ldots, x_\ell$ be a regular sequence in S and $\boldsymbol{y} = y_1, y_2, \ldots, y_\ell$ a sequence of elements in S. We denote by $\mathbb{K} = \mathbb{K}_\bullet(\boldsymbol{x}, S)$ the Koszul complex of S associated to $x_1, x_2, \ldots, x_\ell$ and by $\mathbb{L} = \mathbb{K}_\bullet(\boldsymbol{y}, S)$ the Koszul complex of S

associated to $\boldsymbol{y} = y_1, y_2, \ldots, y_\ell$. We consider the diagram.

$$\begin{array}{ccccc} \mathbb{K}_2 & \xrightarrow{\partial_2^{\mathbb{K}}} & \mathbb{K}_1 = \mathbb{L}_1 & \xrightarrow{\partial_1^{\mathbb{K}}} & K_0(\boldsymbol{x}; S) = S \\ & & \Big\downarrow {\scriptstyle \partial_1^{\mathbb{L}}} & & \\ & & L_0(\boldsymbol{y}; S) = S & & \end{array}$$

and set $L = \partial_1^{\mathbb{L}}(\mathrm{Ker}\ \partial_1^{\mathbb{K}})$. Then because $\mathrm{Ker}\ \partial_1^{\mathbb{K}} = \mathrm{Im}\ \partial_2^{\mathbb{K}}$, we get the following.

Lemma 1.7.9. $L = (x_\alpha y_\beta - x_\beta y_\alpha \mid 1 \leq \alpha < \beta \leq \ell) = \mathrm{I}_2\left(\begin{smallmatrix} y_1 & y_2 & \cdots & y_\ell \\ x_1 & y_2 & \cdots & x_\ell \end{smallmatrix}\right)$.

Let us now check the last assertion in Theorem 1.7.8. We maintain the notation in the proof of implication (1) $\Rightarrow$ (2). Let $a \in S$. Then $a \in \mathfrak{a}$ if and only if $af_1 = 0$, because $\mathfrak{a} = (0) :_S \mathrm{K}_R$. The latter condition is equivalent to saying that

$$a\mathbf{e_1} = \sum_{2 \leq i \leq r,\ 1 \leq j \leq \ell} c_{ij}(y_{ij}\mathbf{e_i} + x_{ij}\mathbf{e_i}) + \sum_{k=1}^{m} d_k(z_k\mathbf{e_1})$$

for some $c_{ij},\ d_k \in S$, that is

$$a = \sum_{i=2}^{r}\left(\sum_{j=1}^{\ell} c_{ij}y_{ij}\right) + \sum_{k=1}^{m} d_k z_k \ \text{ and } \ \sum_{j=1}^{\ell} c_{ij}x_{ij} = 0 \ \text{ for all } \ 2 \leq i \leq r.$$

If $\sum_{j=1}^{\ell} c_{ij}x_{ij} = 0$, then by Lemma 1.7.9 we get

$$\sum_{j=1}^{\ell} c_{ij}y_{ij} \in \mathrm{I}_2\left(\begin{smallmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_{i1} & y_{i2} & \cdots & x_{i\ell} \end{smallmatrix}\right),$$

because $x_{i1}, x_{i2}, \ldots, x_{i\ell}$ is an S-sequence. Hence

$$\mathfrak{a} \subseteq (z_1, z_2, \ldots, z_m) + \sum_{i=2}^{r} \mathrm{I}_2\left(\begin{smallmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_{i1} & y_{i2} & \cdots & x_{i\ell} \end{smallmatrix}\right).$$

To see the reverse inclusion, notice that $z_k \in \mathfrak{a}$ for every $1 \leq k \leq m$, because $z_k f_1 = 0$. Let $2 \leq i \leq r$ and $1 \leq \alpha < \beta \leq \ell$. Then since

$$(x_{i\alpha}y_{i\beta} - x_{i\beta}y_{i\alpha})\mathbf{e}_1 = x_{i\alpha}(y_{i\beta}\mathbf{e}_1 + x_{i\beta}\mathbf{e}_i) - x_{i\beta}(y_{i\alpha}\mathbf{e}_1 + x_{i\alpha}\mathbf{e}_i) \in \mathrm{Ker}\,\varepsilon,$$

we get $(x_{i\alpha}y_{i\beta} - x_{i\beta}x_{i\alpha})f_1 = 0$, so that $x_{i\alpha}y_{i\beta} - x_{i\beta}y_{i\alpha} \in \mathfrak{a}$. Thus $(z_1, z_2, \ldots, z_m) + \sum_{i=2}^{r} \mathrm{I}_2\left(\begin{smallmatrix} y_{i1} & y_{i2} & \cdots & y_{i\ell} \\ x_{i1} & y_{i2} & \cdots & x_{i\ell} \end{smallmatrix}\right) \subseteq \mathfrak{a}$, which completes the proof of Theorem 1.7.8. $\square$

Corollary 1.7.10. *With the notation of Theorem* 1.7.8 *suppose that assertion* (1) *holds true. We then have the following.*

(1) *If $n = 2$, then $r = 2$ and $q = 3$, so that ${}^t\mathbb{M} = \left(\begin{smallmatrix} y_{21} & y_{22} & y_{23} \\ x_{21} & x_{22} & x_{23} \end{smallmatrix}\right)$.*

(2) *Suppose that $\mathfrak{a} \subseteq \mathfrak{n}^2$. If R has maximal embedding dimension, then $r = n$ and $q = n^2 - 1$, so that $m = 0$.*

Proof. (1) Since $n = 2$, we get $q = r + 1 \geq (r-1)\ell$. Hence $2 \geq (r-1)(\ell-1) = 2(r-1)$, as $\ell = 3$. Hence $r = 2$ and $q = 3$.

(2) We set $v = \ell_R(\mathfrak{m}/\mathfrak{m}^2)$ $(= \dim S)$, $e = \mathrm{e}^0_{\mathfrak{m}}(R)$, and $d = \dim R$ $(= v - n)$. Since $v = e + d - 1$, we then have $r = v - d = n$, while $q = (e-2)\cdot\binom{e}{e-1}$ by [62]. Hence $q = n^2 - 1 = (r-1)\ell$, so that $m = 0$. $\square$

One cannot expect $m = 0$ in general, although assertion (1) in Theorem 1.7.8 holds true. Let us note one example.

Example 1.7.11. Let $V = k[[t]]$ be the formal power series ring over a field k and set $R = k[[t^5, t^6, t^7, t^9]]$. Let $S = k[[X, Y, Z, W]]$ be the formal power series ring and let $\varphi : S \to R$ be the k-algebra map defined by $\varphi(X) = t^5, \varphi(Y) = t^6, \varphi(Z) = t^7$, and $\varphi(W) = t^9$. Then R has a minimal S-free resolution of the form

$$0 \to S^2 \overset{\mathbb{M}}{\to} S^6 \to S^5 \to S \to R \to 0,$$

where ${}^t\mathbb{M} = \left(\begin{smallmatrix} W & X^2 & XY & YZ & Y^2 - XZ & Z^2 - XW \\ X & Y & Z & W & 0 & 0 \end{smallmatrix}\right)$. Hence R is semi-Gorenstein with $\mathrm{r}(R) = 2$ and

$$\operatorname{Ker}\varphi = (Y^2 - XZ, Z^2 - XW) + \mathrm{I}_2\left(\begin{smallmatrix} W & X^2 & XY & YZ \\ X & Y & Z & W \end{smallmatrix}\right).$$

1.8 Almost Gorenstein graded rings

We now explore graded rings. In this section let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian graded ring such that $k = R_0$ is a local ring. Let $d = \dim R$ and let $\mathfrak{M}$ be the unique graded maximal ideal of R. Assume that R is a Cohen-Macaulay ring, admitting the graded canonical module K_R. The latter condition is equivalent to saying that $k = R_0$ is a homomorphic image of a Gorenstein ring ([40, 41]). We put $a = \mathrm{a}(R)$. Hence $a = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^d_{\mathfrak{M}}(R)]_n \neq (0)\} = -\min\{n \in \mathbb{Z} \mid [\mathrm{K}_R]_n \neq (0)\}$.

Definition 1.8.1. We say R is an almost Gorenstein graded ring, if there exists an exact sequence $0 \to R \to \mathrm{K}_R(-a) \to C \to 0$ of graded $R-$modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$. Here $\mathrm{K}_R(-a)$ denotes the graded R-module whose underlying R-module is the same as that of K_R and the grading is given by $[\mathrm{K}_R(-a)]_n = [\mathrm{K}_R]_{n-a}$ for all $n \in \mathbb{Z}$.

The ring R is an almost Gorenstein graded ring, if R is a Gorenstein ring. As $(\mathrm{K}_R)_{\mathfrak{M}} = \mathrm{K}_{R_{\mathfrak{M}}}$, the ring $R_{\mathfrak{M}}$ is an almost Gorenstein local ring, once R is an almost Gorenstein graded ring.

The condition stated in Definition 1.8.1 is rather strong, as we show in the following. Firstly we note:

Theorem 1.8.2 ([23])**.** *Suppose that A is a Gorenstein local ring and let I ($\neq A$) be an ideal of A. If $\mathrm{gr}_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ is an almost Gorenstein graded ring, then $\mathrm{gr}_I(A)$ is a Gorenstein ring.*

We secondly explore the almost Gorenstein property in the Rees algebras of parameter ideals. Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 3$ and $Q = (a_1, a_2, \dots, a_d)$ a parameter ideal of A. We set $\mathcal{R} = \mathcal{R}(Q) = A[Qt] \subseteq A[t]$, where t is an indeterminate. We then have the following.

Theorem 1.8.3. *The Rees algebra $\mathcal{R} = \mathcal{R}(Q)$ of Q is an almost Gorenstein graded ring if and only if $Q = \mathfrak{m}$ (and hence A is a regular local ring).*

To prove Theorem 1.8.3, we need some preliminary steps. Let $B = A[X_1, X_2, \dots, X_d]$ be the polynomial ring and let $\varphi : B \to \mathcal{R}$ be the homomorphism of A-algebras defined by $\varphi(X_i) = a_i t$ for all $1 \leq i \leq d$. Then φ preserves grading and $\mathrm{Ker}\,\varphi = I_2\left(\begin{smallmatrix} X_1 & X_2 & \cdots & X_d \\ a_1 & a_2 & \cdots & a_d \end{smallmatrix}\right)$ is a perfect ideal of B, since $\dim \mathcal{R} = d+1$. Therefore $\mathcal{R}$ is a Cohen-Macaulay ring. Because $\mathfrak{M} = \sqrt{(X_1, \{X_{i+1} - a_i\}_{1\leq i\leq d-1}, -a_d)R}$ and

$$\mathcal{R}/(X_1, \{X_{i+1} - a_i\}_{1\leq i\leq d-1}, -a_d)\mathcal{R} \cong A/[(a_1) + (a_2, a_3, \dots, a_d)^2],$$

we get $\mathrm{r}(\mathcal{R}) = \mathrm{r}(A/[(a_1) + (a_2, a_3, \dots, a_d)^2]) = d - 1 \geq 2$. Hence $\mathcal{R}$ is not a Gorenstein ring.

Let $G = \mathrm{gr}_Q(A)$ and choose the canonical Q-filtration $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$ of A which satisfies the following conditions ([23, HTZ]).

(i) $\omega_n = A$, if $n < d$ and $\omega_n = Q^{n-d}\omega_d$, if $n \geq d$.

(ii) $[\mathcal{R}(\omega)]_+ \cong \mathrm{K}_{\mathcal{R}}$ and $\mathrm{gr}_\omega(A)(-1) \cong K_G$ as graded $\mathcal{R}$−modules, where $\mathcal{R}(\omega) = \sum_{n\geq 0} \omega_n t^n$ and $\mathrm{gr}_\omega(A) = \bigoplus_{n\geq 0} \omega_n/\omega_{n+1}$.

On the other hand, since $G = (A/Q)[T_1, T_2, \ldots, T_d]$ is the polynomial ring, we get $K_G \cong G(-d)$. Therefore $\omega_{d-1}/\omega_d \cong A/Q$ by condition (ii) and hence $\omega_d = Q$, because $\omega_{d-1} = A$ by condition (i). Consequently $\omega_n = Q^{n-d+1}$ for all $n \geq d$. Therefore $\mathrm{K}_R = \sum_{n=1}^{d-2} At^n + \mathcal{R}t^{d-1}$, so that we get the exact sequence $0 \to \mathcal{R} \xrightarrow{\psi} \mathrm{K}_{\mathcal{R}}(1) \to C \to 0$ of graded $\mathcal{R}$-modules, where $\psi(1) = t$. Hence $\mathfrak{a}(\mathcal{R}) = -1$, because $C_n = (0)$, if $n \leq 0$.

Let $f = \overline{t^{d-1}}$ denote the image of t^{d-1} in C and put $D = C/\mathcal{R}f$. Then it is standard to check that $(0) :_{\mathcal{R}} C = (0) :_{\mathcal{R}} f = Q^{d-2}\mathcal{R}$. Hence $D = \sum_{n=0}^{d-3} D_n$ is a finitely graded $\mathcal{R}$-module and $\ell_A(D) < \infty$.

Let $\overline{\mathcal{R}} = \mathcal{R}/Q^{d-2}\mathcal{R}$ and look at the exact sequence

$$0 \to \overline{\mathcal{R}}(2-d) \xrightarrow{\varphi} C \to D \to 0 \qquad (\sharp)$$

of graded $\mathcal{R}$-modules, where $\varphi(1) = f$. Then since $\overline{\mathcal{R}}_0 = A/Q^{d-2}$ is an Artinian local ring, the ideal $\mathfrak{M}\overline{\mathcal{R}}$ of $\overline{\mathcal{R}}$ contains $[\overline{\mathcal{R}}]_+ = (a_1t, a_2t, \ldots, a_dt)\overline{\mathcal{R}}$ as a reduction. Therefore thanks to exact sequence ($\sharp$), we get $\mathrm{e}^0_{\mathfrak{M}}(C) = \mathrm{e}^0_{\mathfrak{M}\overline{\mathcal{R}}}(C) = \mathrm{e}^0_{[\overline{\mathcal{R}}]_+}(C) = \mathrm{e}^0_{[\overline{\mathcal{R}}]_+}(\overline{\mathcal{R}})$, because $\ell_A(D) < \infty$ but $\dim_{\overline{\mathcal{R}}} C = \dim \overline{\mathcal{R}} = d$. In order to compute $\mathrm{e}^0_{[\overline{\mathcal{R}}]_+}(\overline{\mathcal{R}})$, it suffices to see the Hilbert function $\ell_A([\overline{\mathcal{R}}]_n)$. Since

$$\ell_A([\overline{\mathcal{R}}]_n) = \ell_A(A/Q^{n+d-2}) - \ell_A(A/Q^n) = \ell_A(A/Q)\cdot \left[\binom{n+2d-3}{d} - \binom{n+d-1}{d}\right]$$

for all $n \geq 0$, we readily get $\mathrm{e}^0_{\mathfrak{M}}(C) = \mathrm{e}^0_{[\overline{\mathcal{R}}]_+}(\overline{\mathcal{R}}) = \ell_A(A/Q)(d-2)$. Summarizing the above observations, we get the following, because $\mu_{\mathcal{R}}(C) = \mathrm{r}(R) - 1 = d-2$.

Lemma 1.8.4. *$\mu_{\mathcal{R}}(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$ if and only if $\ell_A(A/Q) = 1$, i.e., $Q = \mathfrak{m}$.*

We are now ready to prove Theorem 1.8.3.

Proof of Theorem 1.8.3. If $Q = \mathfrak{m}$, then $\mu_{\mathcal{R}}(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$ by Lemma 1.8.4. Let us show the *only if* part. Since $\mathcal{R}$ is an almost Gorenstein graded ring, we get an exact sequence $0 \to R \xrightarrow{\rho} \mathrm{K}_R(1) \to X \to 0$ of graded $\mathcal{R}$-modules such that $\mu_{\mathcal{R}}(X) = \mathrm{e}^0_{\mathfrak{M}}(X)$. Let $\xi = \rho(1) \in [\mathrm{K}_{\mathcal{R}}]_1 = At$ and remember that $\xi \notin \mathfrak{M}\mathrm{K}_{\mathcal{R}}$ (Corollary 1.3.10). We then have $\xi = \varepsilon t$ for some $\varepsilon \in \mathrm{U}(A)$ and therefore $X \cong C = (\mathrm{K}_{\mathcal{R}}/\mathcal{R}t)(1)$ as a graded $\mathcal{R}$-module. Hence $Q = \mathfrak{m}$ by Lemma 1.8.4, because $\mu_R(X) = \mathrm{e}^0_{\mathfrak{M}}(X)$. $\square$

We thirdly explore the almost Gorenstein property in the polynomial extensions.

Theorem 1.8.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue class field. Let $S = R[X_1, X_2, \ldots, X_n]$ be the polynomial ring and consider S to be a $\mathbb{Z}$-graded ring such that $S_0 = R$ and $\deg X_i = 1$ for every $1 \leq i \leq n$. Then the following conditions are equivalent.*

(1) *R is an almost Gorenstein local ring.*

(2) *S is an almost Gorenstein graded ring.*

Proof. We put $\mathfrak{M} = \mathfrak{m}S + S_+$.

(2) $\Rightarrow$ (1) This follows from Theorem 1.3.9, because $S_\mathfrak{M}$ is an almost Gorenstein local ring and the fiber ring $S_\mathfrak{M}/\mathfrak{m}S_\mathfrak{M}$ is a regular local ring.

(1) $\Rightarrow$ (2) We may assume that R is not Gorenstein. Hence $d = \dim R > 0$. Choose an exact sequence

$$(\sharp_1) \quad 0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules so that $\mu_R(C) = \mathrm{e}^0_\mathfrak{m}(C)$ and consider R to be a $\mathbb{Z}$-graded ring trivially. Then, tensoring sequence $(\sharp_1)$ by S, we get the exact sequence

$$(\sharp_2) \quad 0 \to S \to S \otimes_R \mathrm{K}_R \to S \otimes_R C \to 0$$

of graded S-modules. Let $D = S \otimes_R C$. Then D is a Cohen-Macaulay graded S-module of $\dim D = \dim_R C + n = \dim S - 1$. We choose elements $f_1, f_2, \ldots, f_{d-1} \in \mathfrak{m}$ so that $\mathfrak{m}C = \mathfrak{q}C$, where $\mathfrak{q} = (f_1, f_2, \ldots, f_{d-1})$. Then

$$\mathfrak{M}D = (\mathfrak{m}S)D + S_+D = (\mathfrak{q}S)D + S_+D,$$

so that $\mu_S(D) = \mathrm{e}^0_\mathfrak{M}(D)$. Therefore, exact sequence $(\sharp_2)$ shows S to be an almost Gorenstein graded ring, because $\mathrm{a}(S) = -n$ and $S \otimes_R \mathrm{K}_R = \mathrm{K}_S(n)$. $\square$

Corollary 1.8.6. *Let $(R, \mathfrak{m})$ be an Artinian local ring and assume that the residue class field $R/\mathfrak{m}$ of R is infinite. If the polynomial ring $R[X_1, X_2, \ldots, X_n]$ is an almost Gorenstein graded ring for some $n \geq 1$, then R is a Gorenstein ring.*

The last assertion of the following result is due to S.-i. Iai [48, Theorem 1.1].

Corollary 1.8.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $d = \dim R > 0$ and infinite residue class field. Assume that R is a homomorphic image of a Gorenstein local ring.*

We choose a system $a_1, a_2, \ldots, a_d$ of parameters of R. Let $1 \leq r \leq d$ be an integer and set $\mathfrak{q} = (a_1, a_2, \ldots, a_r)$. If $\mathrm{gr}_{\mathfrak{q}}(R)$ is an almost Gorenstein graded ring, then R is an almost Gorenstein local ring. In particular, R is a Gorenstein local ring, if $r = d$ and $\mathrm{gr}_{\mathfrak{q}}(R)$ is an almost Gorenstein graded ring.

Proof. The ring R is Cohen-Macaulay, because the associated graded ring $\mathrm{gr}_{\mathfrak{q}}(R)$ of $\mathfrak{q}$ is Cohen-Macaulay. Hence $a_1, a_2, \ldots, a_r$ forms an R-regular sequence, so that $\mathrm{gr}_{\mathfrak{q}}(R) = (R/\mathfrak{q})[X_1, X_2, \ldots, X_r]$ is the polynomial ring. Therefore by Theorem 1.8.5, $R/\mathfrak{q}$ is an almost Gorenstein local ring, whence R is almost Gorenstein. Remember that if $r = d$, then R is Gorenstein by Corollary 1.8.6, because $\dim R/\mathfrak{q} = 0$. $\square$

Unfortunately, even though $R_{\mathfrak{M}}$ is an almost Gorenstein local ring, R is not necessarily an almost Gorenstein graded ring. We explore one example.

Example 1.8.8. Let $U = k[s, t]$ be the polynomial ring over a field k and look at the Cohen-Macaulay graded subring $R = k[s, s^3t, s^3t^2, s^3t^3]$ of U. Then $R_{\mathfrak{M}}$ is almost Gorenstein. In fact, let $S = k[X, Y, Z, W]$ be the weighted polynomial ring such that $\deg X = 1$, $\deg Y = 4$, $\deg Z = 5$, and $\deg W = 6$. Let $\psi : S \to R$ be the k-algebra map defined by $\psi(X) = s$, $\psi(Y) = s^3t$, $\psi(Z) = s^3t^2$, and $\psi(W) = s^3t^3$. Then $\mathrm{Ker}\,\psi = I_2\left(\begin{smallmatrix} X^3 & Y & Z \\ Y & Z & W \end{smallmatrix}\right)$ and the graded S-module R has a graded minimal free resolution

$$0 \to S(-13) \oplus S(-14) \xrightarrow{\begin{pmatrix} X^3 & Y \\ Y & Z \\ Z & W \end{pmatrix}} S(-10) \oplus S(-9) \oplus S(-8) \xrightarrow{(\Delta_1\ \Delta_2\ \Delta_3)} S \xrightarrow{\psi} R \to 0,$$

where$\Delta_1 = Z^2 - YW$, $\Delta_2 = X^3W - YZ$, and$\Delta_3 = Y^2 - X^3Z$. Therefore, because $\mathrm{K}_S \cong S(-16)$, taking the K_S-dual of the resolution, we get the presentation

$$S(-6) \oplus S(-7) \oplus S(-8) \xrightarrow{\begin{pmatrix} X^3 & Y & Z \\ Y & Z & W \end{pmatrix}} S(-3) \oplus S(-2) \xrightarrow{\varepsilon} K_R \to 0 \qquad (\sharp)$$

of the graded canonical module K_R of R. Hence $\mathfrak{a}(R) = -2$. Let $\xi = \varepsilon(\binom{1}{0}) \in [K_R]_3$ and we have the exact sequence $0 \to R \xrightarrow{\varphi} K_R(3) \to S/(Y, Z, W)(1) \to 0$ of graded R-modules, where $\varphi(1) = \xi$. Hence $R_{\mathfrak{M}}$ is a semi-Gorenstein local ring. On the other hand, thanks to presentation ($\sharp$) of K_R, we know $[\mathrm{K}_R]_2 = k\eta \neq (0)$, where $\eta = \varepsilon(\binom{0}{1})$. Hence if R is an almost Gorenstein graded ring, we must have $\mu_R(\mathrm{K}_R/R\eta) = \mathrm{e}^0_{\mathfrak{M}}(\mathrm{K}_R/R\eta)$, which is impossible, because $\mathrm{K}_R/R\eta \cong [S/(X^3, Y, Z)](-3)$.

This example 1.8.8 seems to suggest a correct definition of almost Gorenstein graded rings might be the following: *there exists an exact sequence* $0 \to R \to \mathrm{K}_R(n) \to C \to 0$ *of graded R-modules for some* $n \in \mathbb{Z}$ *such that* $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$. We would like to leave further investigations to readers as an open problem.

1.9 Almost Gorenstein associated graded rings

The purpose of this section is to explore how the almost Gorenstein property of base local rings is inherited from that of the associated graded rings. Our goal is the following.

Theorem 1.9.1. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring with infinite residue class field, possessing the canonical module* K_R. *Let* I *be an* $\mathfrak{m}$*-primary ideal of* R *and let* $\mathrm{gr}_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ *be the associated graded ring of* I. *If* $\mathrm{gr}_I(R)$ *is an almost Gorenstein graded ring with* $\mathrm{r}(\mathrm{gr}_I(R)) = \mathrm{r}(R)$, *then* R *is an almost Gorenstein local ring.*

Theorem 1.9.1 is reduced, by induction on $\dim R$, to the case where $\dim R = 1$. Let us start from the key result of dimension one. Our setting is the following.

Setting 1.9.2. Let R be a Cohen-Macaulay local ring of dimension one. We consider a filtration $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ of ideals of R. Therefore $\{I_n\}_{n\in\mathbb{Z}}$ is a family of ideals of R which satisfies the following three conditions: (1) $I_n = R$ for all $n \leq 0$ but $I_1 \neq R$, (2) $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{Z}$, and (3) $I_m I_n \subseteq I_{m+n}$ for all $m, n \in \mathbb{Z}$. Let t be an indeterminate and we set

$$\mathcal{R} = \mathcal{R}(\mathcal{F}) = \sum_{n\geq 0} I_n t^n \subseteq R[t],$$

$$\mathcal{R}' = \mathcal{R}'(\mathcal{F}) = \mathcal{R}[t^{-1}] = \sum_{n\in\mathbb{Z}} I_n t^n \subseteq R[t, t^{-1}], \quad \text{and}$$

$$G = G(\mathcal{F}) = \mathcal{R}'(\mathcal{F})/t^{-1}\mathcal{R}'(\mathcal{F}).$$

We call them respectively the Rees algebra, the extended Rees algebra, and the associated graded ring of $\mathcal{F}$. We assume the following three conditions are satisfied:

1. R is a homomorphic image of a Gorenstein ring,

2. $\mathcal{R}$ is a Noetherian ring, and

3. G is a Cohen-Macaulay ring.

Let $\mathrm{a}(G) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^1_{\mathfrak{M}}(G)]_n \neq (0)\}$ ([40]), where $\{[\mathrm{H}^1_{\mathfrak{M}}(G)]_n\}_{n\in\mathbb{Z}}$ stands for the homogeneous components of the first graded local cohomology module $\mathrm{H}^1_{\mathfrak{M}}(G)$ of G with respect to the graded maximal ideal $\mathfrak{M} = \mathfrak{m}G + G_+$ of G. We set $c = \mathrm{a}(G) + 1$ and $K = \mathrm{K}_R$. Then by [23, Theorem 1.1] we have a unique family $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$ of R-submodules of K satisfying the following four conditions:

(i) $\omega_n \supseteq \omega_{n+1}$ for all $n \in \mathbb{Z}$,

(ii) $\omega_n = K$ for all $n \leq -c$,

(iii) $I_m\omega_n \subseteq \omega_{m+n}$ for all $m, n \in \mathbb{Z}$, and

(iv) $\mathrm{K}_{\mathcal{R}'} \cong \mathcal{R}'(\omega)$ and $\mathrm{K}_G \cong G(\omega)(-1)$ as graded $\mathcal{R}'$-modules,

where $\mathcal{R}'(\omega) = \sum_{n\in\mathbb{Z}} \omega_n t^n \subset K[t, t^{-1}]$ and $G(\omega) = \mathcal{R}'(\omega)/t^{-1}\mathcal{R}'(\omega)$, and $\mathrm{K}_{\mathcal{R}'}$ and K_G denote respectively the graded canonical modules of $\mathcal{R}'$ and G. Notice that $[G(\omega)]_n = (0)$ if $n < -c$ (see condition (ii)).

With this notation we have the following.

Lemma 1.9.3. *There exist integers $d > 0$ and $k \geq 0$ such that $\omega_{dn-c} = I_d^{n-k}\omega_{dk-c}$ for all $n \geq k$.*

Proof. Let $L = \mathcal{R}(\omega)(-c)$, where $\mathcal{R}(\omega) = \sum_{n\geq 0} \omega_n t^n \subseteq K[t]$. Then L is a finitely generated graded $\mathcal{R}$-module such that $L_n = (0)$ for $n < 0$. We choose an integer $d \gg 0$ so that the Veronesean subring $\mathcal{R}^{(d)} = \sum_{n\geq 0} \mathcal{R}_{dn}$ of $\mathcal{R}$ with order d is standard, whence $\mathcal{R}^{(d)} = R[\mathcal{R}_d]$. Then, because $L^{(d)} = \sum_{n\geq 0} L_{dn}$ is a finitely generated graded $\mathcal{R}^{(d)}$-module, we may choose a homogeneous system $\{f_i\}_{1\leq i\leq \ell}$ of generators of $L^{(d)}$ so that for each $1 \leq i \leq \ell$

$$f_i \in [L^{(d)}]_{k_i} = [\mathcal{R}(\omega)]_{dk_i - c}$$

with $k_i \geq \frac{c}{d}$. Setting $k = \max\{k_i \mid 1 \leq i \leq \ell\}$, for all $n \geq k$ we get

$$\omega_{dn-c} \subseteq \sum_{i=1}^{\ell} I_{d(n-k_i)}\omega_{dk_i-c} \subseteq I_d^{n-k}\omega_{dk-c},$$

as asserted. □

Let us fix an element $f \in K$ and let $\xi = \overline{ft^{-c}} \in G(\omega)(-c)$ denote the image of $ft^{-c} \in \mathcal{R}'(\omega)$ in $G(\omega)$. Assume $(0) :_G \xi = (0)$ and consider the following short exact sequence

$$(E) \quad 0 \to G \xrightarrow{\psi} G(\omega)(-c) \to C \to 0,$$

of graded G-modules, where $\psi(1) = \xi$. Then $C_\mathfrak{p} = (0)$ for all $\mathfrak{p} \in AssG$, because $[G(\omega)]_\mathfrak{p} \cong [\mathrm{K}_G]_\mathfrak{p} \cong \mathrm{K}_{G_\mathfrak{p}}$ as $G_\mathfrak{p}$-modules by condition (iv) above and $\ell_{G_\mathfrak{p}}(G_\mathfrak{p}) = \ell_{G_\mathfrak{p}}(\mathrm{K}_{G_\mathfrak{p}})$ ([43, Korollar 6.4]). Therefore $\ell_R(C) = \ell_G(C) < \infty$ since $\dim G = 1$, so that C is finitely graded. We now consider the exact sequence

$$\mathcal{R}' \xrightarrow{\varphi} \mathcal{R}'(\omega)(-c) \to D \to 0$$

of graded $\mathcal{R}'$-modules defined by $\varphi(1) = ft^{-c}$. Then $C \cong D/uD$ as a G-module, where $u = t^{-1}$. Notice that $\dim \mathcal{R}'/\mathfrak{p} = 2$ for all $\mathfrak{p} \in \operatorname{Ass} \mathcal{R}'$, because $\mathcal{R}'$ is a Cohen-Macaulay ring of dimension 2. We then have $D_\mathfrak{p} = (0)$ for all $\mathfrak{p} \in \operatorname{Ass} \mathcal{R}'$, since $\dim_{\mathcal{R}'} D \leq 1$. Hence the homomorphism φ is injective, because $\mathcal{R}'(\omega) \cong \mathrm{K}_{\mathcal{R}'}$ by condition (iv) and $\ell_{\mathcal{R}'_\mathfrak{p}}([\mathcal{R}'(\omega)]_\mathfrak{p}) = \ell_{\mathcal{R}'\mathfrak{p}}([\mathrm{K}_{\mathcal{R}'}]_\mathfrak{p}) = \ell_{\mathcal{R}'\mathfrak{p}}(\mathrm{K}_{\mathcal{R}'_\mathfrak{p}}) = \ell_{\mathcal{R}'_\mathfrak{p}}(\mathcal{R}'_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass} \mathcal{R}'$. The snake lemma shows u acts on D as a non-zerodivisor, since u acts on $\mathcal{R}'(\omega)$ as a non-zerodivisor.

Let us suppose that $C \neq (0)$ and set $S = \{n \in \mathbb{Z} \mid C_n \neq (0)\}$. We write $S = \{n_1 < n_2 < \cdots < n_\ell\}$, where $\ell = \sharp S > 0$. We then have the following.

Lemma 1.9.4. *$D_n = (0)$ if $n > n_\ell$ and $D_n \cong K/Rf$ if $n \leq 0$. Consequently, $\ell_R(K/Rf) = \ell_R(C)$.*

Proof. Let $n > n_\ell$. Then $C_n = (0)$. By exact sequence (E) above, we get $I_n/I_{n+1} \cong \omega_{n-c}/\omega_{n+1-c}$, whence $\omega_{n-c} = I_n f + \omega_{n+1-c}$. Therefore $\omega_n - c \subseteq I_n f + \omega_q$ for all $q \in \mathbb{Z}$. By Lemma 1.9.3 we may choose integers $d \gg 0$ and $k \geq 0$ so that

$$\omega_{n-c} \subseteq I_n f + \omega_{dm-c} \subseteq I_n f + I_d^{m-k} f$$

for all $m \geq k$. Consequently, $\omega_{n-c} = I_n f$. Hence $D_n = (0)$ for all $n \geq n_\ell$. If $n \leq 0$, then $D_n \cong [\mathcal{R}'(\omega)(-c)]_n/\mathcal{R}'_n f \cong K/Rf$ (see condition (ii) above). To see the last assertion, notice that because $S = \{n_1 < n_2 < \cdots < n_\ell\}$, $D_n = u^{n-n_1} D_{n_1} \cong D_{n_1}$ if $n \leq n_1$ and $D_n = u^{n_{i+1}-n} D_{n_{i+1}} \cong D_{n_{i+1}}$ if $1 \leq i < \ell$ and $n_i < n \leq n_{i+1}$. Therefore since $D_n = (0)$ for $n > n_\ell$, we get

$$\ell_R(K/Rf) = \ell_R(D_0) = \ell_R(D_{n_1}) = \sum_{i=1}^{\ell} \ell_R(C_{n_i}) = \ell_R(C).$$

$\square$

Exact sequence (E) above now shows the following estimations. Remember that $\mathrm{r}(R) \le \mathrm{r}(G)$, because $\mathrm{K}_G[t] = K[t, t^{-1}]$ so that $\mu_R(K) \le \mu_G(\mathrm{K}_G)$.

Proposition 1.9.5. $\mathrm{r}(R) - 1 \le \mathrm{r}(G) - 1 \le \mu_G(C) \le \ell_G(C) = \ell_R(C) = \ell_R(K/Rf)$.

We are now back to a general situation of Setting 1.9.2.

Theorem 1.9.6. *Let G be as in Setting 2 and assume that G is an almost Gorenstein graded ring with $\mathrm{r}(G) = \mathrm{r}(R)$. Then R is an almost Gorenstein local ring.*

Proof. We may assume that G is not a Gorenstein ring. We choose an exact sequence

$$0 \to G \xrightarrow{\psi} G(\omega)(-c) \to C \to 0$$

of graded G-modules so that $C \neq (0)$ and $\mathfrak{M}C = (0)$. Then $\mu_G(C) = \ell_G(C)$. We set $\xi = \psi(1)$ and write $\xi = \overline{ft^{-c}}$ with $f \in K$. Hence $(0) :_G \xi = (0)$. We now look at the estimations stated in Proposition 1.9.5. If $\mathrm{r}(R) - 1 = \ell_G(C)$, then $\ell_R(K/Rf) = \mu_R(K/Rf)$ because $\mathrm{r}(R) - 1 = \mu_R(K) - 1 \le \mu_R(K/Rf) \le \ell_R(K/Rf) = \ell_G(C)$, so that $\mathfrak{m}{\cdot}(K/Rf) = (0)$. Consequently, we get the exact sequence

$$0 \to R \xrightarrow{\varphi} K \to K/Rf \to 0$$

of R-modules with $\varphi(1) = f$, whence R is an almost Gorenstein local ring. If $\mathrm{r}(R) - 1 < \ell_G(C)$, then $\psi(1) \in \mathfrak{M}{\cdot}[G(\omega)(-c)]$ because $\mathrm{r}(G) - 1 < \mu_G(C)$, so that $G_{\mathfrak{M}}$ is a discrete valuation ring. This is impossible, since G is not a Gorenstein ring. $\square$

The converse of Theorem 1.9.6 is also true when G satisfies some additional conditions. To see this, we need the following. Recall that our graded ring G is said to be level, if $\mathrm{K}_G = G{\cdot}[\mathrm{K}_G]_{-a}$, where $a = \mathfrak{a}(G)$. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R.

Lemma 1.9.7. *Suppose that $\mathrm{Q}(\widehat{R})$ is a Gorenstein ring and the field $R/\mathfrak{m}$ is infinite. Let us choose a canonical ideal K of R so that $R \subseteq K \subseteq \overline{R}$. Let $a \in \mathfrak{m}$ be a regular element of R such that $I = aK \subsetneq R$. We then have the following.*

(1) *Suppose that G is an integral domain. Then there is an element $f \in K \setminus \omega_{1-c}$ so that $af \in I$ generates a minimal reduction of I. Hence $(0) :_G \xi = (0)$, where $\xi = \overline{ft^{-c}} \in G(\omega)(-c)$.*

(2) *Suppose that* $\mathrm{Q}(G)$ *is a Gorenstein ring and* G *is a level ring. Then there is an element* $f \in K$ *such that* $af \in I$ *generates a reduction of* I *and* $G_\mathfrak{p}{\cdot}\frac{\xi}{1} = [G(\omega)(-c)]_\mathfrak{p} \cong G_\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Ass} G$*, where* $\xi = \overline{ft^{-c}} \in G(\omega)(-c)$*. Hence* $(0) :_G \xi = (0)$*.*

Proof. (1) Let $L = \omega_{1-c}$. Then $aL \subsetneq aK = I$, since $L \subsetneq K$. We write $I = (x_1, x_2, \ldots, x_n)$ such that each x_i generates a minimal reduction of I. Choose $f = x_i$ so that $x_i \notin L$, which is the required one.

(2) Let $M = G(\omega)(-c)$. Then since $M = G{\cdot}M_0$ and $M_\mathfrak{p} \neq (0)$, $M_0 \nsubseteq \mathfrak{p}M_\mathfrak{p} \cap M$ for any $\mathfrak{p} \in \operatorname{Ass} G$. Choose an element $f \in K$ so that af generates a reduction of I and $\xi = \overline{ft^{-c}} \notin \mathfrak{p}M_\mathfrak{p} \cap M$ for any $\mathfrak{p} \in \operatorname{Ass} G$. Then $G_\mathfrak{p}{\cdot}\frac{\xi}{1} = M_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} G$, because $M_\mathfrak{p} \cong G_\mathfrak{p}$. □

Theorem 1.9.8. *Suppose that* R *is an almost Gorenstein local ring and the field* $R/\mathfrak{m}$ *is infinite. Assume that one of the following conditions is satisfied*:

(1) G *is an integral domain;*

(2) $\mathrm{Q}(G)$ *is a Gorenstein ring and* G *is a level ring.*

Then G *is an almost Gorenstein graded ring with* $\mathrm{r}(G) = \mathrm{r}(R)$*.*

Proof. The ring $\mathrm{Q}(\widehat{R})$ is Gorenstein, since R is an almost Gorenstein local ring. Let K be a canonical ideal of R such that $R \subseteq K \subseteq \overline{R}$. We choose an element $f \in K$ and $a \in \mathfrak{m}$ as in Lemma 1.9.7. Then $\mu_R(K/Rf) = \mathrm{r}(R) - 1$, since f is a part of a minimal system of generators of K (recall that af generates a minimal reduction of $I = aK$). Therefore by Proposition 1.9.5, $\mathrm{r}(G) = \mathrm{r}(R)$ and $\mathfrak{M}{\cdot}C = (0)$, whence G is an almost Gorenstein graded ring. □

Suppose that $(R, \mathfrak{m})$ is a complete local domain of dimension one and let $V = \overline{R}$. Hence V is a discrete valuation ring. Let $\mathfrak{n}$ be the maximal ideal of V and set $I_n = \mathfrak{n}^n \cap R$ for each $n \in \mathbb{Z}$. Then $\mathcal{F} = \{I_n\}_{n \in \mathbb{Z}}$ is a filtration of ideals of R. We have $I_1 = \mathfrak{m}$ and $G(\mathcal{F})$ ($\subseteq \mathrm{gr}_\mathfrak{n}(V) = \bigoplus_{n \geq 0} \mathfrak{n}^n/\mathfrak{n}^{n+1}$) is an integral domain. The ring $\mathcal{R}(\mathcal{F})$ is Noetherian, since $\mathfrak{n}^n \subseteq R$ for all $n \gg 0$. Therefore, applying Theorems 1.9.6 and 1.9.8 to this setting, we readily get the following, where the implication (2) $\Rightarrow$ (1) is not true in general, unless $\mathrm{r}(G) = \mathrm{r}(R)$ (see Example 1.9.13).

Corollary 1.9.9 (cf. [8, Proposition 29])**.** *Let R and $\mathcal{F}$ be as above and consider the following conditions.*

(1) *R is an almost Gorenstein local ring;*

(2) *G is an almost Gorenstein graded ring and $\mathrm{r}(G) = \mathrm{r}(R)$.*

Then the implication (2) $\Rightarrow$ (1) holds true. If the field $R/\mathfrak{m}$ is infinite, the converse is also true.

To prove Theorem 1.9.1, we need one more result.

Proposition 1.9.10. *Let $G = G_0[G_1]$ be a Noetherian standard graded ring. Assume that G_0 is an Artinian local ring with infinite residue class field. If G is an almost Gorenstein graded ring with $\dim G \geq 2$, then $G/(x)$ is an almost Gorenstein graded ring for some non-zerodivisor $x \in G_1$.*

Proof. We may assume that G is not a Gorenstein ring. Let $\mathfrak{m}$ be the maximal ideal of G_0 and set $\mathfrak{M} = \mathfrak{m}G + G_+$. We consider the sequence

$$0 \to G \to \mathrm{K}_G(-a) \to C \to 0$$

of graded G-modules such that $\mu_G(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $a = \mathrm{a}(G)$ is the a-invariant of G. Then because the field $G_0/\mathfrak{m}$ is infinite and the ideal $G_+ = (G_1)$ of G is a reduction of $\mathfrak{M}$, we may choose an element $x \in G_1$ so that x is G-regular and superficial for C with respect to $\mathfrak{M}$. We set $\overline{G} = G/(x)$ and remember that x is C-regular, as $\dim_G C = \dim G - 1 > 0$. We then have the exact sequence

$$0 \to G/(x) \to (\mathrm{K}_G/x\mathrm{K}_G)(-a) \to C/xC \to 0$$

of graded $\overline{G}$-modules. We now notice that $\mathrm{a}(\overline{G}) = a + 1$ and that

$$(\mathrm{K}_G/x\mathrm{K}_G)(-a) \cong \mathrm{K}_{\overline{G}}(-(a+1))$$

as a graded $\overline{G}$-module, while we see

$$\mathrm{e}^0_{\mathfrak{M}/(x)}(C/xC) = \mathrm{e}^0_{\mathfrak{M}}(C) = \mu_G(C) = \mu_G(C/xC),$$

since x is superficial for C with respect to $\mathfrak{M}$. Thus $\overline{G}$ is an almost Gorenstein graded ring. $\square$

We are now ready to prove Theorem 1.9.1.

Proof of Theorem 1.9.1. We set $d = \dim R$ and $G = \mathrm{gr}_I(R)$. We may assume that G is not a Gorenstein ring. Hence $d = \dim G \geq 1$. By Theorem 1.9.6 we may also assume that $d > 1$ and that our assertion holds true for $d-1$. Let us consider an exact sequence

$$0 \to G \to \mathrm{K}_G(-a) \to C \to 0$$

of graded G-modules with $\mu_G(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $\mathfrak{M} = \mathfrak{m}G + G_+$ and $a = \mathrm{a}(G)$. We choose an element $a \in I$ so that the initial form $a^* = a + I^2 \in G_1 = I/I^2$ of a is G-regular and $G/a^*G = \mathrm{gr}_{I/(a)}(R/(a))$ is an almost Gorenstein graded ring (this choice is possible; see Proposition 1.9.10). Then the hypothesis on d shows $R/(a)$ is an almost Gorenstein local ring. Therefore R is an almost Gorenstein local ring, because a is R-regular. $\square$

In general, the local rings $R_{\mathfrak{p}}$ ($\mathfrak{p} \in \operatorname{Spec} R$) of an almost Gorenstein local ring R are not necessarily almost Gorenstein, as we will show by Example 1.9.13. To do this, we assume that R is a Cohen-Macaulay local ring of dimension $d \geq 0$, possessing the canonical module K_R. Let $\mathcal{F} = \{I_n\}_{n \in \mathbb{Z}}$ be a filtration of ideals of R such that $I_0 = R$ but $I_1 \neq R$. Smilarly as Setting 2, we consider the R-algebras

$$\mathcal{R} = \sum_{n \geq 0} I_n t^n \subseteq R[t], \quad \mathcal{R}' = \sum_{n \in \mathbb{Z}} I_n t^n \subseteq R[t, t^{-1}], \quad \text{and} \quad G = \mathcal{R}'/t^{-1}\mathcal{R}'$$

associated to $\mathcal{F}$, where t is an indeterminate. Notice that $\mathcal{R}' = \mathcal{R}[t^{-1}]$ and that $G = \oplus_{n \geq 0} I_n/I_{n+1}$. Let $\mathfrak{N}$ denote the graded maximal ideal of $\mathcal{R}'$. We then have the following.

Theorem 1.9.11. *Suppose that $R/\mathfrak{m}$ is infinite and that $\mathcal{R}$ is a Noetherian ring. If $G_{\mathfrak{N}}$ is a pseudo-Gorenstein local ring, then R is pseudo-Gorenstein.*

Proof. By Theorem 1.3.7 (1) ${\mathcal{R}'}_{\mathfrak{N}}$ is an almost Gorenstein ring with $\mathrm{r}(G_{\mathfrak{N}}) = \mathrm{r}({\mathcal{R}'}_{\mathfrak{N}}) \leq 2$, as $G_{\mathfrak{N}} = {\mathcal{R}'}_{\mathfrak{N}}/t^{-1}{\mathcal{R}'}_{\mathfrak{N}}$ and t^{-1} is $\mathcal{R}'$-regular. Let $\mathfrak{p} = \mathfrak{m}{\cdot}R[t, t^{-1}]$ and set $P = \mathfrak{p} \cap \mathcal{R}'$. Then $P \subseteq \mathfrak{N}$, so that by Proposition 1.7.2 $R[t, t^{-1}]_{\mathfrak{p}}$ is an almost Gorenstein ring, because $R[t, t^{-1}]_{\mathfrak{p}} = {\mathcal{R}'}_P = ({\mathcal{R}'}_{\mathfrak{N}})_{P{\mathcal{R}'}_{\mathfrak{N}}}$. Thus by Theorem 1.3.9 R is an almost Gorenstein ring with $\mathrm{r}(R) \leq 2$, because $R/\mathfrak{m}$ is infinite and the composite homomorphism $R \to R[t, t^{-1}] \to R[t, t^{-1}]_{\mathfrak{p}}$ is flat. $\square$

Remark 1.9.12. The following example 1.9.13 is given by V. Barucci, D. E. Dobbs, and M. Fontana [7, Example II.1.19]. Because $R[t,t^{-1}]_{\mathfrak{p}} = \mathcal{R}'_P = (\mathcal{R}'_{\mathfrak{N}})_{P\mathcal{R}'_{\mathfrak{N}}}$ with the notation of the proof of Theorem 1.9.11, Theorem 1.3.9 and Example 1.9.13 show that $(\mathcal{R}'_{\mathfrak{N}})_{P\mathcal{R}'_{\mathfrak{N}}}$ is *not* an almost Gorenstein local ring (here we assume the field k is infinite). Hence, in general, local rings $R_{\mathfrak{p}}$ ($\mathfrak{p} \in \operatorname{Spec} R$) of an almost Gorenstein local ring R are not necessarily almost Gorenstein. Notice that $\mathcal{R}'_{\mathfrak{N}}$ is *not* a semi-Gorenstein local ring (remember that the local rings of a semi-Gorenstein local ring are semi-Gorenstein; see Proposition 1.7.2). Therefore the example also shows that a local ring R is *not necessarily* semi-Gorenstein, even if $R/(f)$ is a semi-Gorenstein ring for some non-zerodivisor f of R.

Example 1.9.13. Let k be a field with $\operatorname{ch} k \neq 2$ and let $R = k[[x^4, x^6 + x^7, x^{10}]] \subseteq V$, where $V = k[[x]]$ denotes the formal power series ring over k. Then $V = \overline{R}$. Let v denote the discrete valuation of V and set $H = \{v(a) \mid 0 \neq a \in R\}$, the value semigroup of R. We consider the filtration $\mathcal{F} = \{(xV)^n \cap R\}_{n \in \mathbb{Z}}$ of ideals of R and set $G = \mathcal{R}'/t^{-1}\mathcal{R}'$, where $\mathcal{R}' = \mathcal{R}'(\mathcal{F})$ is the extended Rees algebra of $\mathcal{F}$. We then have:

(1) $H = \langle 4, 6, 11, 13 \rangle$.

(2) $G \cong k[x^4, x^6, x^{11}, x^{13}]$ ($\subseteq k[x]$) as a graded k-algebra and $G_{\mathfrak{N}}$ is an almost Gorenstein local ring with $\mathrm{r}(G_{\mathfrak{N}}) = 3$, where $\mathfrak{N}$ is the graded maximal ideal of $\mathcal{R}'$.

(3) R is not an almost Gorenstein local ring and $\mathrm{r}(R) = 2$.

1.10 Almost Gorenstein homogeneous rings

In this section let $R = k[R_1]$ be a Cohen-Macaulay homogeneous ring over an infinite field $k = R_0$. We assume $d = \dim R > 0$. Let $\mathfrak{M} = R_+$ and $a = \mathfrak{a}(R)$. For each finitely generated graded R-module X, let $[\![\mathrm{X}]\!] = \sum_{n=0}^{\infty} \dim_k X_n {\cdot} \lambda^n \in \mathbb{Z}[\lambda]$ be the Hilbert series of X, where X_n ($n \in \mathbb{Z}$) denotes the homogeneous component of X with degree n. Then as it is well-known, writing $[\![R]\!] = \frac{F(\lambda)}{(1-\lambda)^d}$ with $F(\lambda) \in \mathbb{Z}[\lambda]$, we have $\deg F(\lambda) = a + d \geq 0$ and $[\![\mathrm{K}_R(-a)]\!] = \frac{F(\frac{1}{\lambda}) \cdot \lambda^{a+d}}{(1-\lambda)^d}$. Let $f_1, f_2, \ldots, f_d$ be a linear system of parameters of R. Then, because $\mathfrak{a}(R/(f_1, f_2, \ldots, f_d)) = a + d$, we have $a = 1 - d$ if and only if $\mathfrak{M}^2 = (f_1, f_2, \ldots, f_d)\mathfrak{M}$ and $\mathfrak{M} \neq (f_1, f_2, \ldots, f_d)$. Conversely, we get the following. Remember that the graded ring R is said to be level, if $\mathrm{K}_R = R{\cdot}[\mathrm{K}_R]_{-a}$.

Proposition 1.10.1. *Suppose that $a = 1 - d$. Then R is a level ring with $[\![R]\!] = \frac{1+c\lambda}{(1-\lambda)^d}$ and $[\![\mathrm{K}_R(-a)]\!] = \frac{c+\lambda}{(1-\lambda)^d}$, where $c = \dim_k R_1 - d$.*

The following lemma shows that the converse of Proposition 1.10.1 is also true, if R is an almost Gorenstein graded ring.

Lemma 1.10.2. *Suppose that R is an almost Gorenstein graded ring and assume that R is not a Gorenstein ring. If R is a level ring, then $a = 1 - d$.*

Proof. Suppose that R is a level ring and take an exact sequence $0 \to R \to \mathrm{K}_R(-a) \to C \to 0$ of graded R-modules so that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$. Then $C \neq (0)$ and hence C is a Cohen-Macaulay R-module of dimension $d-1$ (Lemma 1.3.1 (2)). We have $C = RC_0$ and $\mu_R(C) = \mathrm{r}(R) - 1$ (Corollary 1.3.10). Remember that $\mathfrak{M}C = (f_2, f_3, \ldots, f_d)C$ for some $f_2, f_3, \ldots, f_d \in R_1$ (see Proposition 1.2.2 (2)) and we have $[\![C]\!] = \frac{r-1}{(1-\lambda)^{d-1}}$, where $r = \mathrm{r}(R)$. Consequently, $[\![\mathrm{K}_R(-a)]\!] - [\![R]\!] = \frac{r-1}{(1-\lambda)^{d-1}}$, so that $F(\frac{1}{\lambda}) \cdot \lambda^{a+d} - F(\lambda) = (r-1)(1-\lambda)$. Let us write $F(\lambda) = \sum_{i=0}^{a+d} c_i \lambda^i$ with $c_i \in \mathbb{Z}$. Then the equality $\sum_{i=0}^{a+d} c_{a+d-i}\lambda^i = \sum_{i=0}^{a+d} c_i\lambda^i + (r-1)(1-\lambda)$ forces that if $a + d \geq 2$, then $c_0\lambda^{a+d} = c_{a+d}\lambda^{a+d}$ and $c_{a+d} = c_0 + (r-1)$, which is impossible, since $c_0 = 1$ and $r > 1$. Therefore we have $a + d = 1$, because $0 \leq a + d$ and R is not a Gorenstein ring. $\square$

The following is a key in our argument.

Lemma 1.10.3. *Suppose that R is a level ring and $\mathrm{Q}(R)$ is a Gorenstein ring. Then there exists an exact sequence $0 \to R \to \mathrm{K}_R(-a) \to C \to 0$ of graded R-modules with $\dim_R C < d$.*

Proof. Let $V = [\mathrm{K}_R]_{-a}$. Then $\mathrm{K}_R = RV$. For each $\mathfrak{p} \in \operatorname{Ass} R$, let $L(\mathfrak{p})$ be the kernel of the composite of two canonical homomorphism $h(\mathfrak{p}) : \mathrm{K}_R \to (\mathrm{K}_R)_\mathfrak{p} \to (\mathrm{K}_R)_\mathfrak{p}/\mathfrak{p}(\mathrm{K}_R)_\mathfrak{p}$. Then $V \not\subseteq L(\mathfrak{p})$, because $\mathrm{K}_R = RV$ and $(\mathrm{K}_R)_\mathfrak{p} \cong R_\mathfrak{p}$. Let us choose $\xi \in V \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass} R} L(\mathfrak{p})$ and let $\varphi : R \to \mathrm{K}_R(-a)$ be the homomorphism of graded R-modules defined by $\varphi(1) = \xi$. Look now at the exact sequence $R \xrightarrow{\varphi} \mathrm{K}_R(-a) \to C \to 0$ of graded R-modules. Then, because $(\mathrm{K}_R)_\mathfrak{p} = R_\mathfrak{p}\frac{\xi}{1}$ for all $\mathfrak{p} \in \operatorname{Ass} R$ (remember that $(\mathrm{K}_R)_\mathfrak{p} \cong R_\mathfrak{p}$), we have $C_\mathfrak{p} = (0)$ for all $\mathfrak{p} \in \operatorname{Ass} R$. Hence $\dim_R C < d$ and therefore φ is injective (see Lemma 1.3.1 (1)). $\square$

We now come to the main result of this section.

Theorem 1.10.4. *Suppose that* $\mathrm{Q}(R)$ *is a Gorenstein ring. If* $a = 1 - d$*, then* R *is an almost Gorenstein graded ring.*

Proof. We may assume that R is not a Gorenstein ring. Thanks to Lemma 1.10.3, we can choose an exact sequence $0 \to R \to \mathrm{K}_R(-a) \to C \to 0$ of graded R-modules so that C is a Cohen-Macaulay R-module of dimension $d-1$ (see Lemma 1.3.1 (2) also). Then Proposition 1.10.1 implies $[\![C]\!] = [\![\mathrm{K}_R(-a)]\!] - [\![R]\!] = \frac{c-1}{(1-\lambda)^{d-1}}$, where $c = \dim_k R_1 - d$. Let $f_1, f_2, \dots, f_{d-1}$ ($\in R_1$) be a linear system of parameters for C. Then because $[\![C/(f_2, \dots, f_d)C]\!] = (1-\lambda)^{d-1}[\![C]\!] = c - 1$, we get $C/(f_1, f_2, \dots, f_{d-1})C = [C/(f_1, f_2, \dots, f_{d-1})C]_0$, which shows that $\mathfrak{M}C = (f_1, f_2, \dots, f_{d-1})C$. Thus $\mu_R C = \mathrm{e}^0_{\mathfrak{M}}(C)$ and hence R is an almost Gorenstein graded ring. □

Example 1.10.5. Let $S = k[X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n]$ ($n \geq 2$) be the polynomial ring over an infinite field k and let $R = S/\mathrm{I}_2\left(\begin{smallmatrix} X_1 & X_2 & \cdots & X_n \\ Y_1 & Y_2 & \cdots & Y_n \end{smallmatrix}\right)$. Then R is a Cohen-Macaulay normal ring with $\dim R = n+1$. We have $\mathfrak{a}(R) = 1 - \dim R$, because $\mathfrak{M}^2 = (X_1, \{X_{i+1} - Y_i\}_{1 \leq i \leq n-1}, Y_n)\mathfrak{M}$. Hence by Theorem 1.10.4, R is an almost Gorenstein graded ring.

We explore almost Gorenstein Veronesean subrings.

Corollary 1.10.6. *Suppose that* $d = 2$*,* R *is reduced, and* $\mathfrak{a}(R) < 0$*. Then the Veronesean subrings* $R^{(n)} = k[R_n]$ *of* R *are almost Gorenstein graded rings for all* $n \geq 1$*.*

Proof. Let $S = R^{(n)}$. Then $S = k[S_1]$ is a Cohen-Macaulay reduced ring and $\dim S = 2$. Since $\mathrm{H}^2_{\mathfrak{N}}(S) = [\mathrm{H}^2_{\mathfrak{M}}(R)]^{(n)}$ (here $\mathfrak{N} = S_+$), we get $\mathfrak{a}(S) < 0$. Hence it suffices to show that R is an almost Gorenstein graded ring, which readily follows from Theorem 1.10.5, because $\mathrm{Q}(R)$ is a Gorenstein ring and $\mathfrak{a}(R) \leq 1 - \dim R$ (remember that R is the polynomial ring, if $\mathfrak{a}(R) = -\dim R$). □

Let us explore a few concrete examples.

Example 1.10.7. Let $R = k[X, Y, Z]/(Z^2 - XY)$, where $k[X, Y, Z]$ is the polynomial ring over an infinite field k. Then $R^{(n)}$ is an almost Gorenstein graded ring for all $n \geq 1$

Proof. The assertion follows from Corollary 1.10.6, since R is normal with $\dim R = 2$ and $\mathrm{a}(R) = -1$. □

Example 1.10.8. Let $R = k[X_1, X_2, \dots, X_d]$ ($d \geq 1$) be the polynomial ring over an infinite field k. Let $n \geq 1$ be an integer and look at the Veronesean subring $R^{(n)} = k[R_n]$ of R. Then the following hold.

(1) $R^{(n)}$ is an almost Gorenstein graded ring, if $d \leq 2$.

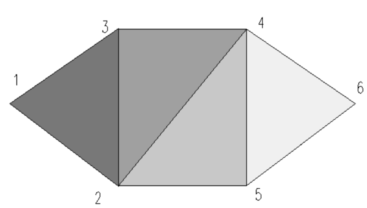
(2) Suppose that $d \geq 3$. Then $R^{(n)}$ is an almost Gorenstein graded ring if and only if either $n \mid d$, or $d = 3$ and $n = 2$.

Proof. Assertion (1) follows from Corollary 1.10.6. Suppose that $d \geq 3$ and consider assertion (2). The ring $R^{(n)}$ is a Gorenstein ring if and only if $n \mid d$ ([53]). Assume that $n \nmid d$ and put $S = R^{(n)}$. If $d = 3$ and $n = 2$, then $S = k[X_1^2, X_2^2, X_3^2, X_1X_2, X_2X_3, X_3X_1]$ with $[S_+]^2 = (X_1^2, X_2^2, X_3^2)S_+$. Hence S is an almost Gorenstein graded ring by Theorem 1.10.4. Conversely, suppose that S is an almost Gorenstein graded ring. Let $L = \{(\alpha_1, \alpha_2, \dots, \alpha_d) \mid 0 \leq \alpha_i \in \mathbb{Z}\}$. We put $|\alpha| = \sum_{i=1}^d \alpha_i$ and $X^\alpha = \prod_{i=1}^d X_i^{\alpha_i}$ for each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in L$. Let $s = \min\{s \in \mathbb{Z} \mid sn \geq q\}$ where $q = (n-1)(d-1)$ and put $\mathfrak{a} = (X_1^n, X_2^n, \dots, X_d^n)$. We then have

$$\mathfrak{a} : S_+ = \mathfrak{a} + (X^\alpha X^\beta \mid \alpha, \beta \in L \text{ such that } |\alpha| = q, \ |\beta| = sn - q),$$

which shows the homogeneous Cohen-Macaulay ring S is a level ring with $\mathfrak{a}(S) = 1 - d$. Therefore, because $\mathrm{H}_{S_+}^d(S) \cong [\mathrm{H}_{R_+}^d(R)]^{(n)}$ and because $[\mathrm{H}_{R_+}^d(R)]_i \neq (0)$ if and only if $i \leq -d$, we have $-n(d-1) < -d < -n(d-2)$, which forces $n = 2$ and $d = 3$, because $n \geq 2$ (remember that $n \nmid d$). $\square$

Example 1.10.9. Let $n \geq 1$ be an integer and let Δ be a simplicial complex with vertex set $[n] = \{1, 2, \dots, n\}$. Let $R = k[\Delta]$ denote the Stanley-Reisner ring of Δ over an infinite field k. Then $\mathrm{e}_{R_+}^0(R)$ is equal to the number of facets of Δ. If R is Cohen-Macaulay and $n = \mathrm{e}_{R_+}^0(R) + \dim R - 1$, then R is an almost Gorenstein graded ring. For example, look at the simplicial complex Δ:



with $n = 6$. Then $R = k[\Delta]$ is an almost Gorenstein graded ring of dimension 3. Note that R_P is not a Gorenstein local ring for $P = (X_1, X_3, X_4, X_5, X_6)R$.

1.11 Two-dimensional rational singularities are almost Gorenstein

Throughout this section let $(R, \mathfrak{m})$ denote a Cohen-Macaulay local ring of dimension $d \geq 0$, admitting the canonical module K_R. We assume that $R/\mathfrak{m}$ is infinite. Let $\mathrm{v}(R) = \mu_R(\mathfrak{m})$ and $\mathrm{e}(R) = \mathrm{e}^0_{\mathfrak{m}}(R)$. We set $G = \mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ be the associated graded ring of $\mathfrak{m}$ and put $\mathfrak{M} = G_+$. The purpose of this section is to study the question of when G is an almost Gorenstein graded ring, in connection with the almost Gorensteinness of the base local ring R. Remember that, thanks to [62], $\mathrm{v}(R) = \mathrm{e}(R) + d - 1$ if and only if $\mathfrak{m}^2 = Q\mathfrak{m}$ for some (and hence any) minimal reduction Q of $\mathfrak{m}$. When this is the case, G is a Cohen-Macaulay ring and $\mathrm{a}(G) = 1 - d$, provided R is not a regular local ring.

Our result is stated as follows.

Theorem 1.11.1. *The following assertions hold true.*

(1) *Suppose that R is an almost Gorenstein local ring with* $\mathrm{v}(R) = \mathrm{e}(R) + d - 1$. *Then G is an almost Gorenstein level graded ring.*

(2) *Suppose that G is an almost Gorenstein level graded ring. Then R is an almost Gorenstein local ring.*

Proof. (1) If R is a Gorenstein ring, then $\mathrm{e}(R) \leq 2$ and R is an abstract hypersurface, since $\mathfrak{m}^2 = Q\mathfrak{m}$ for some minimal reduction Q of $\mathfrak{m}$. Therefore G is also a hypersurface and hence a Gorenstein ring.

Assume that R is not a Gorenstein local ring. Hence $d > 0$ and $\mathrm{a}(G) = 1 - d$. We show that G is an almost Gorenstein graded ring by induction on d. First we consider the case $d = 1$. Let $\overline{R}$ denote the integral closure of R in $\mathrm{Q}(R)$. Choose an R-submodule K of $\overline{R}$ so that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ as an R-module (this choice is possible; see [26, Corollary 2.8]). We have $\mathfrak{m}K \subseteq R$ by [26, Theorem 3.11] as R is an almost Gorenstein local ring. Hence $\mathfrak{m}K = \mathfrak{m}$ (see Corollary 1.3.10), and $\mathfrak{m}^n K = \mathfrak{m}^n$ for all $n \geq 1$. Let $C = K/R$ and consider the $\mathfrak{m}$-adic filtrations of R, K, and C. We then have the exact sequence

$$0 \to G \to \mathrm{gr}_{\mathfrak{m}}(K) \to \mathrm{gr}_{\mathfrak{m}}(C) \to 0 \qquad (\sharp)$$

of graded G-modules induced from the canonical exact sequence $0 \to R \to K \to C \to 0$ of filtered R-modules. Notice that $\mathrm{gr}_{\mathfrak{m}}(C) = [\mathrm{gr}_{\mathfrak{m}}(C)]_0$, since $\mathfrak{m}C = (0)$. Thanks to exact sequence ($\sharp$) above, the following claim yields that G is an almost Gorenstein graded ring.

Claim 1.11.2. *$\mathrm{gr}_{\mathfrak{m}}(K) \cong \mathrm{K}_G$ as a graded G-module.*

Proof of Claim 1.11.2. Since $\mathrm{a}(G) = 0$ and G is a graded submodule of $\mathrm{gr}_{\mathfrak{m}}(K)$, it suffices to show that $\mathrm{depth}_G \,\mathrm{gr}_{\mathfrak{m}}(K) > 0$ and $\mathrm{r}_G(\mathrm{gr}_{\mathfrak{m}}(K)) = 1$. Choose $a \in \mathfrak{m}$ so that $\mathfrak{m}^2 = a\mathfrak{m}$ and let $f = \overline{a}$ ($\in \mathfrak{m}/\mathfrak{m}^2$) denotes the image of a in G. Let $x \in \mathfrak{m}^n K$ ($n \geq 0$) and assume that $ax \in \mathfrak{m}^{n+2}K = \mathfrak{m}^{n+2}$. Then, since $\mathfrak{m}^{n+2} = a\mathfrak{m}^{n+1}$, we readily get $x \in \mathfrak{m}^{n+1} = \mathfrak{m}^{n+1}K$. Thus f is $\mathrm{gr}_{\mathfrak{m}}(K)$-regular, $[\mathrm{gr}_{\mathfrak{m}}(K)/f\mathrm{gr}_{\mathfrak{m}}(K)]_0 = K/\mathfrak{m}$, $[\mathrm{gr}_{\mathfrak{m}}(K)/f\mathrm{gr}_{\mathfrak{m}}(K)]_1 = \mathfrak{m}/aK$, and $\mathrm{gr}_{\mathfrak{m}}(K)/f\mathrm{gr}_{\mathfrak{m}}(K) = K/\mathfrak{m} \oplus \mathfrak{m}/aK$. Let $x \in K \setminus \mathfrak{m}$ and assume that $\mathfrak{M}\overline{x} \subseteq f\mathrm{gr}_{\mathfrak{m}}(K)$, where $\overline{x}$ ($\in K/\mathfrak{m}$) denotes the image of x in $\mathrm{gr}_{\mathfrak{m}}(K)$. Then, since $\mathfrak{m}x \subseteq aK$ and $\mathrm{r}_R(K) = 1$ (remember that $K \cong \mathrm{K}_R$ as an R-module), we get $aK :_K \mathfrak{m} = aK + Rx$. Therefore $\mathfrak{m}(K/aK) = (0)$; otherwise $\mathfrak{m}(K/aK) \supseteq [(0) :_{K/aK} \mathfrak{m}]$ and hence $x \in \mathfrak{m}K = \mathfrak{m}$. Consequently, because $K/aK = \mathrm{E}_{R/(a)}(R/\mathfrak{m})$ (the injective envelope of the $R/(a)$-module $R/\mathfrak{m}$; see [43, Korollar 6.4]) is a faithful $R/(a)$-module, we have $\mathfrak{m} = (a)$. This is, however, impossible, because R is not a discrete valuation ring. Let $x \in \mathfrak{m} \setminus aK$ and assume that $\mathfrak{M}\overline{x} \subseteq f\mathrm{gr}_{\mathfrak{m}}(K)$, where $\overline{x}$ ($\in \mathfrak{m}K/\mathfrak{m}^2K$) denotes the image of x in $\mathrm{gr}_{\mathfrak{m}}(K)$. Then $\mathfrak{m}x \subseteq aK$ and hence $aK :_K \mathfrak{m} = aK + Rx$, which proves $\ell_{R/\mathfrak{m}}((0) :_{\mathrm{gr}_{\mathfrak{m}}(K)/f\mathrm{gr}_{\mathfrak{m}}(K)} \mathfrak{M}) = 1$. Thus $\mathrm{r}_G(\mathrm{gr}_{\mathfrak{m}}(K)) = 1$, so that $\mathrm{gr}_{\mathfrak{m}}(K) \cong \mathrm{K}_G$ as a graded G-module. □

Assume now that $d > 1$ and that our assertion holds true for $d-1$. Let $0 \to R \to \mathrm{K}_R \to C \to 0$ be an exact sequence of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$. Choose $a \in \mathfrak{m}$ so that a is a part of a minimal reduction of $\mathfrak{m}$ and a is superficial for C with respect to $\mathfrak{m}$. Let $f = \overline{a}$ ($\in \mathfrak{m}/\mathfrak{m}^2$) denote the image of a in $G = \mathrm{gr}_{\mathfrak{m}}(R)$. We then have $G/fG = \mathrm{gr}_{\mathfrak{m}}(R/(a))$ and $\mathrm{v}(R/(a)) = \mathrm{e}(R/(a)) + d - 2$. By the hypothesis of induction, G/fG is an almost Gorenstein graded ring, because $R/(a)$ is an almost Gorenstein local ring (see Proof of Theorem 1.3.9). Choose an exact sequence $0 \to G/fG \to \mathrm{K}_{G/fG}(d-2) \to X \to 0$ of graded G/fG-modules so that $\mu_{G/fG}(X) = \mathrm{e}^0_{[G/fG]_+}(X)$. Recall that $\mathrm{K}_{G/fG}(d-2) \cong \mathrm{K}_{G/f\mathrm{K}_G}(d-1)$ as a graded G-module and we get an exact sequence $0 \to G \to \mathrm{K}_G(d-1) \to Y \to 0$ of graded G-modules, similarly as in Proof of Theorem 1.3.7, such that $\mu_G(Y) = \mathrm{e}^0_{\mathfrak{M}}(Y)$. Hence G is an almost Gorenstein graded ring.

(2) We may assume G is not a Gorenstein ring. Hence $d > 0$ and $a = 1 - d$ (Lemma 1.10.2). Suppose that $d = 1$ and choose an exact sequence

$$0 \to G \to \mathrm{K}_G(-a) \to C \to 0 \tag{$\sharp$}$$

of graded G-modules so that $\mathfrak{M}C = (0)$, where $a = \mathrm{a}(G)$.

We now take the canonical $\mathfrak{m}$-filtration $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$ of K_R so that $\omega_n = \mathrm{K}_R$ if $n \le -a$ and $\mathrm{K}_G = \mathrm{gr}_\omega(\mathrm{K}_R)$ (see, e.g., [23] for the existence of canonical filtrations). Let $f \in \mathrm{K}_R = \omega_{-a}$ such that $\varphi(1) = \overline{f}$, where $\overline{f}$ $(\in \mathrm{K}_R/\omega_{-a+1})$ denotes the image of f in $\mathrm{K}_G(-a)$. Let α be the R-linear map $R \to \mathrm{K}_R$ defined by $\alpha(1) = f$ and consider R to be filtered with filtration $\mathbb{F} = \{\mathfrak{m}^{n+a}\}_{n\in\mathbb{Z}}$. Then the homomorphism $\alpha : R \to \mathrm{K}_R$ preserves filtrations and $G(a) \cong \mathrm{gr}_{\mathbb{F}}(R)$ as an graded G-module. Consequently, exact sequence ($\sharp$) turns into $0 \to \mathrm{gr}_{\mathbb{F}}(R)(-a) \to \mathrm{gr}_\omega(\mathrm{K}_R)(-a) \to C \to 0$ with $\varphi(1) = \overline{f}$. We now notice that $C = C_0$, because G is a level ring. Hence $\omega_n = \mathfrak{m}^{n+a}f + \omega_{n+1}$ for $n > -a$. Therefore $\omega_{-a+1} \subseteq \mathfrak{m}f + \omega_\ell$ for all $\ell \in \mathbb{Z}$ and hence $\omega_{-a+1} = \mathfrak{m}f$. Thus $\omega_{-a+1} = \mathfrak{m}f = \mathfrak{m}\mathrm{K}_R$, because $\mathfrak{m}\mathrm{K}_R = \mathfrak{m}\omega_{-a} \subseteq \omega_{-a+1}$. Consequently, in the exact sequence $R \overset{\alpha}{\to} \mathrm{K}_R \to X \to 0$, we have $\mathfrak{m}X = (0)$. Thus R is an almost Gorenstein local ring (see Lemma 1.3.1).

Now suppose that $d > 1$ and that our assertion holds true for $d - 1$. Look at the exact sequence $0 \to G \to \mathrm{K}_G(d-1) \to C \to 0$ of graded G-modules with $\mu_G(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$. Choose $f \in G_1$ so that f is G-regular and $\mathfrak{M}C = (f, f_2, \ldots, f_{d-1})C$ for some $f_2, f_3, \ldots, f_{d-1} \in G_1$ (Proposition 1.2.2 (2)). We then have the exact sequence $0 \to G/fG \to (\mathrm{K}_G/f\mathrm{K}_G)(d-1) \to C/fC \to 0$ of graded G/fG-modules, which guarantees that G/fG is an almost Gorenstein graded ring (remember that $(\mathrm{K}_G/f\mathrm{K}_G)(d-1) \cong \mathrm{K}_{G/fG}(d-2)$). Consequently, thanks to the hypothesis of induction, the local ring $R/(a)$ (here $a \in \mathfrak{m}$ such that $f = \overline{a}$ in $\mathfrak{m}/\mathfrak{m}^2 = [\mathrm{gr}_{\mathfrak{m}}(R)]_1$) is an almost Gorenstein local ring and therefore by Theorem 1.3.7, R is an almost Gorenstein local ring, because a is R-regular. $\square$

When $\mathrm{v}(R) = \mathrm{e}(R) + d - 1$, the almost Gorensteinness of R is equivalent to the Gorensteinness of $\mathrm{Q}(G)$, as we show in the following.

Corollary 1.11.3. *Suppose that* $\mathrm{v}(R) = \mathrm{e}(R)+d-1$. *Then the following are equivalent.*

(1) *R is an almost Gorenstein local ring,*

(2) *G is an almost Gorenstein graded ring,*

(3) $\mathrm{Q}(G)$ *is a Gorenstein ring.*

Proof. Since G is a Cohen-Macaulay level graded ring (Proposition 1.10.1 and [62]), the equivalence (1) $\Leftrightarrow$ (2) follows from Theorem 1.11.1. See Theorem 1.10.4 (resp. Lemma 1.3.1 (1)) for the implication (3) $\Rightarrow$ (2) (resp. (2) $\Rightarrow$ (3)). □

We say that $\mathfrak{m}$ is a normal ideal, if $\mathfrak{m}^n$ is an integrally closed ideal for all $n \geq 1$.

Corollary 1.11.4. *Suppose that $\mathrm{v}(R) = \mathrm{e}(R) + d - 1$ and that R is a normal ring. If $\mathfrak{m}$ is a normal ideal, then R is an almost Gorenstein local ring.*

Proof. Let $\mathcal{R}' = \mathcal{R}'(\mathfrak{m}) = R[\mathfrak{m}t, t^{-1}]$ be the extended Rees algebra of $\mathfrak{m}$, where t is an indeterminate. Then $\mathcal{R}'$ is a normal ring, because R is a normal local ring and $\mathfrak{m}$ is a normal ideal. Hence the total ring of fractions of $G = \mathcal{R}'/t^{-1}\mathcal{R}'$ is a Gorenstein ring, so that R is almost Gorenstein by Corollary 1.11.3. □

We now reach the goal of this section.

Corollary 1.11.5. *Every 2-dimensional rational singularity is an almost Gorenstein local ring.*

Auslander's theorem [3] says that every two-dimensional Cohen-Macaulay complete local ring R of finite Cohen-Macaulay representation type is a rational singularity, provided R contains a field of characteristic 0. Hence by Corollary 1.11.5 we get the following.

Corollary 1.11.6. *Every two-dimensional Cohen-Macaulay complete local ring R of finite Cohen-Macaulay representation type is an almost Gorenstein local ring, provided R contains a field of characteristic 0.*

1.12 One-dimensional Cohen-Macaulay local rings of finite CM-representation type are almost Gorenstein

The purpose of this section is to prove the following.

Theorem 1.12.1. *Let A be a Cohen-Macaulay complete local ring of dimension one and assume that the residue class field of A is algebraically closed of characterisic 0. If A has finite Cohen-Macaulay representation type, then A is an almost Gorenstein local ring.*

Let A be a Cohen-Macaulay complete local ring of dimension one with algebraically closed residue class field k of characterisic 0. Assume that A has finite Cohen-Macaulay representation type. Then by [79, (9.2)] one obtains a *simple singularity* R so that $R \subseteq A \subseteq \overline{R}$, where $\overline{R}$ denotes the integral closure of R in the total ring $\mathrm{Q}(R)$ of fractions. We remember that $R = S/(F)$, where $S = k[[X, Y]]$ is the formal power series ring over k and F is one of the following polynomials ([79, (8.5)]).

(A_n) $X^2 - Y^{n+1} \quad (n \geq 1)$

(D_n) $X^2Y - Y^{n-1} \quad (n \geq 4)$

(E_6) $X^3 - Y^4$

(E_7) $X^3 - XY^3$

(E_8) $X^3 - Y^5$.

Our purpose is, therefore, to show that all the intermediate local rings $R \subseteq A \subseteq \overline{R}$ are almost Gorenstein.

Let us begin with the analysis of overrings of local rings with multiplicity 2.

Lemma 1.12.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$ and $\mathrm{e}^0_{\mathfrak{m}}(R) = 2$. Let $R \subseteq S \subseteq \mathrm{Q}(R)$ be an intermediate ring such that S is a finitely generated R-module. Then the following assertions hold true.*

(1) *If S is a local ring with maximal ideal $\mathfrak{n}$, then $\mathrm{e}^0_{\mathfrak{n}}(S) \leq 2$. We have $\mathrm{e}^0_{\mathfrak{n}}(S) = 2$ and $R/\mathfrak{m} \cong S/\mathfrak{n}$, if $S \subsetneq \overline{R}$.*

(2) *Suppose that S is not a local ring. Then $\widehat{R} \otimes_R S \cong V_1 \times V_2$ with discrete valuation rings V_1 and V_2, where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R. Hence S is a regular ring and $S = \overline{R}$.*

(3) *Let $A = R : \mathfrak{m}$ in $\mathrm{Q}(R)$. Then $A = \mathfrak{m} : \mathfrak{m}$ and if $R \subsetneq S$, then $A \subseteq S$.*

Proof. (1), (2) We have $\mathfrak{m}^2 = f\mathfrak{m}$ for some $f \in \mathfrak{m}$, since $\mathrm{e}^0_{\mathfrak{m}}(R) = 2$ (see [64, Theorem 3.4]). Let $Q = (f)$. Then

$$2 = \mathrm{e}^0_{\mathfrak{m}}(R) = \mathrm{e}_0(Q, R) = \mathrm{e}_0(Q, S) = \ell_R(S/fS) \geq \ell_S(S/fS) \geq \mu_R(S),$$

because $\ell_R(S/R) < \infty$. Therefore assertion (1) follows, since $\ell_S(S/fS) \geq \mathrm{e}^0_{\mathfrak{n}}(S)$. We have $R/\mathfrak{m} = S/\mathfrak{n}$, if $\mathrm{e}^0_{\mathfrak{n}}(S) = 2$ (remember that $\ell_R(S/fS) = [S/\mathfrak{n} : R/\mathfrak{m}]{\cdot}\ell_S(S/fS)$). Assume now that S is not a local ring. To see assertion (2), passing to $\widehat{R}$, we may assume that R is complete. Then $\mu_R(S) = 2$. Hence S contains exactly two maximal ideals $\mathfrak{n}_1$ and $\mathfrak{n}_2$, so that $S \cong S_{\mathfrak{n}_1} \times S_{\mathfrak{n}_2}$ with $\mu_R(S_{\mathfrak{n}_i}) = 1$ for $i = 1, 2$. Because $\ell_R(S/fS) = 2$, we get $\ell_R(S_{\mathfrak{n}_i}/fS_{\mathfrak{n}_i}) = 1$, whence the local rings $S_{\mathfrak{n}_i}$ are discrete valuation rings. Therefore S is regular.

(3) Because R is not a discrete valuation ring, we get $A = \mathfrak{m} : \mathfrak{m}$. We also have $\ell_R(A/R) = 1$ ([43, Satz 1.46]), since R is a Gorenstein ring. Therefore $A \subseteq S$, if $R \subsetneq S$. ☐

Proposition 1.12.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with* $\dim R = 1$ *and* $\mathrm{e}^0_{\mathfrak{m}}(R) = 2$. *Let $R \subsetneq S \subseteq \mathrm{Q}(R)$ be an intermediate ring such that S is a finitely generated R-module. Let $n = \ell_R(S/R)$. We then have the following.*

(1) *There exists a unique chain of intermediate rings*

$$R = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_n = S.$$

(2) *Every intermediate ring $R \subseteq A \subseteq S$ appears as one of $\{A_i\}_{0 \leq i \leq n}$.*

Proof. Let $A_1 = R : \mathfrak{m}$. Then by Lemma 1.12.2 (3) $A_1 \subseteq A$ for every intermediate ring $R \subsetneq A \subseteq S$, which enables us to assume that $n > 1$ and that the assertion holds true for $n - 1$. As $R \subsetneq A_1 \subsetneq S$, by Lemma 1.12.2 A_1 is a local ring with $\mathrm{e}^0_{\mathfrak{n}_1}(A_1) = 2$ and $R/\mathfrak{m} \cong A_1/\mathfrak{n}_1$, where $\mathfrak{n}_1$ is the maximal ideal of A_1. Hence $\ell_{A_1}(S/A_1) = \ell_R(S/A_1) = n - 1$, so that the assertion follows from the hypothesis of induction on n. ☐

Corollary 1.12.4. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with* $\dim R = 1$ *and* $\mathrm{e}^0_{\mathfrak{m}}(R) = 2$ *and let $R \subseteq A, B \subseteq \mathrm{Q}(R)$ be intermediate rings such that A and B are finitely generated R-modules. Then $A \subseteq B$ or $B \subseteq A$.*

Proof. Let $S = R[A, B]$. Then $R \subseteq S \subseteq \mathrm{Q}(R)$ and S is a finitely generated R-module. Hence assertion follows from Proposition 1.12.3. ☐

For a Cohen-Macaulay local ring R of dimension one, let $\mathcal{X}_R$ denote the set of intermediate rings $R \subseteq A \subseteq \mathrm{Q}(R)$ such that A is a finitely generated R-module.

Corollary 1.12.5. *The following assertions hold true.*

(1) *Let $V = k[[t]]$ be the formal power series ring over a field k and $R = k[[t^2, t^{2\ell+1}]]$ where $\ell > 0$. Then $\mathcal{X}_R = \{k[[t^2, t^{2q+1}]] \mid 0 \le q \le \ell\}$.*

(2) *Let $S = k[[X, Y]]$ be the formal power series ring over a field k of $\mathrm{ch}k \neq 2$ and $R = S/(X^2 - Y^{2\ell})$ with $\ell > 0$. Let x, y denote the images of X, Y in R, respectively. Then $\mathcal{X}_R = \{R[\frac{x}{y^q}] \mid 0 \le q \le \ell\}$.*

Proof. (1) Let $R_q = k[[t^2, t^{2q+1}]]$ for $0 \le q \le \ell$. Then we have the tower

$$R = R_\ell \subsetneq R_{\ell-1} \subsetneq \ldots \subsetneq R_0 = V$$

of intermediate rings. Hence the result follows from Proposition 1.12.3.

(2) Let $R_q = R[\frac{x}{y^q}]$ for $0 \le q \le \ell$. We then have a tower

$$R = R_0 \subsetneq R_1 \subsetneq \ldots \subsetneq R_\ell \subseteq \overline{R}$$

of intermediate rings. Since $\ell_R(\overline{R}/R) = \ell$ (remember that $\overline{R} = S/(X+Y^\ell) \oplus S/(X-Y^\ell)$, because $\mathrm{ch}k \neq 2$), the assertion follows Proposition 1.12.3. ∎

We need one more result.

Proposition 1.12.6. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 1$ and assume that there is an element $f \in \mathfrak{m}$ such that fR is a reduction of $\mathfrak{m}$. Let $R \subseteq A \subseteq \mathrm{Q}(R)$ be an intermediate ring and assume that $\ell_R(A/R) = 1$ and that A is a local ring with maximal ideal $\mathfrak{n}$. Then the following assertions hold.*

(1) *If A has maximal embedding dimension and fA is a reduction of $\mathfrak{n}$, then A is an almost Gorenstein local ring.*

(2) *If $\mathrm{e}^0_{\mathfrak{m}}(R) = 3$, then A is an almost Gorenstein local ring.*

Proof. (1) We get $A = R : \mathfrak{m}$, since $R \subsetneq A \subseteq R : \mathfrak{m}$ and $\ell_R([R : \mathfrak{m}]/R) = 1$. Hence $\mathrm{K}_A = R : A = R : (R : \mathfrak{m}) = \mathfrak{m}$ ([43, 5.19, Definition 2.4]). Since fA is a reduction of $\mathfrak{n}$ and A has maximal embedding dimension, we get $\mathfrak{n}^2 = f\mathfrak{n}$, so that $\mathfrak{n}{\cdot}(\mathfrak{m}/fA) = (0)$, because $\mathfrak{m}/fA \subseteq \mathfrak{n}/fA$. Hence A is an almost Gorenstein local ring (Lemma 1.3.1 (1)).

(2) We may assume $\mathrm{v}(A) \geq 3$, where $\mathrm{v}(A)$ denotes the embedding dimension of A. Remember

$$3 = \mathrm{e}_{\mathfrak{m}}^0(R) = \mathrm{e}_0(fR, A) = \ell_R(A/fA) \geq \ell_A(A/fA) = \mathrm{e}_{\mathfrak{n}}^0(A) \geq \mathrm{v}(A) \geq 3,$$

so that $\ell_A(A/fA) = \mathrm{e}_{\mathfrak{n}}^0(A) = \mathrm{v}(A) = 3$. Hence fA is a reduction of $\mathfrak{n}$ by a theorem of Rees [59] and the assertion follows from assertion (1), because A has maximal embedding dimension. □

We shall now check, for the five cases from (A_n) to (D_n) separately, that all the intermediate local rings $R \subseteq A \subseteq \overline{R}$ are almost Gorenstein.

(1) The case (A_n)**.** Let $R \subseteq A \subseteq \overline{R}$ be an intermediate local ring such that A is a finitely generated R-module. Let $\mathfrak{n}$ be the maximal ideal of A. Then $\mathrm{e}_{\mathfrak{n}}^0(A) \leq 2$ by Lemma 1.12.2 (2), so that A is Gorenstein.

(2) The cases (E_6) **and** (E_8)**.** Let $V = k[[t]]$ be the formal power series ring over k. We then have $S/(X^3 - Y^4) \cong k[[t^3, t^4]]$ and $S/(X^3 - Y^5) \cong k[[t^3, t^5]]$. We begin with the following.

Proposition 1.12.7. *The following assertions hold true.*

(1) $\mathcal{X}_{k[[t^3,t^4,t^5]]} = \{k[[t^3, t^4, t^5]], k[[t^2, t^3]], V\}$.

(2) $\mathcal{X}_{k[[t^3,t^4]]} = \{k[[t^3, t^4]], k[[t^3, t^4, t^5]], k[[t^2, t^3]], V\}$.

(3) $\mathcal{X}_{k[[t^3,t^5]]} = \{k[[t^3, t^5]], k[[t^3, t^5, t^7]], k[[t^3, t^4, t^5]], k[[t^2, t^3]], V\}$.

Proof. (1) Let $A = k[[t^3, t^4, t^5]]$ and let $B \in \mathcal{X}_A$ such that $B \neq A$. We choose $f \in B \setminus A$ and write

$$f = c_1 t + c_2 t^2 + g$$

with $c_1, c_2 \in k$ and $g \in A$. If $c_1 \neq 0$, then $fV = tV$, so that $V = k[[f]] \subseteq B$. Suppose $c_1 = 0$. Then $f = c_2 t^2 + g$ and $c_2 \neq 0$, so that $t^2 \in B$. Hence $k[[t^2, t^3]] \subseteq B$, which shows $B = k[[t^2, t^3]]$ or $B = V$, because $\ell_{k[[t^2,t^3]]}(V/k[[t^2, t^3]]) = 1$.

(2) Let $A = k[[t^3, t^4]]$ and let $B \in \mathcal{X}_A$ such that $B \neq A$. We choose $f \in B \setminus A$ and write

$$f = c_1 t + c_2 t^2 + c_5 t^5 + g$$

with $c_i \in k$ and $g \in A$. If $c_1 \neq 0$, then $V = k[[f]] \subseteq B$. Suppose $c_1 = 0$ but $c_2 \neq 0$. Then, rechoosing f so that $c_2 = 1$,we get $t^2 + c_5 t^5 \in B$. Hence $t^3(t^2 + c_5 t^5) = t^5 + c_5 t^8 \in B$.

Therefore $t^5 \in B$, because $\langle 3, 4\rangle \ni n$ for all $n \geq 8$ (here $\langle 3, 4\rangle$ denotes the numerical semigroup generated by 3, 4). Thus $k[[t^3, t^4, t^5]] \subsetneq B$, whence $B = k[[t^2, t^3]]$ or $B = V$. Suppose that $c_1 = c_2 = 0$ for any choice of $f \in B \setminus A$. We then have $t^5 \in B$ since $c_5 \neq 0$, so that $B = k[[t^3, t^4, t^5]]$.

(3) Let $A = k[[t^3, t^5]]$ and let $B \in \mathcal{X}_A$ such that $B \neq A$. We choose $f \in B \setminus A$ and write

$$f = c_1 t + c_2 t^2 + c_4 t^4 + c_7 t^7 + g$$

with $c_i \in k$ and $g \in A$. If $c_1 \neq 0$, then $B = V$. Suppose $c_1 = 0$ but $c_2 \neq 0$, say $c_2 = 1$. Then, since $t^5 f = t^7 + c_4 t^9 + c_7 t^{12} + t^5 g \in B$, we get $t^7 \in B$ (remember that $\langle 3, 5\rangle \ni n$ for all $n \geq 8$). Hence $(t^2 + c_4 t^4)^2 \in B$, so that $t^4 \in B$. Therefore $k[[t^2, t^3]] \subseteq B$. If $c_1 = c_2 = 0$ but $c_4 = 1$, then $t^4 + c_7 t^7 \in B$, so that $t^7 \in B$ because $t^3(t^4 + c_7 t^7) \in B$. Hence $k[[t^3, t^4]] \subseteq B$. Suppose that $c_1 = c_2 = c_7 = 0$ for any choice of $f \in B \setminus A$. We then have $t^7 \in B$, whence $B = k[[t^3, t^5, t^7]]$. $\square$

It is standard to check that $k[[t^3, t^4, t^5]]$ and $k[[t^3, t^5, t^7]]$ are almost Gorenstein local rings, which proves the case (E_6) or (E_8).

(3) The case (E_7)**.** We consider $F = X^3 - XY^3$. Let $f = X^2 - Y^3$. Then X, f is a system of parameters of $S = k[[X, Y]]$. Therefore $(F) = (X) \cap (f)$. Let $k[[t]]$ be the formal power series ring and we get a tower

$$R = S/(F) \subseteq S/(X) \oplus S/(f) = k[[Y]] \oplus k[[t^2, t^3]] \subseteq k[[Y]] \oplus k[[t]] = \overline{R}$$

of rings, where we naturally identify $S/(X) = k[[Y]]$ and $S/(f) = k[[t^2, t^3]] \subseteq k[[t]]$. Let $R \subsetneq A \subseteq \overline{R} = k[[Y]] \oplus k[[t]]$ be an intermediate local ring. Let $p_2 : k[[Y]] \oplus k[[t]] \to k[[t]]$, $(a, b) \mapsto b$ be the projection and set $C = p_2(A)$. Then $k[[t^2, t^3]] \subseteq C \subseteq V$, whence $C = k[[t^2, t^3]]$ or $C = V$ (Corollary 1.12.5).

We firstly consider the case where $C = V$. Let $\mathfrak{n}$ denote the maximal ideal of A.

Claim 1.12.8. *There exists an element $z \in A$ such that $z = (0, t)$*

Proof of Claim 1.12.8. Since $t \in C$, there exists $z \in A$ such that $z = (g, t)$ with $g \in k[[Y]]$. Then $z \in \mathfrak{n}$. Suppose $g \neq 0$ and write $g = Y^n \varepsilon$, where $n > 0$ and $\varepsilon \in \mathrm{U}(k[[Y]])$. Let $\overline{g}$ denotes the image of $g \in S = k[[X, Y]]$ in A. Then since $\overline{g} = (g, g(t^2))$ in $S/(X) \oplus k[[t]]$, we have $z - \overline{g} = (0, t - g(t^2))$ and $t - g(t^2) = t - t^{2n}{\cdot}\varepsilon(t^2) = t u_2$ with $u_2 = 1 - t^{2n-1}{\cdot}\varepsilon(t^2)$. Hence because u_2 is a unit of $C = k[[t]]$, we may choose a unit

$u \in A$ so that $p_2(u) = u_2$. We then have $(0,t) = u^{-1}(z - \overline{g}) \in A$. Thus $z' = u^{-1}(z - \overline{g})$ is a required element of A. $\square$

Let $z = (0,t)$. Let $x = \overline{X}$ and $y = \overline{Y}$ denote the images of X, Y in A, respectively. Hence $x = (0, t^3)$, $y = (Y, t^2)$, and therefore $x = z^3$ and $z(y - z^2) = 0$. We consider the k-algebra map $\psi : k[[Y, Z]] \to A$ defined by $\psi(Y) = y$, $\psi(Z) = z$. Then $Z(Y - Z^2) \in \operatorname{Ker}\psi$. We now consider the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \overline{R} = k[[Y]] \oplus V & \longrightarrow & \overline{R}/A & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \bar{\psi}} & & \uparrow{\scriptstyle \cong} & & & & \\
0 & \longrightarrow & \frac{k[[Y,Z]]}{(Z(Y-Z^2))} & \longrightarrow & \frac{k[[Y,Z]]}{(Z)} \oplus \frac{k[[Y,Z]]}{(Y-Z^2)} & \longrightarrow & \frac{k[[Y,Z]]}{(Y,Z)} & \longrightarrow & 0,
\end{array}
$$

where the rows are canonical exact sequences and $\bar{\psi} : k[[Y, Z]] \to A$ is the homomorphism derived from ψ. Then the induced homomorphism $\rho : k[[Y, Z]]/(Y, Z) \to \overline{R}/A$ has to be bijective, because $\overline{R}/A \neq (0)$ (remember that A is a *local* ring) and ρ is surjective. Consequently, $\bar{\psi} : k[[Y, Z]]/(Z(Y - Z^2)) \to A$ is an isomorphism, so that A is a Gorenstein ring.

Next we consider the case where $C = k[[t^2, t^3]]$. Hence $R \subsetneq A \subsetneq k[[Y]] \oplus k[[t^2, t^3]]$. We set $B = k[[t^2, t^3]]$ and $T = k[[Y]] \oplus B$. Remember that $\ell_R(T/R) = 3$, whence $\ell_R(A/R) = 1$ or 2. If $\ell_R(A/R) = 1$, then by Proposition 1.12.6 (2) A is an almost Gorenstein local ring.

Suppose that $\ell_R(A/R) = 2$. Hence $\ell_R(T/A) = 1$. Therefore as

$$\ell_R(T/A) = [A/\mathfrak{n} : R/\mathfrak{m}] {\cdot} \ell_A(T/A),$$

we get $R/\mathfrak{m} \cong A/\mathfrak{n}$ and $\ell_A(T/A) = 1$, whence $\mathfrak{n} = (0) :_A T/A$ is an ideal of T. Let J denote the Jacobson radical of T and consider the exact sequence

$$0 \to A/\mathfrak{n} \to T/\mathfrak{n} \to T/A \to 0$$

of A-modules. We then have $\ell_A(T/\mathfrak{n}) = 2$, so that $\mathfrak{n} = J$, because $\mathfrak{n} \subseteq J$ and $\ell_A(T/J) = \ell_R(T/J) = 2$. Hence $A = k + J$ and $\mathfrak{n} = ((0, t^3), (Y, 0), (0, t^2))$. Let $\psi : k[[X, Y, Z]] \to A$ be the k-algebra map defined by $\psi(X) = (0, t^3)$, $\psi(Y) = (Y, 0)$, $\psi(Z) = (0, t^2)$. Then

$X^2 - Z^3, XY, YZ \in \operatorname{Ker}\psi$ and we get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & T = k[[Y]] \oplus k[[t^2,t^3]] & \longrightarrow & T/A & \longrightarrow & 0 \\ & & \uparrow \bar{\psi} & & \uparrow \cong & & & & \\ 0 & \longrightarrow & \frac{k[[X,Y,Z]]}{(X,Z)\cap(X^2-Z^3,Y)} & \longrightarrow & \frac{k[[X,Y,Z]]}{(X,Z)} \oplus \frac{k[[X,Y,Z]]}{(X^2-Z^3,Y)} & \longrightarrow & \frac{k[[X,Y,Z]]}{(X,Y,Z)} & \longrightarrow & 0. \end{array}$$

For the same reason as above, the induced homomorphism $k[[X,Y,Z]]/(X,Y,Z) \to T/A$ has to be bijective, so that $A \cong k[[X,Y,Z]]/(X,Z) \cap (X^2 - Z^3, Y)$. Notice now that

$$(X,Z) \cap (X^2 - Z^3, Y) = \mathrm{I}_2 \begin{pmatrix} Z^2 & X & Y \\ X & Z & 0 \end{pmatrix} = \mathrm{I}_2 \begin{pmatrix} Z^2 & X & Y \\ X+Z^2 & X+Z & Y \end{pmatrix}.$$

Then thanks to the theorem of Hilbert-Burch and Theorem 1.7.8, A is an almost Gorenstein local ring, because $X + Z^2, X + Z, Y$ is a regular system of parameters of $k[[X,Y,Z]]$. This completes the proof of the case (E_7).

(4) The case (D_n)**.**

(i) The case where $n = 2\ell + 1$ **with** $\ell \geq 1$**.** We consider $F = Y(X^2 - Y^{2\ell+1})$. Let $f = X^2 - Y^{2\ell+1}$. Then X, f is a system of parameters of $S = k[[X,Y]]$. Therefore $(F) = (Y) \cap (f)$ and we get a tower

$$R = S/(F) \subseteq S/(Y) \oplus S/(f) = k[[X]] \oplus k[[t^2, t^{2\ell+1}]] \subseteq k[[X]] \oplus k[[t]] = \overline{R}$$

of rings, where we naturally identify $S/(Y) = k[[X]]$ and $S/(f) = k[[t^2, t^{2\ell+1}]] \subseteq k[[t]]$. Let $R \subsetneq A \subseteq \overline{R}$ be an intermediate ring and assume that $(A, \mathfrak{n})$ is a local ring. Let $p_2 : \overline{R} \to V$ be the projection and set $B = p_2(A)$. Then since $k[[t^2, t^{2\ell+1}]] \subseteq B \subseteq V$, by Corollary 1.12.5 (1) $B = k[[t^2, t^{2q+1}]]$ for some $0 \leq q \leq \ell$. We choose an element $z \in A$ so that $z = (g, t^{2q+1})$ in $\overline{R} = k[[X]] \oplus k[[t]]$ with $g \in k[[X]]$. Suppose $g \neq 0$ and write $g = X^n \varepsilon$ ($n > 0, \varepsilon \in \mathrm{U}(k[[X]])$). Denote by $\overline{g}$ the image of $g \in S = k[[X,Y]]$ in A. We have

$$\begin{aligned} z - \overline{g} &= z - (g, (t^{2\ell+1})^n {\cdot} \varepsilon(t^{2\ell+1})) \\ &= (0, t^{2q+1}(1 - t^{(2\ell+1)n-(2q+1)} {\cdot} \varepsilon(t^{2\ell+1}))). \end{aligned}$$

Here we notice that $(2\ell + 1)n - (2q + 1) \geq 0$ and that $(2\ell + 1)n - (2q + 1) = 0$ if and only if $n = 1$ and $\ell = q$.

If $(2\ell+1)n - (2q+1) > 0$, we set $u_2 = 1 - t^{(2\ell+1)n-(2q+1)} {\cdot} \varepsilon(t^{2\ell+1})$. Then $u_2 \in \mathrm{U}(B)$. We choose an element $u \in \mathrm{U}(A)$ so that $u_2 = p_2(u)$. Then

$$z' = u^{-1}(z - \overline{g}) = (0, t^{2q+1}).$$

Therefore, replacing z with z', we may assume without loss of generality that $z \in A$ such that $z = (0, t^{2q+1})$. Let $x = \overline{X}$ and $y = \overline{Y}$, where $\overline{X}, \overline{Y}$ respectively denote the images of $X, Y \in S = k[[X, Y]]$ in A. Then $x = (X, t^{2q+1})$ and $y = (0, t^2)$, so that

$$y^{2q+1} = z^2, \quad y(x - y^{\ell-q}z) = 0, \quad \text{and} \quad z(x - y^{\ell-q}z) = 0.$$

Let $\psi : k[[X, Y, Z]] \to A$ be the k-algebra map defined by $\psi(X) = x$, $\psi(Y) = y$, $\psi(Z) = z$. Then $\operatorname{Ker}\psi \supseteq (Y, Z) \cap (Z^2 - Y^{2q+1}, X - Y^{\ell-q}Z)$. Therefore, considering the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & T = k[[X]] \oplus k[[t^2, t^{2q+1}]] & \longrightarrow & T/A & \longrightarrow & 0 \\
 & & \uparrow \bar{\psi} & & \uparrow \cong & & & & \\
0 & \longrightarrow & \frac{k[[X,Y,Z]]}{(Y,Z)\cap(Z^2 - Y^{2\ell+1}, X - Y^{\ell-q}Z)} & \longrightarrow & \frac{k[[X,Y,Z]]}{(Y,Z)} \oplus \frac{k[[X,Y,Z]]}{(Z^2 - Y^{2\ell+1}, X - Y^{\ell-q}Z)} & \longrightarrow & \frac{k[[X,Y,Z]]}{(X,Y,Z)} & \longrightarrow & 0,
\end{array}$$

we see that

$$A \cong k[[X, Y, Z]]/(Y, Z) \cap (X - Y^{\ell-q}Z, Z^2 - Y^{2q+1}),$$

because $T/A \neq (0)$. Notice now that

$$(X, Z) \cap (X^2 - Z^3, Y) = \mathrm{I}_2 \begin{pmatrix} Y^{2q} & Z & X - Y^{\ell-q} \\ Z & Y & 0 \end{pmatrix} = \mathrm{I}_2 \begin{pmatrix} Y^{2q} & Z & X - Y^{\ell-q}Z \\ Z - Y^{2q} & Y - Z & Y^{\ell-q}Z - X \end{pmatrix}.$$

Then by Theorem 1.7.8 A is an almost Gorenstein local ring, because $Z - Y^{2q}, Y - Z, Y^{\ell-q}Z - X$ is a regular system of parameters of $k[[X, Y, Z]]$.

If $n = 1$ and $\ell = q$, then $R \subsetneq A \subsetneq k[[X]] \oplus k[[t^2, t^{2\ell+1}]]$, so that $\ell_R(A/R) = 1$ (remember that $\ell_R((k[[X]] \oplus k[[t^2, t^{2\ell+1}]])/R) = 2$). Hence A is an almost Gorenstein local ring by Proposition 1.12.6 (2).

(ii) The case where $n = 2\ell$ with $\ell \geq 1$. Let $f = X^2 - Y^{2\ell} = (X + Y^\ell)(X - Y^\ell)$ and $T = S/(f)$. Since $\operatorname{ch} k \neq 2$, $X + Y^\ell$, $X - Y^\ell$ is a system of parameters of $S = k[[X, Y]]$, so we get the exact sequence

$$0 \to T \xrightarrow{\alpha} S/(X + Y^\ell) \oplus S/(X - Y^\ell) \xrightarrow{\beta} S/(X, Y^\ell) \to 0.$$

Hence $\ell_T(\overline{T}/T) = \ell$. We look at the tower

$$R \subseteq k[[X]] \oplus T \subseteq k[[X]] \oplus \overline{T} = \overline{R}$$

of rings and consider an intermediate ring $R \subsetneq A \subsetneq \overline{R}$ such that $(A, \mathfrak{n})$ is a local ring. Let $p_2 : k[[X]] \oplus \overline{T} \to \overline{T}$ be the projection and set $B = p_2(A)$. We denote by x, y the images of X, Y in $T = S/(f)$, respectively. Then by Corollary 1.12.5 (2) $B = T[\frac{x}{y^q}]$ for some $0 \le q \le \ell$. Here we notice that $q < \ell$, since $A \ne \overline{R}$. We choose an element $z \in A$ so that $z = (g, \frac{x}{y^q})$, where $g \in k[[X]]$. If $g \ne 0$, then we write $g = X^n \varepsilon$ $(n > 0, \varepsilon \in \mathrm{U}(k[[X]])$. Let $\overline{g}$ be the image of $g \in S$ in A. We then have

$$z - \overline{g} = (g, \frac{x}{y^q}) - (g, x^n \cdot \varepsilon(x)) = (0, \frac{x}{y^q} \cdot (1 - (\frac{x}{y^q})^{n-1}) \cdot y^{nq} \cdot \varepsilon(x)).$$

Suppose now that $1 - (\frac{x}{y^q})^{n-1}) y^{nq} \varepsilon(x) \in \mathrm{U}(B)$ (this is the case if $n > 1$ or if $n = 1$ and $q > 0$). We then have $(0, \frac{x}{y^q}) \in A$ for the same reason as above. Let x_1, y_1 be the images of $X, Y \in S$ in A, respectively. Hence $x_1 = (X, x)$ and $y_1 = (0, y)$, so that

$$z^2 - y_1^{2(\ell - q)}, \quad y_1(x_1 - y_1^q z) = 0, \quad \text{and} \quad z(x_1 - y_1^q z) = 0.$$

Let $\psi : k[[X, Y, Z]] \to A$ be the k-algebra map defined by $\psi(X) = x_1$, $\psi(Y) = y_1$, $\psi(Z) = z$. Then $\operatorname{Ker} \psi \supseteq (Y, Z) \cap (Z^2 - Y^{2(\ell - q)}, X - Y^q Z)$, and by the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & k[[X]] \oplus B & \longrightarrow & D & \longrightarrow & 0 \\
 & & \uparrow \bar{\psi} & & \uparrow \cong & & & & \\
0 & \longrightarrow & \frac{k[[X,Y,Z]]}{(Y,Z) \cap (Z^2 - Y^{2(\ell-q)}, X - Y^q Z)} & \longrightarrow & \frac{k[[X,Y,Z]]}{(Y,Z)} \oplus \frac{k[[X,Y,Z]]}{(Z^2 - Y^{2(\ell-q)}, X - Y^q Z)} & \longrightarrow & \frac{k[[X,Y,Z]]}{(X,Y,Z)} & \longrightarrow & 0
\end{array}$$

we get

$$A \cong k[[X, Y, Z]]/(Y, Z) \cap (X - Y^q Z, Z^2 - Y^{2(\ell - q)}).$$

Notice that

$$(Y, Z) \cap (Z^2 - Y^{2\ell - q}, X - Y^q Z) = \mathrm{I}_2 \begin{pmatrix} Y & Z & 0 \\ Z & Y^{2\ell - q) - 1} & X - Y^q Z \end{pmatrix} = \mathrm{I}_2 \begin{pmatrix} Y & Z & 0 \\ Z + Y & Z + Y^{2(\ell - q) - 1} & X - Y^q Z \end{pmatrix}.$$

If $\ell - q = 1$, then $Z, Y^{2\ell - q) - 1}, X - Y^q Z$ is a regular system of parameters of $k[[X, Y, Z]]$ and if $\ell - q \ge 2$, then $Z + Y, Y^{2(\ell - q) - 1} + Z, X - Y^q Z$ is a regular system of parameters of $k[[X, Y, Z]]$, so that A is an almost Gorenstein local ring in any case.

If $n = 1$ and $q = 0$, then $R \subsetneq A \subsetneq k[[X]] \oplus T$, so that $\ell_R(A/R) = 1$, because $\ell_R((k[[X]] \oplus T)/R) = 2$. Therefore A is an almost Gorenstein local ring by Proposition 1.12.6 (2). This completes the proof of Theorem 1.12.1 as well as the proof of the case (D_n).

Chapter 2

The almost Gorenstein Rees algebras of parameters

2.1 Introduction

This chapter purposes to study the question of when the Rees algebras of given ideals are almost Gorenstein rings. Almost Gorenstein rings are newcomers, which form a class of Cohen-Macaulay rings that are not necessarily Gorenstein but still good, hopefully next to the Gorenstein rings. The notion of this kind of *local* rings dates back to the article [8] of V. Barucci and R. Fröberg in 1997. They introduced almost Gorenstein rings in the case where the local rings are of dimension one and analytically unramified. One can refer to [8] for a beautiful theory of almost symmetric numerical semigroups. Nevertheless, since the notion given by [8] was not flexible for the analysis of analytically ramified case, in 2013 S. Goto, N. Matsuoka and T. T. Phuong [26] extended the notion over arbitrary (but still of dimension one) Cohen-Macaulay local rings. The reader may consult [26] for concrete examples of analytically ramified almost Gorenstein local rings as well as generalizations/repairs of results given in [8]. It was 2015 when S. Goto, R. Takahashi and N. Taniguchi [36] finally gave the definition of almost Gorenstein *graded/local* rings of higher dimension. We recall here the precise definitions which we need throughout this chapter.

Definition 2.1.1. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring possessing the canonical module K_R. Then we say that R is *an almost Gorenstein local ring*, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$, where $\mu_R(C)$ (resp. $\mathrm{e}^0_{\mathfrak{m}}(C)$) stands for the number of elements in a minimal system of generators for C (resp. the multiplicity of C with respect to $\mathfrak{m}$).

Definition 2.1.2. Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay graded ring with R_0 a local ring. Suppose that R possesses the graded canonical module K_R. Then R is called *an almost Gorenstein graded ring*, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $\mathfrak{M}$ is the unique graded maximal ideal of R and $a = \mathrm{a}(R)$ denotes the a-invariant of R. Remember that $\mathrm{K}_R(-a)$ stands for the graded R-module whose underlying R-module is the same as that of K_R and whose grading is given by $[\mathrm{K}_R(-a)]_n = [\mathrm{K}_R]_{n-a}$ for all $n \in \mathbb{Z}$.

Definition 2.1.2 means that if R is an almost Gorenstein graded ring, then even though R is not a Gorenstein ring, R can be embedded into the graded R-module $\mathrm{K}_R(-a)$, so that the difference $\mathrm{K}_R(-a)/R$ is a graded Ulrich R-module (see [10], [36, Section 2]) and behaves well. The reader may consult [36] about a basic theory of almost Gorenstein graded/local rings and the relation between the graded theory and the local theory. For instance, it is shown in [36] that certain Cohen-Macaulay local rings of finite Cohen-Macaulay representation type, including two-dimensional rational singularities, are almost Gorenstein local rings. The almost Gorenstein local rings which are not Gorenstein are G-regular ([36, Corollary 4.5]) in the sense of [71] and they are now getting revealed to enjoy good properties. However, in order to develop a more theory, it is still required to find more examples of almost Gorenstein graded/local rings. This observation has strongly motivated the present research.

On the other hand, as for the Rees algebras we nowadays have a satisfactorily developed theory about the Cohen-Macaulay property (see, e.g., [33, 46, 55, 66]). Among them Gorenstein Rees algebras are rather rare ([49]). Nevertheless, as is shown in [34], some of the non-Gorenstein Cohen-Macaulay Rees algebras can be almost Gorenstein graded rings, which we are eager to report also in this chapter.

Let us now state our results, explaining how this chapter is organized. Throughout this chapter let $(R, \mathfrak{m})$ be a Gorenstein local ring with $d = \dim R$. For each ideal I in R let $\mathcal{R}(I) = R[It]$ (t denotes an indeterminate over R) be the Rees algebra of I. We set $\mathcal{R} = \mathcal{R}(I)$ and $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$. We are mainly interested in the almost Gorenstein property of $\mathcal{R}$ and $\mathcal{R}_\mathfrak{M}$ in the following two cases. The first one is the case where $I = Q$ is generated by a part $a_1, a_2, \ldots, a_r$ of a system of parameters for R. The second one is the case where $I = Q : \mathfrak{m}$, that is I is the socle ideal of a full parameter ideal Q of R. In Section 2.2 we study the first case. We will show that $\mathcal{R}_\mathfrak{M}$ is an almost Gorenstein local ring if and only if R is a regular local ring, provided $Q = (a_1, a_2, \ldots, a_r)$ with $r = \mu_R(Q) \geq 3$ (Theorem 2.2.7). The result on the almost Gorensteinness in the ring $\mathcal{R}$ is stated as follows, which is a generalization of [36, Theorem 8.3].

Theorem 2.1.3 (Theorem 2.2.8)**.** *Let R be a Gorenstein local ring, $a_1, a_2, \ldots, a_r$ ($r \geq 3$) a subsystem of parameters for R and set $Q = (a_1, a_2, \ldots, a_r)$. Then the following conditions are equivalent.*

(1) *$\mathcal{R}(Q)$ is an almost Gorenstein graded ring.*

(2) *R is a regular local ring and $a_1, a_2, \ldots, a_r$ form a part of a regular system of parameters for R.*

In Section 2.3 we shall study the second case where $I = Q : \mathfrak{m}$ is the socle ideal of a parameter ideal Q in a regular local ring R. The reader may consult [34] for the case where $\dim R = 2$ and in the present chapter we focus our attention on the case where $\dim R \geq 3$. Then somewhat surprisingly we have the following.

Theorem 2.1.4 (Theorem 2.3.6)**.** *Let $(R, \mathfrak{m})$ be a regular local ring with $d = \dim R \geq 3$ and infinite residue class field. Let Q be a parameter ideal of R such that $Q \neq \mathfrak{m}$ and set $I = Q : \mathfrak{m}$. Then the following conditions are equivalent.*

(1) *$\mathcal{R}(I)$ is an almost Gorenstein graded ring.*

(2) *Either $I = \mathfrak{m}$, or $d = 3$ and $I = (x) + \mathfrak{m}^2$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

Theorems 2.1.3 and 2.1.4 might suggest that when $\dim R \geq 3$, except the case where $I = Q$ the Rees algebras which are almost Gorenstein graded rings are rather rare. We shall continue the quest also in the future to get more evidence.

In what follows, unless otherwise specified, let R stand for a Noetherian local ring with maximal ideal $\mathfrak{m}$. For each finitely generated R-module M let $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators of M (resp. the length of M). We denote by $\mathrm{e}^0_\mathfrak{m}(M)$ the multiplicity of M with respect to $\mathfrak{m}$. Let K_R denote the canonical module of R.

2.2 The case where the ideals are generated by a subsystem of parameters

Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $d = \dim R \geq 3$ and let $a_1, a_2, \ldots, a_r$ $(r \geq 3)$ be a subsystem of parameters for R. We set $Q = (a_1, a_2, \ldots, a_r)$. Let

$$\mathcal{R} = \mathcal{R}(Q) = R[Qt] \subseteq R[t]$$

denote the Rees algebra of Q and set $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$, where t is an indeterminate over R. Remember that $\mathrm{a}(\mathcal{R}) = -1$. In this section we study the almost Gorenstein property of $\mathcal{R}$ and $\mathcal{R}_{\mathfrak{M}}$. To do this we need some machinery.

Let $S = R[X_1, X_2, \ldots, X_r]$ be the polynomial ring over R. We consider S as a graded ring with $\deg X_i = 1$ for each $1 \leq i \leq r$ and set $\mathfrak{N} = \mathfrak{m}S + S_+$. Let $\Psi : S \longrightarrow \mathcal{R}$ be the R-algebra map defined by $\Psi(X_i) = a_i t$ for $1 \leq i \leq r$. We set

$$\mathbb{A} = \begin{pmatrix} X_1 & X_2 & \cdots & X_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}.$$

Then $\operatorname{Ker} \Psi$ is generated by 2×2 minors of the matrix $\mathbb{A}$, that is

$$\operatorname{Ker} \Psi = \mathbf{I}_2 \begin{pmatrix} X_1 & X_2 & \cdots & X_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix},$$

which is a perfect ideal of S with grade $r-1$. Let

$$\mathcal{C}_\bullet \ : \ 0 \to C_{r-1} \overset{d_{r-1}}{\to} C_{r-2} \to \cdots \to C_1 \to C_0$$

be the Eagon-Northcott complex associated with the matrix $\mathbb{A}$ ([17]). Since we are strongly interested in the form of the matrix corresponding to the differentiation $C_{r-1} \overset{d_{r-1}}{\to} C_{r-2}$, let us briefly remind the reader about the construction of the complex.

Now let L be a finitely generated free S-module of rank r with basis $\{T_i\}_{1 \leq i \leq r}$. We denote by $K = \Lambda L$ the exterior algebra of L over S and let $\mathbb{K}_\bullet(X_1, X_2, \ldots, X_r; S)$ (resp. $\mathbb{K}_\bullet(a_1, a_2, \ldots, a_r; S)$) be the Koszul complex of S generated by $X_1, X_2, \ldots, X_r$ (resp. $a_1, a_2, \ldots, a_r$) with differentiations ∂_1 (resp. ∂_2). Let $U = S[Y_1, Y_2]$ be the polynomial ring with two indeterminates Y_1, Y_2 over S. We set $C_0 = S$ and $C_n = K_{n+1} \otimes_S U_{n-1}$ for each $1 \leq n \leq r-1$. Hence C_n is a finitely generated free S-module with free basis

$$\{T_{i_1} T_{i_2} \cdots T_{i_{n+1}} \otimes Y_1^{\nu_1} Y_2^{\nu_2} \mid 1 \leq i_1 < i_2 < \cdots < i_{n+1} \leq r, \nu_1 + \nu_2 = n-1\}.$$

We consider C_n to be a graded S-module so that

$$\deg(T_{i_1}T_{i_2}\dots T_{i_{n+1}} \otimes Y_1^{\nu_1}Y_2^{\nu_2}) = \nu_1 + 1.$$

Then the Eagon-Northcott complex

$$\mathcal{C}_\bullet \quad : \quad 0 \to C_{r-1} \to C_{r-2} \to \cdots \to C_1 \to C_0 \to 0$$

associated with $\mathbb{A}$ is defined to be a complex of graded free S-modules with differentiations

$$d_n(T_{i_1}T_{i_2}\cdots T_{i_{n+1}} \otimes Y_1^{\nu_1}Y_2^{\nu_2}) = \sum_{j=1,2 \text{ and } \nu_j>0} \partial_j(T_{i_1}T_{i_2}\cdots T_{i_{n+1}}) \otimes Y_1^{\nu_1}\cdots Y_j^{\nu_j-1}\cdots Y_2^{\nu_2}$$

for $n \geq 2$ and

$$d_1(T_{i_1}T_{i_2} \otimes 1) = \det\begin{pmatrix} X_{i_1} & X_{i_2} \\ a_{i_1} & a_{i_2}\end{pmatrix}.$$

Hence $d_1(C_1) = \mathbf{I}_2(\mathbb{A}) \subseteq S$. The complex $C_\bullet$ is acyclic and gives rise to a graded minimal S-free resolution of $\mathcal{R}$, since $\mathbf{I}_2(\mathbb{A})$ is perfect of grade $r-1$ and $X_i, a_i \in \mathfrak{N} = \mathfrak{m}S + S_+$ for all $1 \leq i \leq r$ (cf. [17]).

Let $\mathbb{M}$ denote the matrix of the differentiation $C_{r-1} \xrightarrow{d_{r-1}} C_{r-2}$ with respect to the free basis $\{T_1T_2\cdots T_r \otimes Y_1^iY_2^{r-2-i}\}_{0\leq i\leq r-2}$ and $\{T_1\cdots \overset{\vee}{T_j}\cdots T_r \otimes Y_1^kY_2^{r-3-k}\}_{1\leq j\leq r,\ 0\leq k\leq r-3}$ of C_{r-1} and C_{r-2}, respectively. Then a standard computation gives the following.

Proposition 2.2.1.

$$ {}^t\mathbb{M} = \begin{pmatrix} a_1 \ -a_2 \cdots (-1)^{r+1}a_r & 0 & & & \\ X_1 \ -X_2 \cdots (-1)^{r+1}X_r & a_1 \ -a_2 \cdots (-1)^{r+1}a_r & & & \\ & & \ddots & & \\ & & & X_1 \ -X_2 \cdots (-1)^{r+1}X_r & a_1 \ -a_2 \cdots (-1)^{r+1}a_r \\ & & & 0 & X_1 \ -X_2 \cdots (-1)^{r+1}X_r \end{pmatrix}.$$

We take the $S(-r)$-dual of the resolution $\mathcal{C}_\bullet$ to get the following presentation of the graded canonical module $\mathrm{K}_{\mathcal{R}}$ of $\mathcal{R}$, where $\bigoplus_{i=1}^{r-2} S\left(-(i+1)\right)^{\oplus r}$ and $\bigoplus_{i=1}^{r-1} S(-i)$ consist of column vectors, say $\bigoplus_{i=1}^{r-1} S(-i) = {}^t\left[S(-(r-1)) \oplus \cdots \oplus S(-2) \oplus S(-1)\right]$ and $\bigoplus_{i=1}^{r-2} S\left(-(i+1)\right)^{\oplus r} = {}^t\left[S\left(-(r-1)\right)^{\oplus r} \oplus \cdots S\left(-3\right)^{\oplus r} \oplus S\left(-2\right)^{\oplus r}\right]$.

Corollary 2.2.2.

$$\bigoplus_{i=1}^{r-2} S\left(-(i+1)\right)^{\oplus r} \xrightarrow{{}^t\mathbb{M}} \bigoplus_{i=1}^{r-1} S(-i) \xrightarrow{\varepsilon} \mathrm{K}_{\mathcal{R}} \to 0.$$

Hence $\mathrm{r}(\mathcal{R}) = r - 1 \geq 2$, where $\mathrm{r}(\mathcal{R})$ denotes the Cohen-Macaulay type of $\mathcal{R}$.

For each graded S-module M and $q \in \mathbb{Z}$ we denote by $M^{(q)} = \sum_{n \in \mathbb{Z}} M_{nq}$ the Veronesean submodule of M with degree q. Remember that $M^{(q)}$ is a graded $S^{(q)}$-module whose grading is given by $[M^{(q)}]_n = M_{nq}$ for $n \in \mathbb{Z}$. We then have the following. This might be known (see, e.g., [24]). Let us note a brief proof in our context.

Proposition 2.2.3. *$\mathcal{R}(Q^{r-1})$ is a Gorenstein ring.*

Proof. Notice that $\mathcal{R}(Q^{r-1}) = \mathcal{R}^{(r-1)}$. Let $\eta = \varepsilon(\mathbf{f}) \in [\mathrm{K}_{\mathcal{R}}]_{r-1}$ in the presentation given by Corollary 2.2.2 where $\mathbf{f} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \bigoplus_{i=1}^{r-1} S(-i)$, and set $D = \mathrm{K}_{\mathcal{R}}/\mathcal{R}\eta$. Then $D_0 = (0)$, since $[\mathrm{K}_{\mathcal{R}}]_{r-1} = R\eta$ and we get by Proposition 2.2.1 the isomorphism

$$D/\mathfrak{m}D \cong \bigoplus_{i=1}^{r-2} [S/\mathfrak{N}]\,(-i)$$

of graded S-modules, which shows that $\dim_{\mathcal{R}_{\mathfrak{M}}} D_{\mathfrak{M}} \leq d$ and that $D^{(r-1)} = (0)$, because

$$D^{(r-1)}/\mathfrak{m}D^{(r-1)} = [D/\mathfrak{m}D]^{(r-1)} = (0).$$

We now consider the exact sequence

$$(E_{r-1}) \qquad \mathcal{R} \xrightarrow{\psi} \mathrm{K}_{\mathcal{R}}(r-1) \to D \to 0$$

of graded $\mathcal{R}$-modules, where $\psi(1) = \eta$. Then the homomorphism ψ is injective by [36, Lemma 3.1 (1)], so that applying the functor $[\ *\ ]^{(r-1)}$ to sequence (E_{r-1}), we get the isomorphism

$$\mathcal{R}^{(r-1)} \cong [\mathrm{K}_{\mathcal{R}}]^{(r-1)}\,(-1)$$

of graded $\mathcal{R}^{(r-1)}$-modules. Thus $\mathcal{R}(Q^{r-1}) = \mathcal{R}^{(r-1)}$ is a Gorenstein ring, because $[\mathrm{K}_{\mathcal{R}}]^{(r-1)} \cong \mathrm{K}_{\mathcal{R}^{(r-1)}}$ (cf. [40]). $\square$

Before going ahead, let us discuss a little bit more about the presentation

$$\bigoplus_{i=1}^{r-2} S(-(i+1))^{\oplus r} \xrightarrow{{}^t\mathbb{M}} \bigoplus_{i=1}^{r-1} S(-i) \xrightarrow{\varepsilon} \mathrm{K}_{\mathcal{R}} \to 0$$

in Corollary 2.2.2 of the graded canonical module $\mathrm{K}_{\mathcal{R}}$ of $\mathcal{R}$. We set $\xi = \varepsilon(\mathbf{e}) \in [\mathrm{K}_{\mathcal{R}}]_1$ where $\mathbf{e} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \bigoplus_{i=1}^{r-1} S(-i)$, whence $[\mathrm{K}_{\mathcal{R}}]_1 = R\xi$. We set $C = \mathrm{K}_{\mathcal{R}}/\mathcal{R}\xi$. Hence

$$C \cong \operatorname{Coker}\left[\bigoplus_{i=1}^{r-2} S(-(i+1))^{\oplus r} \xrightarrow{\mathbb{N}} \bigoplus_{i=2}^{r-1} S(-i)\right],$$

where $\mathbb{N}$ denotes the matrix obtained from ${}^t\mathbb{M}$ by deleting the bottom row, so that Proposition 2.2.1 gives the following.

Lemma 2.2.4.

$$\begin{aligned} C/S_+C &\cong S/(S_+ + QS) \otimes_S \left[\bigoplus_{i=2}^{r-1} S(-i)\right] \\ &\cong \bigoplus_{i=2}^{r-1} [R/Q](-i) \end{aligned}$$

as graded S-modules, where $R = S/S_+$ is considered trivially to be a graded S-module.

In particular $\dim_{\mathcal{R}_{\mathfrak{M}}} C_{\mathfrak{M}} \leq d$. Therefore by [36, Lemma 3.1 (1)] $\dim_{\mathcal{R}} C = d$ and the homomorphism $\varphi : \mathcal{R} \to \mathrm{K}_{\mathcal{R}}(1)$ defined by $\varphi(1) = \xi$ is injective, so that we get the following.

Corollary 2.2.5. *The sequence*

$$0 \to \mathcal{R} \xrightarrow{\varphi} \mathrm{K}_{\mathcal{R}}(1) \to C \to 0$$

of graded $\mathcal{R}$-modules is exact and $\dim_{\mathcal{R}} C = d$.

We need the following result to prove Theorem 2.2.7 below.

Proposition 2.2.6. *Let $\mathfrak{a}$ be an ideal in a Gorenstein local ring B and suppose that $A = B/\mathfrak{a}$ is an almost Gorenstein local ring. If A is not a Gorenstein ring but* $\mathrm{pd}_B A < \infty$*, then B is a regular local ring.*

Proof. Enlarging it if necessary, we may assume the residue class field of B to be infinite. We choose an exact sequence

$$0 \to A \to \mathrm{K}_A \to C \to 0$$

of A-modules so that $C \neq (0)$ and C is an Ulrich A-module. Then $\mathrm{pd}_B \mathrm{K}_A < \infty$, because B is a Gorenstein ring and $\mathrm{pd}_B A < \infty$. Hence $\mathrm{pd}_B C < \infty$. We take an A-regular sequence $f_1, f_2, \ldots, f_{d-1} \in \mathfrak{n}$ ($d = \dim A$) such that $\mathfrak{n}C = (f_1, f_2, \ldots, f_{d-1})C$ (this choice is possible; see [36, Proposition 2.2 (2)]) and set $\mathfrak{b} = (f_1, f_2, \ldots, f_{d-1})$. Then by [36, Proof of Theorem 3.7] we get an exact sequence

$$0 \to A/\mathfrak{b}A \to \mathrm{K}_A/\mathfrak{b}\mathrm{K}_A \to C/\mathfrak{b}C \to 0,$$

whence B is a regular local ring, because $\mathrm{pd}_B C/\mathfrak{b}C < \infty$ and $C/\mathfrak{b}C$ ($\neq (0)$) is a vector space over $B/\mathfrak{n}$. $\square$

Theorem 2.2.7. *The following conditions are equivalent.*

(1) $\mathcal{R}_{\mathfrak{M}}$ *is an almost Gorenstein local ring.*

(2) R *is a regular local ring.*

Proof. (1) ⇒ (2) This readily follows from Proposition 2.2.6. Remember that $\mathcal{R}$ is a perfect S-module.

(2) ⇒ (1) We maintain the same notaion as in Lemma 2.2.4. Then

$$C/\mathfrak{m}C \cong (S/\mathfrak{N})^{\oplus(r-2)}$$

by Lemma 2.2.4, whence $\mathfrak{M}C = \mathfrak{m}C$. Therefore C is a graded Ulrich $\mathcal{R}$-module, because $\dim_{\mathcal{R}} C = d$ (cf. Corollary 2.2.5) and $\mathfrak{m}$ is generated by d elements. Thus the exact sequence

$$0 \to \mathcal{R}_{\mathfrak{M}} \xrightarrow{\mathcal{R}_{\mathfrak{M}} \otimes \varphi} [\mathrm{K}_{\mathcal{R}}]_{\mathfrak{M}} \to C_{\mathfrak{M}} \to 0$$

derived from the sequence in Corollary 2.2.5 guarantees that $\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring, because $\mathrm{K}_{\mathcal{R}_{\mathfrak{M}}} = [\mathrm{K}_{\mathcal{R}}]_{\mathfrak{M}}$. □

We are now in a position to study the question of when the Rees algebra $\mathcal{R}(Q)$ is an almost Gorenstein graded ring. Our answer is the following.

Theorem 2.2.8. *The following conditions are equivalent.*

(1) $\mathcal{R}$ *is an almost Gorenstein graded ring.*

(2) R *is a regular local ring and* $a_1, a_2, \ldots, a_r$ *form a part of a regular system of parameters for* R.

Proof. (2) ⇒ (1) We maintain the same notation as in Lemma 2.2.4. Firstly choose elements $y_1, y_2, \ldots, y_{d-r} \in \mathfrak{m}$ so that $\mathfrak{m} = Q + \mathfrak{a}$, where $\mathfrak{a} = (y_1, y_2, \ldots, y_{d-r})$. We then have by Lemma 2.2.4

$$C/(S_+ + \mathfrak{a}S)C \cong \bigoplus_{i=2}^{r-1} [R/\mathfrak{m}](-i),$$

so that $\mathfrak{N}\cdot [(C/(S_+ + \mathfrak{a}S)C] = (0)$. Therefore C is a graded Ulrich $\mathcal{R}$-module, whence $\mathcal{R}$ is an almost Gorenstein graded ring by Corollary 2.2.5.

(1) ⇒ (2) Suppose that $\mathcal{R}$ is an almost Gorenstein graded ring and consider the exact sequence

$$0 \longrightarrow \mathcal{R} \xrightarrow{\phi} \mathrm{K}_{\mathcal{R}}(1) \longrightarrow C \longrightarrow 0$$

of graded $\mathcal{R}$-modules such that $\mu_{\mathcal{R}}(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$. We set $\rho = \phi(1)$. Then since $\mathrm{r}(\mathcal{R}) = r-1 \geq 2$, we have $\rho = \phi(1) \notin \mathfrak{m}{\cdot}[\mathrm{K}_{\mathcal{R}}]_1$ by [36, Corporally 3.10]. Hence $[\mathrm{K}_{\mathcal{R}}]_1 = R\rho$ (remember that $[\mathrm{K}_{\mathcal{R}}]_1 \cong R$; see Corollary 2.2.2). Thus $C \neq (0)$, $\dim_{\mathcal{R}} C = d$, and $C_n = (0)$ for every $n \leq 1$. Therefore $C = \sum_{i=2}^{r-1} S\xi_i$ with $\xi_i \in C_i$ by Corollary 2.2.2 and hence $Q^{r-2}C = (0)$, because $Q(C/S_+C) = (0)$ by Lemma 2.2.4. We set $\mathfrak{a} = (0) :_S C$ and $\mathfrak{b} = \mathfrak{a} \cap \mathcal{R}$. Hence $Q^{r-2} \subseteq \mathfrak{b} \subseteq Q$ (see Proposition 2.2.4).

Claim. $\mathrm{e}^0_{\mathfrak{M}}(C) = (r-2){\cdot}\mathrm{e}^0_{\mathfrak{m}/Q}(R/Q)$.

Proof of Claim. We may assume the field $R/\mathfrak{m}$ to be infinite. We set $\overline{S} = S/\mathfrak{a}$, $A = [\overline{S}]_0$ $(= R/\mathfrak{b})$, and $\mathfrak{n}$ the maximal ideal of A. Notice that $\dim A = d-r$, since $Q^{r-2} \subseteq \mathfrak{b} \subseteq Q$. Let $B = A[z_1, z_2, \ldots, z_r]$ be the standard graded polynomial ring and let $\psi : B \to \overline{S}$ be the A-algebra map defined by $\psi(z_i) = \overline{X_i}$ for each $1 \leq i \leq r$, where $\overline{X_i}$ denotes the image of X_i in $\overline{S}$. We regard C to be a graded B-module via ψ. Notice that $\dim_B C = \dim B = d$. Let us choose elements $y_1, y_2, \ldots, y_{d-r}$ of $\mathfrak{m}$ so that their images $\{\overline{y_i}\}_{1 \leq i \leq d-r}$ in $A = R/\mathfrak{b}$ generate a reduction of $\mathfrak{n}$. Then $(\overline{y_i} \mid 1 \leq i \leq d-r)B + B_+$ is a reduction of the unique graded maximal ideal $\mathfrak{n}B + B_+$ of B, while the images of $\{y_i\}_{1 \leq i \leq d-r}$ in R/Q generate a reduction of the maximal ideal $\mathfrak{m}/Q$ of R/Q, since R/Q is a homomorphic image of $A = R/\mathfrak{b}$. Hence setting $\mathfrak{N}_B = \mathfrak{n}B + B_+$, we get

$$\begin{aligned}
\mathrm{e}^0_{\mathfrak{M}}(C) &= \mathrm{e}^0_{\mathfrak{N}_B}(C) \\
&= \ell_B(C/\left[(\overline{y_i} \mid 1 \leq i \leq d-r)B + B_+\right]C) \\
&= \ell_S(C/\left[(y_i \mid 1 \leq i \leq d-r)S + S_+\right]C) \\
&= (r-2){\cdot}\ell_R(R/[Q + (y_i \mid 1 \leq i \leq d-r)]) \quad \text{(by Lemma 2.2.4)} \\
&= (r-2){\cdot}\mathrm{e}^0_{\mathfrak{m}/Q}(R/Q)
\end{aligned}$$

as claimed. □

Since $\mathcal{R}$ is an almost Gorenstein graded ring with $\mathrm{r}(\mathcal{R}) = r-1 \geq 2$, we have $\mathrm{e}^0_{\mathfrak{M}}(C) = r-2$ by [36, Corollary 3.10], so that $\mathrm{e}^0_{\mathfrak{m}/Q}(R/Q) = 1$ by the above claim. Thus R is a regular local ring and $a_1, a_2, \ldots, a_r$ form a part of a regular system of parameters for R. □

Remark 2.2.9. Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay graded ring such that R_0 is a local ring. Assume that R possesses the graded canonical module K_R and let $\mathfrak{M}$ denote the graded maximal ideal of R. Then because $\mathrm{K}_{R_{\mathfrak{M}}} = [\mathrm{K}_R]_{\mathfrak{M}}$, $R_{\mathfrak{M}}$ is by definition an almost Gorenstein local ring, once R is an almost Gorenstein graded ring. Theorems 2.2.7 and 2.2.8 show that the converse is not true in general. This phenomenon is already recognized by [36, Example 8.8]. See [36, Section 11] for the interplay between the graded theory and the local theory.

Before closing this section, let us discuss a bit about the case where $r = 2$.

Proposition 2.2.10. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and let a, b be a subsystem of parameters for R. We set $Q = (a, b)$, $\mathcal{R} = \mathcal{R}(Q)$, and $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$. If $\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring, then R is a Gorenstein ring, so that $\mathcal{R}$ is a Gorenstein ring.*

Proof. Let $S = R[x, y]$ be the polynomial ring over R and consider the R-algebra map $\Psi : S \to \mathcal{R}$ defined by $\Psi(x) = at$, $\Psi(y) = bt$. Then $\operatorname{Ker} \Psi = (bx - ay)$ and $bx - ay \in \mathfrak{N}^2$, where $\mathfrak{N} = \mathfrak{m}S + S_+$. Therefore since $\mathcal{R}_{\mathfrak{M}} = S_{\mathfrak{N}}/(bx - ay)S_{\mathfrak{N}}$ is an almost Gorenstein local ring, by [36, Theorem 3.7 (1)] $S_{\mathfrak{N}}$ must be a Gorenstein local ring, whence so is R. □

Remark 2.2.11. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and let Q be an ideal of R generated by a subsystem $a_1, a_2, \ldots, a_r$ of parameters for R. We set $\mathcal{R} = \mathcal{R}(Q)$ and $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$. With this setting the authors do not know whether R is necessarily a Gorenstein ring and hence a regular local ring, if $\mathcal{R}$ (resp. $\mathcal{R}_{\mathfrak{M}}$) is an almost Gorenstein graded (resp. local) ring, provided $r \geq 3$.

2.3 The case where the ideals are socle ideals of parameters

In this section we explore the question of when the Rees algebras of socle ideals are almost Gorenstein. In what follows, let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 3$ with infinite residue class field. Let I be an $\mathfrak{m}$-primary ideal of R. We assume that our ideal I contains a parameter ideal $Q = (a_1, a_2, \ldots, a_d)$ of R such that $I^2 = QI$. We set $J = Q : I$, $\mathcal{R} = R[It] \subseteq R[t]$ (t an indeterminate over R), and $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$.

Notice that $\mathcal{R}$ is a Cohen-Macaulay ring ([33]) and $\mathrm{a}(\mathcal{R}) = -1$. We are interested in the question of when $\mathcal{R}$ (resp. $\mathcal{R}_{\mathfrak{M}}$) is an almost Gorenstein graded (resp. local) ring.

Let us note the following.

Theorem 2.3.1 ([75, Theorem 2.7]).

$$\mathrm{K}_{\mathcal{R}}(1) \cong \sum_{i=0}^{d-3} \mathcal{R}{\cdot}t^i + \mathcal{R}{\cdot}Jt^{d-2}$$

as a graded $\mathcal{R}$-module.

As a direct consequence we get the following.

Corollary 2.3.2. $\mathrm{r}(\mathcal{R}) = (d-2) + \mu_R(J/I)$.

Here $\mathrm{r}(\mathcal{R})$ denotes the Cohen-Macaulay type of $\mathcal{R}$. Consequently, $\mathcal{R}$ is a Gorenstein ring if and only if $d = 3$ and $I = J$, that is I is a good ideal in the sense of [25].

Let us begin with the following.

Lemma 2.3.3. *Suppose that $Q \subseteq \mathfrak{m}^2$. Then $\mu_R(\mathfrak{m}Q) = d{\cdot}\mu_R(\mathfrak{m})$.*

Proof. Let $\sigma : \mathfrak{m} \otimes_R Q \to \mathfrak{m}Q$ be the R-linear map defined by $\sigma(x \otimes y) = xy$ for all $x \in \mathfrak{m}$ and $y \in Q$. To see $\mu_R(\mathfrak{m}Q) = d{\cdot}\mu_R(\mathfrak{m})$, it is enough to show that

$$\operatorname{Ker} \sigma \subseteq \mathfrak{m}{\cdot} \left[\mathfrak{m} \otimes_R Q\right].$$

Let $x \in \operatorname{Ker} \sigma$ and write $x = \sum_{i=1}^{d} f_i \otimes a_i$ with $f_i \in \mathfrak{m}$. Then since $\sum_{i=1}^{d} a_i f_i = 0$ and $a_1, a_2, \dots, a_d$ form an R-regular sequence, for each $1 \leq i \leq d$ we have $f_i \in (a_1, \dots, \overset{\vee}{a_i}, \dots, a_d) \subseteq \mathfrak{m}^2$. Hence $x \in \mathfrak{m}{\cdot} \left[\mathfrak{m} \otimes_R Q\right]$ as required. $\square$

Theorem 2.3.4. *If $J = \mathfrak{m}$ and $I \subseteq \mathfrak{m}^2$, then $\mathcal{R}_{\mathfrak{M}}$ is not an almost Gorenstein local ring.*

Proof of Theorem 2.3.4. We set $A = \mathcal{R}_{\mathfrak{M}}$ and suppose that A is an almost Gorenstein local ring. Notice that A is not a Gorenstein ring, since $J \neq I$ (Corollary 2.3.2). We choose an exact sequence

$$0 \to A \xrightarrow{\varphi} \mathrm{K}_A \to C \to 0$$

of A-modules with $C \neq (0)$ and C an Ulrich A-module. Let $\mathfrak{n}$ denote the maximal ideal of A and choose elements $f_1, f_2, \ldots, f_d \in \mathfrak{n}$ so that $\mathfrak{n}C = (f_1, f_2, \ldots, f_d)C$. Let $\xi = \varphi(1)$. Then because $\xi \notin \mathfrak{n}\mathrm{K}_A$ by [36, Corollary 3.10], we get

$$\mu_A(\mathfrak{n}C) \leq d{\cdot}(r-1),$$

where $r = \mathrm{r}(A) = (d-2) + \mu_R(J/I)$ (Corollary 2.3.2). As $\xi \notin \mathfrak{n}\mathrm{K}_A$, we also have the exact sequence

$$0 \to \mathfrak{n}\xi \to \mathfrak{n}\mathrm{K}_A \to \mathfrak{n}C \to 0.$$

Therefore because $\mathrm{K}_A = [\mathrm{K}_{\mathcal{R}}]_{\mathfrak{M}}$, we get the estimation

$$\begin{aligned} \mu_{\mathcal{R}}(\mathfrak{M}\mathrm{K}_{\mathcal{R}}) = \mu_A(\mathfrak{n}\mathrm{K}_A) &\leq \mu_A(\mathfrak{n}C) + \mu_A(\mathfrak{n}) \\ &\leq d{\cdot}\left[(d-2) + \mu_R(J/I) - 1\right] + \left[\mu_R(\mathfrak{m}) + \mu_R(I)\right]. \end{aligned}$$

On the other hand, since $\mathfrak{M} = (\mathfrak{m}, It)\mathcal{R}$ and $\mathrm{K}_{\mathcal{R}}(1) = \displaystyle\sum_{i=0}^{d-3} \mathcal{R}{\cdot}t^i + \mathcal{R}{\cdot}Jt^{d-2}$ by Theorem 2.3.1), it is straightforward to check that

$$\mu_{\mathcal{R}}(\mathfrak{M}\mathrm{K}_{\mathcal{R}}) = (d-2){\cdot}\mu_R(\mathfrak{m}) + \mu_R(I + \mathfrak{m}J) + \mu_R(IJ/I^2).$$

Therefore

$$\left[\mu_R(I + \mathfrak{m}J) + \mu_R(IJ/I^2)\right] - \left[\mu_R(I) + d{\cdot}\mu_R(J/I)\right] \leq (d-3){\cdot}\left[d - \mu_R(\mathfrak{m})\right] \leq 0,$$

whence

$$(*) \qquad \mu_R(I + \mathfrak{m}J) + \mu_R(IJ/I^2) \leq \mu_R(I) + d{\cdot}\mu_R(J/I).$$

We now use the hypothesis that $J = \mathfrak{m}$ and $I \subseteq \mathfrak{m}^2$. Notice that $\mathfrak{m}I = \mathfrak{m}Q$, since $I = Q : \mathfrak{m}$ and Q is a minimal reduction of I. Then by the above estimation $(*)$ we get

$$\mu_R(\mathfrak{m}^2) + \mu_R(\mathfrak{m}Q) \leq \mu_R(I) + d{\cdot}\mu_R(\mathfrak{m}),$$

whence $\mu_R(\mathfrak{m}^2) \leq \mu_R(I)$ by Lemma 2.3.3. Therefore

$$\binom{d+1}{2} \leq \mu_R(\mathfrak{m}^2) \leq \mu_R(I) = d+1,$$

which is impossible, because $d \geq 3$. $\square$

Corollary 2.3.5. *Let Q be a parameter ideal of R such that $Q \subseteq \mathfrak{m}^2$. Then $\mathcal{R}_{\mathfrak{M}}$ is not an almost Gorenstein local ring, where $\mathcal{R} = \mathcal{R}(Q : \mathfrak{m})$ and $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$.*

Proof. Let $I = Q : \mathfrak{m}$. Then $I^2 = QI$ and $I \subseteq \mathfrak{m}^2$ by [78, Theorem 1.1], while $Q : I = Q : (Q : \mathfrak{m}) = \mathfrak{m}$, since R is a Gorenstein ring. $\square$

Let us study the case where R is a regular local ring. The goal is the following.

Theorem 2.3.6. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 3$ with infinite residue class field. Let Q be a parameter ideal of R. Assume that $Q \neq \mathfrak{m}$ and set $I = Q : \mathfrak{m}$. Then the following conditions are equivalent.*

(1) *$\mathcal{R}(I)$ is an almost Gorenstein graded ring.*

(2) *Either $I = \mathfrak{m}$, or $d = 3$ and $I = (x) + \mathfrak{m}^2$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

We divide the proof of Theorem 2.3.6 into several steps. Let us begin with the case where $Q \not\subseteq \mathfrak{m}^2$. Our setting is the following.

Setting 2.3.7. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 3$ with infinite residue class field. We write $\mathfrak{m} = (x_1, x_2, \ldots, x_d)$. Let Q be a parameter ideal of R and let $1 \leq i \leq d-2$ be an integer. For the ideal Q and the integer i we assume that*

$$(x_j \mid 1 \leq j \leq i) \subseteq Q \subseteq (x_j \mid 1 \leq j \leq i) + \mathfrak{m}^2.$$

We set $\mathfrak{a} = (x_j \mid 1 \leq j \leq i)$, $\mathfrak{b} = (x_j \mid i+1 \leq j \leq d)$, and $I = Q : \mathfrak{m}$. Hence $Q = \mathfrak{a} + (a_j \mid i+1 \leq j \leq d)$ with $a_j \in \mathfrak{b}^2$, so that we have the presentation

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ a_{i+1} \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ & 0 & & \alpha_{jk} \\ & & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ x_{i+1} \\ \vdots \\ x_d \end{pmatrix}$$

with $\alpha_{jk} \in \mathfrak{b}$ for every $i+1 \leq j, k \leq d$. Let $\Delta = \det(\alpha_{jk})$. Then$\Delta \in \mathfrak{b}^2$ and $Q : \Delta = \mathfrak{m}$ by [42, Theorem 3.1], whence $I = Q + (\Delta)$. Consequently we have the following. See Corollary 2.3.2 for assertion (4).

Proposition 2.3.8. *The following assertions hold true.*

(1) $I^2 = QI$.

(2) $Q : I = \mathfrak{m}$.

(3) $I \subseteq \mathfrak{a} + \mathfrak{b}^2$.

(4) $\mu_R(\mathfrak{m}/I) = d - i$ *and* $\mathrm{r}(\mathcal{R}) = 2d - (i + 2)$.

Proposition 2.3.9. $\mu_R(\mathfrak{m}Q/I^2) = d(d - i)$.

Proof. Since $\mathfrak{m} = \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \subseteq Q$, we get

$$\mathfrak{m}Q/[I^2 + \mathfrak{m}^2Q] \cong \mathfrak{b}Q/[\mathfrak{b}Q \cap (Q^2 + \mathfrak{m}\mathfrak{b}Q)],$$

while $Q \cap \mathfrak{b} \subseteq \mathfrak{m}\mathfrak{b}$, since $Q = \mathfrak{a} + (a_j \mid i+1 \leq j \leq d)$ and $a_j \in \mathfrak{b}^2$ for every $i+1 \leq j \leq d$. Hence $Q^2 \cap \mathfrak{b}Q \subseteq \mathfrak{m}\mathfrak{b}Q$, so that

$$\mathfrak{b}Q \cap (Q^2 + \mathfrak{m}\mathfrak{b}Q) = \mathfrak{m}\mathfrak{b}Q.$$

Therefore $\mu_R(\mathfrak{m}Q/I^2) = \mu_R(\mathfrak{b}Q)$. The method in the proof of Lemma 2.3.3 works to get $\mu_R(\mathfrak{b}Q) = \mu_R(\mathfrak{b} \otimes_R Q) = d(d - i)$ as claimed. $\square$

The following is the heart of the proof.

Proposition 2.3.10. *Suppose that* $\mathcal{R}(I)$ *is an almost Gorenstein graded ring. Then* $d = 3$ *and* $I = (x_1) + \mathfrak{m}^2$.

Proof. Since $\mathrm{r}(\mathcal{R}) = 2d - (i + 2) \geq 3$ by Proposition 2.3.8 (4), $\mathcal{R}$ is not a Gorenstein ring. We take an exact sequence

$$0 \to \mathcal{R} \xrightarrow{\varphi} \mathrm{K}_{\mathcal{R}}(1) \to C \to 0$$

of graded $\mathcal{R}$-modules so that $C \neq (0)$ and $\mu_{\mathcal{R}}(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$. Since $[\mathrm{K}_{\mathcal{R}}]_1 \cong R$ (see Corollary 2.3.2) and $\xi = \varphi(1) \notin \mathfrak{M}{\cdot}[\mathrm{K}_{\mathcal{R}}(1)]$, ξ is a unit of R. Therefore the isomorphism of Theorem 2.3.1 shows

$$C \cong \left[\sum_{i=1}^{d-3} \mathcal{R}{\cdot}t^i + \mathcal{R}{\cdot}\mathfrak{m}t^{d-2}\right] / \mathcal{R}_+,$$

from which by a direct computation we get the following.

Fact 2.3.11.

$$\mu_{\mathcal{R}}(\mathfrak{M}C) = \begin{cases} \mu_R(\mathfrak{m}^2/I \cap \mathfrak{m}^2) + \mu_R(\mathfrak{m}Q/I^2) & (d = 3), \\ (d-i) + d(d-4) + \mu_R(I + \mathfrak{m}^2) + \mu_R(\mathfrak{m}Q/I^2) & (d \geq 4). \end{cases}$$

On the other hand we have $\mu_{\mathcal{R}_{\mathfrak{M}}}(C_{\mathfrak{M}}) = \mathrm{r}(\mathcal{R}) - 1 = 2d - (i+3)$ by [36, Corollary 3.10]. Consequently

$$\mu_{\mathcal{R}}(\mathfrak{M}C) = \mu_{\mathcal{R}_{\mathfrak{M}}}(\mathfrak{M}C_{\mathfrak{M}}) \leq d\cdot(2d - (i+3)),$$

because $C_{\mathfrak{M}}$ is an Ulrich $\mathcal{R}_{\mathfrak{M}}$-module with $\dim_{\mathcal{R}_{\mathfrak{M}}} C_{\mathfrak{M}} = d$ ([36, Proposition 2.2 (2)]). Assume now that $d \geq 4$. Then by Fact 2.3.11 and Proposition 2.3.9 we have

$$(d-i) + d(d-4) + \mu_R(I + \mathfrak{m}^2) + d(d-i) \leq d(2d - (i+3)),$$

so that $\mu_R(I + \mathfrak{m}^2) \leq i$. This is impossible, because

$$\mu_R(I + \mathfrak{m}^2) = \mu_R(\mathfrak{a} + \mathfrak{m}^2) = i + \binom{d-i+1}{2} > i.$$

Therefore we get $d = 3$ and $i = 1$, whence

$$\mu_R(\mathfrak{m}^2/[I \cap \mathfrak{m}^2]) + \mu_R(\mathfrak{m}Q/I^2) \leq 6,$$

so that we have $\mathfrak{m}^2 = I \cap \mathfrak{m}^2 \subseteq I$, because $\mu_R(\mathfrak{m}Q/I^2) = 6$ by Proposition 2.3.9. Thus $I = (x_1) + \mathfrak{m}^2$ by Proposition 2.3.8 (3). □

We are now in a position to finish the proof of Theorem 2.3.6.

Proof of Theorem 2.3.6. (1) ⇒ (2) If Q is integrally closed in R, then by [21, Theorem 3.1] $Q = (x_1, x_2, \ldots, x_{d-1}, x_d^q)$ for some regular system $\{x_i\}_{1 \leq i \leq d}$ of parameters of R and for some integer $q \geq 1$. Therefore

$$I = Q : \mathfrak{m} = Q : x_d = (x_1, x_2, \ldots, x_{d-1}, x_d^{q-1}),$$

so that we have $q = 2$ by Theorem 2.2.8, that is $I = \mathfrak{m}$. Suppose that Q is not integrally closed in R. Then $Q \nsubseteq \mathfrak{m}^2$ by Corollary 2.3.5. Let $\{x_j\}_{1 \leq j \leq d}$ be a regular system of parameters for R and take the integer $1 \leq i \leq d-2$ so that $(x_j \mid 1 \leq j \leq i) \subseteq Q \subseteq (x_j \mid 1 \leq j \leq i) + \mathfrak{m}^2$ (cf. [21, Theorem 3.1]). We then have $d = 3$ and $I = (x_1) + \mathfrak{m}^2$ by Proposition 2.3.10.

(2) ⇒ (1) This follows from Theorem 2.2.8 and the following proposition. □

Proposition 2.3.12. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 3$ and infinite residue class field. Let Q be a parameter ideal of R. Assume that $Q \neq \mathfrak{m}$ and set $I = Q : \mathfrak{m}$. If $I^2 = QI$ and $\mathfrak{m}^2 \subseteq I$, then $\mathcal{R}(I)$ is an almost Gorenstein graded ring.*

Proof. We have $\mathrm{K}_{\mathcal{R}}(1) = \mathcal{R} + \mathcal{R}{\cdot}\mathfrak{m}t$. Consider the exact sequence

$$0 \to \mathcal{R} \xrightarrow{\varphi} \mathrm{K}_{\mathcal{R}}(1) \to C \to 0$$

of graded $\mathcal{R}$-modules with $\varphi(1) = 1$. Then since $\mathfrak{m}^2 \subseteq I$ and $\mathfrak{m}I = \mathfrak{m}Q$, we readily see

$$\mathfrak{M}\,[\mathcal{R}{\cdot}\mathfrak{m}t] \subseteq \mathcal{R} + Qt\,[\mathcal{R}{\cdot}\mathfrak{m}t]$$

which shows C is a graded Ulrich $\mathcal{R}$-module (see [36, Proposition 2.2 (2)]. Thus $\mathcal{R}$ is an almost Gorenstein graded ring. □

Let us note one example.

Example 2.3.13. Let $R = k[[x, y, z]]$ be the formal power series ring over an infinite field k. We set $\mathfrak{m} = (x, y, z)$, $Q = (x, y^2, z^n)$ with $n \geq 2$, and $I = Q : \mathfrak{m}$. Then $I = (x, y^2, yz^{n-1}, z^n)$ and $I^2 = QI$.

(1) If $n = 2$, then $I = (x) + \mathfrak{m}^2$, so that $\mathcal{R}(I)$ is an almost Gorenstein graded ring.

(2) Suppose $n \geq 3$. Then $I \neq \mathfrak{m}$, $Q \neq \mathfrak{m}$, and $I \neq (f) + \mathfrak{m}^2$ for any $f \in \mathfrak{m} \setminus \mathfrak{m}^2$. Hence $\mathcal{R}(I)$ is not an almost Gorenstein graded ring.

Chapter 3

The almost Gorenstein Rees algebras over two-dimensional regular local rings

3.1 Introduction

The purpose of this chapter is to study the problem of when the Rees algebras of ideals and modules over two-dimensional regular local rings $(R, \mathfrak{m})$ are almost Gorenstein graded rings. Almost Gorenstein rings in our sense are newcomers and different from those rings studied in [47]. They form a new class of Cohen-Macaulay rings, which are not necessarily Gorenstein, but still good, possibly next to the Gorenstein rings. The notion of these local rings dates back to the paper [8] of V. Barucci and R. Fröberg in 1997, where they dealt with one-dimensional analytically unramified local rings and developed a beautiful theory. Because their notion is not flexible enough to analyze analytically ramified rings, in 2013 S. Goto, N. Matsuoka, and T. T. Phuong [26] extended the notion to arbitrary (but still of dimension one) Cohen-Macaulay local rings. The reader may consult [26] for examples of analytically ramified almost Gorenstein local rings. S. Goto, R. Takahashi, and N. Taniguchi [36] finally gave the definition of almost Gorenstein local/graded rings in our sense. Here let us recall it, which we shall utilize throughout this chapter.

Definition 3.1.1 ([36, Definition 3.3])**.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring which possesses the canonical module K_R. Then we say that R is an almost Gorenstein *local* ring, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $\mu_R(C)$ denotes the number of elements in a minimal system of generators of C and $\mathrm{e}^0_{\mathfrak{m}}(C)$ is the multiplicity of C with respect to $\mathfrak{m}$.

Definition 3.1.2 ([36, Definition 8.1])**.** Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay graded ring such that R_0 is a local ring. Suppose that R possesses the graded canonical module K_R. Let $\mathfrak{M}$ be the unique graded maximal ideal of R and $a = \mathrm{a}(R)$ the a-invariant of R. Then we say that R is an almost Gorenstein *graded* ring, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R-modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$, where $\mu_R(C)$ denotes the number of elements in a minimal system of generators of C and $\mathrm{e}^0_{\mathfrak{M}}(C)$ is the multiplicity of C with respect to $\mathfrak{M}$. Here $\mathrm{K}_R(-a)$ stands for the graded R-module whose underlying R-module is the same as that of K_R and whose grading is given by $[\mathrm{K}_R(-a)]_n = [\mathrm{K}_R]_{n-a}$ for all $n \in \mathbb{Z}$.

Definition 3.1.1 (resp. Definition 3.1.2) means that if R is an almost Gorenstein local (resp. graded) ring, then even though R is not a Gorenstein ring, R can be embedded into the canonical module K_R (resp. $\mathrm{K}_R(-a)$), so that the difference K_R/R (resp. $\mathrm{K}_R(-a)/R$) is an Ulrich R-module ([10]) and behaves well. The reader may consult [36] about the basic theory of almost Gorenstein local/graded rings and the relation between the graded theory and the local theory, as well.

It is shown in [36] that every two-dimensional rational singularity is an almost Gorenstein local ring and all the known examples of Cohen-Macaulay local rings of finite Cohen-Macaulay representation type are almost Gorenstein local rings. The almost Gorenstein local rings which are not Gorenstein are G-regular ([36, Corollary 4.5]) in the sense of [71], that is every totally reflexive module is free, so that the Gorenstein dimension of a finitely generated module is equal to its projective dimension, while over Gorenstein local rings the totally reflexive modules are exactly the maximal Cohen-Macaulay modules. The local rings $R_{\mathfrak{p}}$ ($\mathfrak{p} \in \operatorname{Spec} R$) of almost Gorenstein local rings R are not necessarily almost Gorenstein (see [36, Remark 9.12] for a counterexample). These are particular discrepancies between Gorenstein local rings and almost Gorenstein local rings.

In this chapter we are interested in the almost Gorenstein property of Rees algebras and our main result is sated as follows.

Theorem 3.1.3. *Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue class field and I an $\mathfrak{m}$-primary integrally closed ideal in R. Then the Rees algebra $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ of I is an almost Gorenstein graded ring.*

As a direct consequence we have the following.

Corollary 3.1.4. *Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue class field. Then $\mathcal{R}(\mathfrak{m}^\ell)$ is an almost Gorenstein graded ring for every integer $\ell > 0$.*

The proof of Theorem 3.1.3 depends on a result of J. Verma [76] which guarantees the existence of joint reductions with joint reduction number zero. Therefore our method of proof works also for two-dimensional rational singularities, which we shall discuss in the forthcoming paper [39].

Possessing in [33] one of its roots, the theory of Rees algebras has been satisfactorily developed and nowadays one knows many Cohen-Macaulay Rees algebras (see, e.g. [46, 55, 66]). Among them Gorenstein Rees algebras are rather rare ([49]). Nevertheless, although they are not Gorenstein, some of Cohen-Macaulay Rees algebras are still good and could be *almost Gorenstein graded* rings, which we would like to report in this chapter and also in the forthcoming papers [34, 35]. Except [36, Theorems 8.2, 8.3] our Theorem 3.1.3 is the first attempt to answer the question of when the Rees algebras are almost Gorenstein graded rings.

We now briefly explain how this chapter is organized. The proof of Theorem 3.1.3 shall be given in Section 3.2. For the Rees algebras of modules over two-dimensional regular local rings we have a similar result, which we give in Section 3.2 (Corollary 3.2.7). In Section 3.3 we explore the case where the ideals are linearly presented over power series rings. The result (Theorem 3.3.1) seems to suggest that almost Gorenstein Rees algebras are still rather rare, when the dimension of base rings is greater than two, which we shall discuss also in the forthcoming paper [35]. In Section 3.4 we explore the Rees algebra of the socle ideal $I = Q : \mathfrak{m}$, where Q is a parameter ideal in a two-dimensional regular local ring $(R, \mathfrak{m})$, and show that the Rees algebra $\mathcal{R}(I)$ is an almost Gorenstein graded ring if and only if the order of Q is at most two (Theorem 3.4.1).

We should note here that for every almost Gorenstein graded ring R with graded maximal ideal $\mathfrak{M}$ the local ring $R_{\mathfrak{M}}$ of R at $\mathfrak{M}$ is by definition an almost Gorenstein local ring, because $[\mathrm{K}_R]_{\mathfrak{M}} \cong \mathrm{K}_{R_{\mathfrak{M}}}$. The converse is not true in general. The typical examples are the Rees algebras $\mathcal{R}(Q)$ of parameter ideals Q in a regular local ring $(R, \mathfrak{m})$ with $\dim R \geq 3$. For this algebra $\mathcal{R} = \mathcal{R}(Q)$ the local ring $\mathcal{R}_{\mathfrak{M}}$ is always an almost Gorenstein *local* ring ([35, Theorem 2.7]) but $\mathcal{R}$ is an almost Gorenstein *graded* ring if and only if $Q = \mathfrak{m}$ ([36, Theorem 8.3]). On the other hand the converse is also true in certain special cases like Theorems 3.3.1 and 3.4.1 of the present chapter. These facts seem to suggest the property of being an almost Gorenstein graded ring is a rather rigid condition for Rees algebras.

In what follows, unless otherwise specified, let $(R, \mathfrak{m})$ denote a Cohen-Macaulay local ring. For each finitely generated R-module M let $\mu_R(M)$ (resp. $\mathrm{e}^0_{\mathfrak{m}}(M)$) denote the number of elements in a minimal system of generators for M (resp. the multiplicity of M with respect to $\mathfrak{m}$). Let K_R stand for the canonical module of R.

3.2 Proof of Theorem 3.1.3

The purpose of this section is to prove Theorem 3.1.3. Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 2$ and let $I \subsetneq R$ be an $\mathfrak{m}$-primary ideal of R. Assume that I contains a parameter ideal $Q = (a, b)$ of R such that $I^2 = QI$. We set $J = Q : I$. Let

$$\mathcal{R} = R[It] \subseteq R[t] \quad \text{and} \quad T = R[Qt] \subseteq R[t],$$

where t stands for an indeterminate over R. Remember that the Rees algebra $\mathcal{R}$ of I is a Cohen-Macaulay ring ([33]) with $\mathrm{a}(\mathcal{R}) = -1$ and $\mathcal{R} = T + T{\cdot}It$, while the Rees algebra T of Q is a Gorenstein ring of dimension 3 and $\mathrm{a}(T) = -1$ (remember that $T \cong R[x, y]/(bx - ay)$). Hence $\mathrm{K}_T(1) \cong T$ as a graded T-module, where K_T denotes the graded canonical module of T.

Let us begin with the following, which is a special case of [75, Theorem 2.7 (a)]. We note a brief proof.

Proposition 3.2.1. $\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$ *as a graded* $\mathcal{R}$*-module.*

Proof. Since $\mathcal{R}$ is a module-finite extension of T, we get

$$\mathrm{K}_{\mathcal{R}}(1) \cong \mathrm{Hom}_T(\mathcal{R}, \mathrm{K}_T)(1) \cong \mathrm{Hom}_T(\mathcal{R}, T) \cong T :_F \mathcal{R}$$

as graded $\mathcal{R}$-modules, where $F = \mathrm{Q}(T) = \mathrm{Q}(\mathcal{R})$ is the total ring of fractions. Therefore $T :_F \mathcal{R} = T :_T It$, since $\mathcal{R} = T + T{\cdot}It$. Because $Q^n \cap [Q^{n+1} : I] = Q^n[Q : I]$ for all $n \geq 0$, we have $T :_T It = JT$. Hence $T :_F \mathcal{R} = JT$, so that $JT = J\mathcal{R}$. Thus $\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$ as a graded $\mathcal{R}$-module. □

Corollary 3.2.2. *The ideal $J = Q : I$ in R is integrally closed, if $\mathcal{R}$ is a normal ring.*

Proof. Since $J\mathcal{R} \cong \mathrm{K}_{\mathcal{R}}(1)$, the ideal $J\mathcal{R}$ of $\mathcal{R}$ is unmixed and of height one. Therefore, if $\mathcal{R}$ is a normal ring, $J\mathcal{R}$ must be integrally closed in $\mathcal{R}$, whence J is integrally closed in R because $\overline{J} \subseteq J\mathcal{R}$, where $\overline{J}$ denotes the integral closure of J. □

Let us give the following criterion for $\mathcal{R}$ to be a special kind of almost Gorenstein graded rings. Notice that Condition (2) in Theorem **??** requires the existence of joint reductions of $\mathfrak{m}$, I, and J with reduction number zero (cf. [76]).

Theorem 3.2.3. *With the same notation as above, set $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$, the graded maximal ideal of $\mathcal{R}$. Then the following conditions are equivalent.*

(1) *There exists an exact sequence*

$$0 \to \mathcal{R} \to \mathrm{K}_{\mathcal{R}}(1) \to C \to 0$$

of graded $\mathcal{R}$-modules such that $\mathfrak{M}C = (\xi,\eta)C$ for some homogeneous elements ξ,η of $\mathfrak{M}$.

(2) *There exist elements $f \in \mathfrak{m}$, $g \in I$, and $h \in J$ such that*

$$IJ = gJ + Ih \quad \text{and} \quad \mathfrak{m}J = fJ + \mathfrak{m}h.$$

When this is the case, $\mathcal{R}$ is an almost Gorenstein graded ring.

Proof. (2) $\Rightarrow$ (1) Notice that $\mathfrak{M}{\cdot}J\mathcal{R} \subseteq (f, gt){\cdot}J\mathcal{R} + \mathcal{R}h$, since $IJ = gJ + Ih$ and $\mathfrak{m}J = fJ + \mathfrak{m}h$. Consider the exact sequence

$$\mathcal{R} \xrightarrow{\varphi} J\mathcal{R} \to C \to 0$$

of graded $\mathcal{R}$-modules where $\varphi(1) = h$. We then have $\mathfrak{M}C = (f, gt)C$, so that $\dim_{\mathcal{R}_{\mathfrak{M}}} C_{\mathfrak{M}} \leq 2$. Hence by [36, Lemma 3.1] the homomorphism φ is injective and $\mathcal{R}$ is an almost Gorenstein graded ring.

(1) $\Rightarrow$ (2) Suppose that $\mathcal{R}$ is a Gorenstein ring. Then $\mu_R(J) = 1$, since $\mathrm{K}_{\mathcal{R}}(1) \cong J\mathcal{R}$. Hence $J = R$ as $\mathfrak{m} \subseteq \sqrt{J}$, so that choosing $h = 1$ and $f = g = 0$, we get $IJ = gJ + Ih$ and $\mathfrak{m}J = fJ + \mathfrak{m}h$.

Suppose that $\mathcal{R}$ is not a Gorenstein ring and consider the exact sequence

$$0 \to \mathcal{R} \xrightarrow{\varphi} J\mathcal{R} \to C \to 0$$

of graded $\mathcal{R}$-modules with $C \neq (0)$ and $\mathfrak{M}C = (\xi,\eta)C$ for some homogeneous elements ξ,η of $\mathfrak{M}$. Hence $\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring in the sense of [36, Definition 3.3]. We set $h = \varphi(1) \in J$, $m = \deg \xi$, and $n = \deg \eta$; hence $C = J\mathcal{R}/\mathcal{R}h$. Remember that $h \notin \mathfrak{m}J$, since $\mathcal{R}_{\mathfrak{M}}$ is not a regular local ring (see [36, Corollary 3.10]). If $\min\{m,n\} > 0$, then $\mathfrak{M}C \subseteq \mathcal{R}_+C$, whence $\mathfrak{m}C_0 = (0)$ (notice that $[\mathcal{R}_+C]_0 = (0)$, as $C = \mathcal{R}C_0$). Therefore $\mathfrak{m}J \subseteq (h)$, so that we have $J = (h) = R$. Thus $\mathcal{R}h = J\mathcal{R}$ and $\mathcal{R}$ is a Gorenstein ring, which is impossible. Assume $m = 0$. If $n = 0$, then $\mathfrak{M}C = \mathfrak{m}C$ since $\xi,\eta \in \mathfrak{m}$, so that

$$C_1 \subseteq \mathcal{R}_+C_0 \subseteq \mathfrak{m}C$$

and therefore $C_1 = (0)$ by Nakayama's lemma. Hence $IJ = Ih$ as $[J\mathcal{R}]_1 = \varphi(\mathcal{R}_1)$, which shows (h) is a reduction of J, so that $(h) = R = J$. Therefore $\mathcal{R}$ is a Gorenstein ring, which is impossible. If $n \geq 2$, then because

$$\mathfrak{M}{\cdot}J\mathcal{R} \subseteq \xi{\cdot}J\mathcal{R} + \eta{\cdot}J\mathcal{R} + \mathcal{R}h,$$

we get $IJ \subseteq \xi IJ + Ih$, whence $IJ = Ih$. This is impossible as we have shown above. Hence $n = 1$. Let us write $\eta = gt$ with $g \in I$ and take $f = \xi$. We then have

$$\mathfrak{M}{\cdot}J\mathcal{R} \subseteq (f, gt){\cdot}J\mathcal{R} + \mathcal{R}h,$$

whence $\mathfrak{m}J \subseteq fJ + Rh$. Because $h \notin \mathfrak{m}J$, we get $\mathfrak{m}J \subseteq fJ + \mathfrak{m}h$, so that $\mathfrak{m}J = fJ + \mathfrak{m}h$, while $IJ = gJ + Ih$, because $IJ \subseteq fIJ + gJ + Ih$. This completes the proof of Theorem 3.2.3. $\square$

Let us explore two examples to show how Theorem 3.2.3 works.

Example 3.2.4. Let $S = k[[x,y,z]]$ be the formal power series ring over an infinite field k. Let $\mathfrak{n} = (x,y,z)$ and choose $f \in \mathfrak{n}^2 \setminus \mathfrak{n}^3$. We set $R = S/(f)$ and $\mathfrak{m} = \mathfrak{n}/(f)$. Then for every integer $\ell > 0$ the Rees algebra $\mathcal{R}(\mathfrak{m}^\ell)$ of $\mathfrak{m}^\ell$ is an almost Gorenstein graded ring and $\mathrm{r}(\mathcal{R}) = 2\ell + 1$, where $\mathrm{r}(\mathcal{R})$ denotes the Cohen-Macaulay type of $\mathcal{R}$.

Proof. Since $\mathrm{e}^0_{\mathfrak{m}}(R) = 2$, we have $\mathfrak{m}^2 = (a,b)\mathfrak{m}$ for some elements $a, b \in \mathfrak{m}$. Let $\ell > 0$ be an integer and set $I = \mathfrak{m}^\ell$ and $Q = (a^\ell, b^\ell)$. We then have $I^2 = QI$ and $Q : I = I$, so that $\mathcal{R} = \mathcal{R}(I)$ is a Cohen-Macaulay ring and $\mathrm{K}_{\mathcal{R}}(1) \cong I\mathcal{R}$ by Proposition 3.2.1, whence $\mathrm{r}(\mathcal{R}) = \mu_R(I) = 2\ell + 1$. Because $\mathfrak{m}^{\ell+1} = a\mathfrak{m}^\ell + b^\ell\mathfrak{m}$ and $Q : I = I = \mathfrak{m}^\ell$, by Theorem 3.2.3 $\mathcal{R}$ is an almost Gorenstein graded ring. $\square$

Example 3.2.5. Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with $\mathfrak{m} = (x, y)$. Let $1 \le m \le n$ be integers and set $I = (x^m) + \mathfrak{m}^n$. Then $\mathcal{R}(I)$ is an almost Gorenstein graded ring.

Proof. We may assume $m > 1$. We set $Q = (x^m, y^n)$ and $J = Q : I$. Then $Q \subseteq I$ and $I^2 = QI$. Since $I = (x^m) + (x^i y^{n-i} \mid 0 \le i \le m-1)$, we get

$$\begin{aligned} J = Q : (x^i y^{n-i} \mid 0 \le i \le m-1) &= \bigcap_{i=1}^{m-1} \left[(x^m, y^n) : x^i y^{n-i}\right] \\ &= \bigcap_{i=1}^{m-1} (x^{m-i}, y^i) \\ &= \mathfrak{m}^{m-1}. \end{aligned}$$

Take $f = x \in \mathfrak{m}$, $g = x^m \in I$, and $h = y^{m-1} \in J = \mathfrak{m}^{m-1}$. We then have $\mathfrak{m}J = fJ + \mathfrak{m}h$ and $IJ = Ih + gJ$, so that by Theorem 3.2.3 $\mathcal{R}(I)$ is an almost Gorenstein graded ring. $\square$

To prove Theorem 3.1.3 we need a result of J. Verma [76] about joint reductions of integrally closed ideals. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let I and J be ideals of R and let $a \in I$ and $b \in J$. Then we say that a, b are a *joint reduction* of I, J if $aJ + Ib$ is a reduction of IJ. Joint reductions always exist (see, e.g., [**?**]) if the residue class field of R is infinite. We furthermore have the following.

Theorem 3.2.6 ([76, Theorem 2.1])**.** *Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring. Let I and J be $\mathfrak{m}$-primary ideals of R. Assume that a, b are a joint reduction of I, J. Then $IJ = aJ + Ib$, if I and J are integrally closed.*

We are now ready to prove Theorem 3.1.3.

Proof of Theorem 3.1.3. Let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue class field and let I be an $\mathfrak{m}$-primary integrally closed ideal in R. We choose a parameter ideal Q of R so that $Q \subseteq I$ and $I^2 = QI$ (this choice is possible; see [81, Appendix 5] or [45]). Therefore the Rees algebra $\mathcal{R} = \mathcal{R}(I)$ is a Cohen-Macaulay ring

([33]). Because $\mathcal{R}$ is a normal ring ([81]), by Corollary 3.2.2 $J = Q : I$ is an integrally closed ideal in R. Consequently, choosing three elements $f \in \mathfrak{m}$, $g \in I$, and $h \in J$ so that f, h are a joint reduction of $\mathfrak{m}, J$ and g, h are a joint reduction of I, J, we readily get by Theorem 3.2.6 the equalities

$$\mathfrak{m}J = fJ + \mathfrak{m}g \quad \text{and} \quad IJ = gJ + Ih$$

stated in Condition (2) of Theorem 3.2.3. Thus $\mathcal{R} = \mathcal{R}(I)$ is an almost Gorenstein graded ring. $\square$

We now explore the almost Gorenstein property of the Rees algebras of modules. To state the result we need additional notation. For the rest of this section let $(R, \mathfrak{m})$ be a two-dimensional regular local ring with infinite residue class field. Let $M \neq (0)$ be a finitely generated torsion-free R-module and assume that M is non-free. Let $(-)^* = \mathrm{Hom}_R(-, R)$. Then $F = M^{**}$ is a finitely generated free R-module and we get a canonical exact sequence

$$0 \to M \xrightarrow{\varphi} F \to C \to 0$$

of R-modules with $C \neq (0)$ and $\ell_R(C) < \infty$. Let $\mathrm{Sym}(M)$ and $\mathrm{Sym}(F)$ denote the symmetric algebras of M and F respectively and let $\mathrm{Sym}(\varphi) : \mathrm{Sym}(M) \to \mathrm{Sym}(F)$ be the homomorphism induced from $\varphi : M \to F$. Then the Rees algebra $\mathcal{R}(M)$ of M is defined by

$$\mathcal{R}(M) = \mathrm{Im}\left[\mathrm{Sym}(M) \xrightarrow{\mathrm{Sym}(\varphi)} \mathrm{Sym}(F)\right]$$

([66]). Hence $\mathcal{R}(M) = \mathrm{Sym}(M)/T$ where $T = t(\mathrm{Sym}(M))$ denotes the R-torsion part of $\mathrm{Sym}(M)$, so that $M = [\mathcal{R}(M)]_1$ is an R-submodule of $\mathcal{R}(M)$. Let $x \in F$. Then we say that x is integral over M, if it satisfies an integral equation

$$x^n + c_1 x^{n-1} + \cdots + c_n = 0$$

in the symmetric algebra $\mathrm{Sym}(F)$ with $n > 0$ and $c_i \in M^i$ for each $1 \leq i \leq n$. Let $\overline{M}$ be the set of elements of F which are integral over M. Then $\overline{M}$ forms an R-submodule of F, which is called the integral closure of M. We say that M is *integrally closed*, if $\overline{M} = M$.

With this notation we have the following.

Corollary 3.2.7. *Let $\mathfrak{M} = \mathfrak{m}\mathcal{R}(M) + \mathcal{R}(M)_+$ be the unique graded maximal ideal of $\mathcal{R}(M)$ and suppose that M is integrally closed. Then $\mathcal{R}(M)_{\mathfrak{M}}$ is an almost Gorenstein local ring in the sense of [36, Definition 3.3].*

Proof. Let $U = R[x_1, x_2, \dots, x_n]$ be the polynomial ring with sufficiently large $n > 0$ and set $S = U_{\mathfrak{m}U}$. We denote by $\mathfrak{n}$ the maximal ideal of S. Then thanks to [66, Theorem 3.5] and [44, Theorem 3.6], we can find some elements $f_1, f_2, \dots, f_{r-1} \in S \otimes_R M$ ($r = \mathrm{rank}_R F$) and an $\mathfrak{n}$-primary integrally closed ideal I in S, so that $f_1, f_2, \dots, f_{r-1}$ form a regular sequence in $\mathcal{R}(S \otimes_R M)$ and

$$\mathcal{R}(S \otimes_R M)/(f_1, f_2, \dots, f_{r-1}) \cong \mathcal{R}(I)$$

as a graded S-algebra. Therefore, because $\mathcal{R}(I)$ is an almost Gorenstein graded ring by Theorem 3.1.3, $S \otimes_R \mathcal{R}(M) = \mathcal{R}(S \otimes_R M)$ is an almost Gorenstein graded ring (cf. [36, Theorem 3.7 (1)]). Consequently $\mathcal{R}(M)_{\mathfrak{M}}$ is an almost Gorenstein local ring by [36, Theorem 3.9]. □

3.3 Almost Gorenstein property in Rees algebras of ideals with linear presentation matrices

Let $R = k[[x_1, x_2, \dots, x_d]]$ ($d \geq 2$) be the formal power series ring over an infinite field k. Let I be a perfect ideal of R with $\mathrm{grade}_R I = 2$, possessing a linear presentation matrix φ

$$0 \to R^{\oplus(n-1)} \xrightarrow{\varphi} R^{\oplus n} \to I \to 0,$$

that is each entry of the matrix φ is contained in $\sum_{i=1}^{d} kx_i$. We set $n = \mu_R(I)$ and $\mathfrak{m} = (x_1, x_2, \dots, x_d)$; hence $I = \mathfrak{m}^{n-1}$ if $d = 2$. In what follows we assume that $n > d$ and that our ideal I satisfies the condition (G_d) of [4], that is $\mu_{R_\mathfrak{p}}(IR_\mathfrak{p}) \leq \dim R_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{V}(I) \setminus \{\mathfrak{m}\}$. Then thanks to [55, Theorem 1.3] and [46, Proposition 2.3], the Rees algebra $\mathcal{R} = \mathcal{R}(I)$ of I is a Cohen-Macaulay ring with $\mathrm{a}(\mathcal{R}) = -1$ and

$$\mathrm{K}_{\mathcal{R}}(1) \cong \mathfrak{m}^{n-d}\mathcal{R}$$

as a graded $\mathcal{R}$-module.

We are interested in the question of when $\mathcal{R}$ is an almost Gorenstein graded ring. Our answer is the following, which suggests that almost Gorenstein Rees algebras might be rare in dimension greater than two.

Theorem 3.3.1. *With the same notation as above, set $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$, the graded maximal ideal of $\mathcal{R}$. Then the following conditions are equivalent.*

(1) *$\mathcal{R}$ is an almost Gorenstein graded ring*

(2) *$\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring*

(3) *$d = 2$.*

Proof. (1) $\Rightarrow$ (2) This follows from the definition, since $[\mathrm{K}_{\mathcal{R}}]_{\mathfrak{M}} \cong \mathrm{K}_{\mathcal{R}_{\mathfrak{M}}}$.

(3) $\Rightarrow$ (1) We have $I = \mathfrak{m}^{n-1}$ since $d = 2$ and so $\mathcal{R}$ is an almost Gorenstein graded ring (Corollary 3.1.4).

(2) $\Rightarrow$ (3) Let$\Delta_i = (-1)^{i+1} \det \varphi_i$ for each $1 \le i \le n$, where φ_i stands for the $(n-1) \times (n-1)$ matrix which is obtained from φ by deleting the i-th row. Hence $I = (\Delta_1, \Delta_2, \dots, \Delta_n)$ and the ideal I has a presentation

$$(P) \qquad 0 \to R^{\oplus(n-1)} \xrightarrow{\varphi} R^{\oplus n} \xrightarrow{[\Delta_1\ \Delta_2\ \cdots\ \Delta_n]} I \to 0.$$

Notice that $\mathcal{R}$ is not a Gorenstein ring, since $\mathrm{r}(\mathcal{R}) = \mu_R(\mathfrak{m}^{n-d}) = \binom{n-1}{d-1} > 1$. We set $A = \mathcal{R}_{\mathfrak{M}}$ and $\mathfrak{n} = \mathfrak{M}A$; hence $\mathrm{K}_A = [\mathrm{K}_{\mathcal{R}}]_{\mathfrak{M}}$. We take an exact sequence

$$0 \to A \xrightarrow{\phi} \mathrm{K}_A \to C \to 0$$

of A-modules such that $C \neq (0)$ and C is an Ulrich A-module. Let $f = \phi(1)$. Then $f \notin \mathfrak{n}\mathrm{K}_A$ by [36, Corollary 3.10] and we get the exact sequence

$$(E) \qquad 0 \to \mathfrak{n}f \to \mathfrak{n}\mathrm{K}_A \to \mathfrak{n}C \to 0.$$

Because $\mathfrak{n}C = (f_1, f_2, \dots, f_d)C$ for some $f_1, f_2, \dots, f_d \in \mathfrak{n}$ ([36, Proposition 2.2]) and $\mu_A(\mathfrak{n}) = d + n$, we get by the exact sequence (E) that

$$\mu_{\mathcal{R}}(\mathfrak{M}\mathrm{K}_{\mathcal{R}}) = \mu_A(\mathfrak{n}\mathrm{K}_A) \le (d+n) + d\cdot(\mathrm{r}(A) - 1) = d\binom{n-1}{d-1} + n,$$

while

$$\mu_{\mathcal{R}}(\mathfrak{M}\mathrm{K}_{\mathcal{R}}) = \mu_R(\mathfrak{m}^{n-d+1}) + \mu_R(\mathfrak{m}^{n-d}I) = \binom{n}{d-1} + \mu_R(\mathfrak{m}^{n-d}I)$$

since $\mathfrak{M} = (\mathfrak{m}, It)\mathcal{R}$ and $\mathrm{K}_{\mathcal{R}}(1) = \mathfrak{m}^{n-d}\mathcal{R}$. Consequently we have

$$(*) \qquad \mu_R(\mathfrak{m}^{n-d}I) \le d\binom{n-1}{d-1} + n - \binom{n}{d-1}.$$

To estimate the number $\mu_R(\mathfrak{m}^{n-d}I)$ from below, we consider the homomorphism

$$\psi : \mathfrak{m}^{n-d} \otimes_R I \to \mathfrak{m}^{n-d}I$$

defined by $x \otimes y \mapsto xy$ and set $X = \operatorname{Ker}\psi$. Let $x \in X$ and write $x = \sum_{i=1}^{d} x_i \otimes \Delta_i$ with $x_i \in \mathfrak{m}^{n-d}$. Then since $\sum_{i=1}^{d} x_i\Delta_i = 0$ in R and since every entry of the matrix φ is linear, the presentation (P) of I guarantees the existence of elements $y_j \in \mathfrak{m}^{n-d-1}$ ($1 \leq j \leq n-1$) such that

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \varphi \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

Hence X is a homomorphic image of $[\mathfrak{m}^{n-d-1}]^{\oplus(n-1)}$. Therefore in the exact sequence

$$0 \to X \to \mathfrak{m}^{n-d} \otimes_R I \to \mathfrak{m}^{n-d}I \to 0$$

we get

$$\mu_R(X) \leq (n-1)\binom{n-2}{d-1}.$$

Consequently

$$(**) \qquad \mu_R(\mathfrak{m}^{d-n}I) \geq n\binom{n-1}{d-1} - (n-1)\binom{n-2}{d-1},$$

so that combining the estimations $(*)$ and $(**)$, we get

$$\begin{aligned} 0 &\leq \left[d\binom{n-1}{d-1} + n - \binom{n}{d-1}\right] - \left[n\binom{n-1}{d-1} - (n-1)\binom{n-2}{d-1}\right] \\ &= \left[(d-n)\binom{n-1}{d-1} + (n-1)\binom{n-2}{d-1}\right] + \left[n - \binom{n}{d-1}\right] \\ &= n - \binom{n}{d-1}. \end{aligned}$$

Hence $d = 2$, because $n < \binom{n}{d-1}$ if $n > d \geq 3$. $\square$

Before closing this section, let us note one concrete example.

Example 3.3.2. Let $R = k[[x, y, z]]$ be the formal power series ring over an infinite field k. We set $I = (x^2y, y^2z, z^2x, xyz)$ and $Q = (x^2y, y^2z, z^2x)$. Then Q is a minimal reduction of I with $\mathrm{red}_Q(I) = 2$. The ideal I has a presentation of the form

$$0 \to R^{\oplus 3} \xrightarrow{\varphi} R^{\oplus 4} \to I \to 0$$

with $\varphi = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \\ y & z & x \end{pmatrix}$ and it is direct to check that I satisfies all the conditions required for Theorem 3.3.1. Hence Theorem 3.3.1 shows that $\mathcal{R}(I)$ cannot be an almost Gorenstein graded ring, while Q is not a perfect ideal of R but its Rees algebra $\mathcal{R}(Q)$ is an almost Gorenstein graded ring with $\mathrm{r}(\mathcal{R}) = 2$; see [50].

3.4 The Rees algebras of socle ideals $I = (a, b) : \mathfrak{m}$

Throughout this section let $(R, \mathfrak{m})$ denote a two-dimensional regular local ring with infinite residue class field. Let $Q = (a, b)$ be a parameter ideal of R. We set $I = Q : \mathfrak{m}$ and $\mathcal{R} = \mathcal{R}(I)$. For each ideal $\mathfrak{a}$ in R we set $\mathrm{o}(\mathfrak{a}) = \sup\{n \in \mathbb{Z} \mid \mathfrak{a} \subseteq \mathfrak{m}^n\}$. We are interested in the question of when $\mathcal{R}$ is an almost Gorenstein graded ring. Our answer is the following. Notice that the implication (1) $\Rightarrow$ (2) follows from the definition.

Theorem 3.4.1. *With the same notation as above assume that $Q \neq \mathfrak{m}$. Then the following conditions are equivalent.*

(1) *$\mathcal{R}$ is an almost Gorenstein graded ring.*

(2) *$\mathcal{R}_{\mathfrak{M}}$ is an almost Gorenstein local ring*

(3) *$\mathrm{o}(Q) \leq 2$.*

Proof of the implication (3) $\Rightarrow$ (1). If $\mathrm{o}(Q) = 1$, then $Q = (x, y^q)$ $(q \geq 2)$ for some regular system x, y of parameters of R. Hence $I = (x, y^{q-1})$ and $\mathcal{R}$ is a Gorenstein ring. Suppose that $\mathrm{o}(Q) = 2$. Then $\mathrm{o}(I) = \mathrm{o}(Q) = 2$ since $I^2 = QI$ ([78]). Because $\mu_R(I) = 3 = \mathrm{o}(I) + 1$, there exists an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $I : x = I : \mathfrak{m}$ (cf. [45], [81, App. 5]). We set $\overline{R} = R/(x)$. Then $Q\overline{R} = I\overline{R}$, since Q is a reduction of I and $\overline{R}$ is a DVR. We may assume $a\overline{R} = I\overline{R}$, whence $I \subseteq (a, x)$. Let us write $b = af + xg$ with $f, g \in R$. Then $Q = (a, xg)$. Therefore $I\mathfrak{m} = Q\mathfrak{m} = (g\mathfrak{m})x + a\mathfrak{m}$ (remember that $I\mathfrak{m} = Q\mathfrak{m}$; see, e.g. [78]). Notice that $g\mathfrak{m} \subseteq I$, because $g \in Q : x \subseteq I : x = I : \mathfrak{m}$. Therefore $I\mathfrak{m} = Ix + a\mathfrak{m}$, whence $\mathcal{R}$ is an almost Gorenstein graded ring by Theorem 3.2.3. $\square$

To prove the implication (2) $\Rightarrow$ (3) we need Theorem 3.4.2 below. From now we write $\mathfrak{m} = (x, y)$ and let $Q = (a, b)$ be a parameter ideal of R such that $\mathrm{o}(Q) \geq 2$. Hence

$I^2 = QI$ with $\mu_R(I) = 3$. Let $I = (a, b, c)$. Then since $xc, yc \in Q$, we get equations

$$f_1 a + f_2 b + xc = 0 \quad \text{and} \quad g_1 a + g_2 b + yc = 0$$

with $f_i, g_i \in \mathfrak{m}$ $(i = 1, 2)$.

Theorem 3.4.2. *With the notation above, if $(f_1, f_2, g_1, g_2) \subseteq \mathfrak{m}^2$, then $\mathcal{R}_{\mathfrak{M}}$ is not an almost Gorenstein local ring.*

We divide the proof of Theorem 3.4.2 into three steps. Let us begin with the following.

Lemma 3.4.3. *Let $\mathbb{M} = \begin{pmatrix} f_1 & f_2 & x \\ g_1 & g_2 & y \end{pmatrix}$. Then R/I has a minimal free resolution*

$$0 \to R^{\oplus 2} \xrightarrow{{}^t\mathbb{M}} R^{\oplus 3} \xrightarrow{[a\ b\ c]} R \to R/I \to 0.$$

Proof. Let $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ x \end{pmatrix}$ and $\mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ y \end{pmatrix}$. Then $\mathbf{f}, \mathbf{g} \in \mathfrak{m}{\cdot}R^{\oplus 3}$. As $\mathbf{f}, \mathbf{g}$ mod $\mathfrak{m}^2{\cdot}R^{\oplus 3}$ are linearly independent over $R/\mathfrak{m}$, the complex

$$0 \to R^{\oplus 2} \xrightarrow{{}^t\mathbb{M}} R^{\oplus 3} \xrightarrow{[a\ b\ c]} \longrightarrow R \to R/I \to 0$$

is exact and gives rise to a minimal free resolution of R/I. $\square$

Let $\mathcal{S} = R[X, Y, Z]$ be the polynomial ring and let $\varphi : \mathcal{S} \to \mathcal{R} = R[It]$ (t an indeterminate) be the R-algebra map defined by $\varphi(X) = at$, $\varphi(Y) = bt$, and $\varphi(Z) = ct$. Let $K = \operatorname{Ker} \varphi$. Since $c^2 \in QI$, we have a relation of the form

$$c^2 = a^2 f + b^2 g + abh + bci + caj$$

with $f, g, h, i, j \in R$. We set

$$\begin{aligned} F &= Z^2 - \left(fX^2 + gY^2 + hXY + iYZ + jZX\right), \\ G &= f_1 X + f_2 Y + xZ, \\ H &= g_1 X + g_2 Y + yZ. \end{aligned}$$

Notice that $F \in \mathcal{S}_2$ and $G, H \in \mathcal{S}_1$.

Proposition 3.4.4. *$\mathcal{R}$ has a minimal graded free resolution of the form*

$$0 \to \mathcal{S}(-2) \oplus \mathcal{S}(-2) \xrightarrow{{}^t\mathbb{N}} \mathcal{S}(-2) \oplus \mathcal{S}(-1) \oplus \mathcal{S}(-1) \xrightarrow{[F\ G\ H]} \mathcal{S} \to \mathcal{R} \to 0,$$

so that the graded canonical module $\mathrm{K}_{\mathcal{R}}$ of $\mathcal{R}$ has a presentation

$$\mathcal{S}(-1) \oplus \mathcal{S}(-2) \oplus \mathcal{S}(-2) \xrightarrow{\mathbb{N}} \mathcal{S}(-1) \oplus \mathcal{S}(-1) \to \mathrm{K}_{\mathcal{R}} \to 0.$$

Proof. We have $K = \mathcal{S}K_1 + (F)$ (cf., e.g. [54, Theorem 4.1]; use the fact that $I^2 = QI$ and $c^2 \in QI$). Hence $\mathcal{R}$ has a minimal graded free resolution of the form

$$(*) \quad 0 \to \mathcal{S}(-m) \oplus \mathcal{S}(-\ell) \xrightarrow{{}^t\mathbb{N}} \mathcal{S}(-2) \oplus \mathcal{S}(-1) \oplus \mathcal{S}(-1) \xrightarrow{[F\ G\ H]} \mathcal{S} \to \mathcal{R} \to 0$$

with $m,\ell \geq 1$. We take the $\mathcal{S}(-3)$-dual of the resolution $(*)$. Then as $\mathrm{K}_{\mathcal{S}} = \mathcal{S}(-3)$, we get the presentation

$$\mathcal{S}(-1) \oplus \mathcal{S}(-2) \oplus \mathcal{S}(-2) \xrightarrow{\mathbb{N}} \mathcal{S}(m-3) \oplus \mathcal{S}(\ell-3) \to \mathrm{K}_{\mathcal{R}} \to 0$$

of the canonical module $\mathrm{K}_{\mathcal{R}}$ of $\mathcal{R}$. Hence $m,\ell \leq 2$ because $\mathrm{a}(\mathcal{R}) = -1$. Assume that $m = 1$. Then the matrix ${}^t\mathbb{N}$ has the form ${}^t\mathbb{N} = \begin{pmatrix} 0 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix}$ with $\alpha_2, \alpha_3 \in R$. We have $\alpha_2 G + \alpha_3 H = 0$, or equivalently $\alpha_2 \begin{pmatrix} f_1 \\ f_2 \\ x \end{pmatrix} + \alpha_3 \begin{pmatrix} g_1 \\ g_2 \\ y \end{pmatrix} = \mathbf{0}$, whence $\alpha_2 = \alpha_3 = 0$ by Lemma 3.4.3. This is impossible, whence $m = 2$. We similarly have $\ell = 2$ and the assertion follows. $\square$

We are ready to prove Theorem 3.4.2.

Proof of Theorem 3.4.2. Let $\mathbb{N}$ be the matrix given by Proposition 3.4.4 and write $\mathbb{N} = \begin{pmatrix} \alpha & F_1 & F_2 \\ \beta & G_1 & G_2 \end{pmatrix}$. Then Proposition 3.4.4 shows that $F_i, G_i \in \mathcal{S}_1$ $(i = 1, 2)$ and $\alpha,\beta \in \mathfrak{m}$. We write $F_i = \alpha_{i1}X + \alpha_{i2}Y + \alpha_{i3}Z$ and $G_i = \beta_{i1}X + \beta_{i2}Y + \beta_{i3}Z$ with $\alpha_{ij}, \beta_{ij} \in R$. Let Δ_j denote the determinant of the matrix obtained by deleting the j-th column from $\mathbb{N}$. Then by the theorem of Hilbert-Burch we have $G = -\varepsilon\Delta_2$ and $H = \varepsilon\Delta_3$ for some unit ε of R, so that

$$\begin{pmatrix} f_1 \\ f_2 \\ x \end{pmatrix} = (\varepsilon\beta) \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \end{pmatrix} - (\varepsilon\alpha) \begin{pmatrix} \beta_{21} \\ \beta_{22} \\ \beta_{23} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g_1 \\ g_2 \\ y \end{pmatrix} = (\varepsilon\alpha) \begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{pmatrix} - (\varepsilon\beta) \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \end{pmatrix}.$$

Hence

$$x = (\varepsilon\beta)\alpha_{23} - (\varepsilon\alpha)\beta_{23} \quad \text{and} \quad y = (\varepsilon\alpha)\beta_{13} - (\varepsilon\beta)\alpha_{13},$$

which shows $(x,y) = (\varepsilon\alpha, \varepsilon\beta) = \mathfrak{m}$ because $(x,y) \subseteq (\varepsilon\alpha, \varepsilon\beta) \subseteq \mathfrak{m}$.

Since $f_1 = (\varepsilon\beta)\alpha_{21} - (\varepsilon\alpha)\beta_{21}$ and $\varepsilon\alpha, \varepsilon\beta$ is a regular system of parameters of R, we get $\alpha_{21}, \beta_{21} \in \mathfrak{m}$ if $f_1 \in \mathfrak{m}^2$. Therefore if $(f_1, f_2, g_1, g_2) \subseteq \mathfrak{m}^2$, then $\alpha_{ij}, \beta_{ij} \in \mathfrak{m}$ for all $i, j = 1, 2$, whence

$$\mathbb{N} \equiv \begin{pmatrix} \alpha & \alpha_{13}Z & \alpha_{23}Z \\ \beta & \beta_{13}Z & \beta_{23}Z \end{pmatrix} \bmod \mathfrak{N}^2$$

where $\mathfrak{N} = \mathfrak{m}\mathcal{S} + \mathcal{S}_+$ denotes the graded maximal ideal of $\mathcal{S}$. We set $B = \mathcal{S}_{\mathfrak{N}}$. Then it is clear that after any elementary row and column operations the matrix $\mathbb{N}$ over the regular local ring B of dimension 5 is not equivalent to a matrix of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}$$

with $\alpha_1, \alpha_2, \alpha_3$ a part of a regular system of parameters of B. Hence by [36, Theorem 7.8] $\mathcal{R}_{\mathfrak{M}}$ cannot be an almost Gorenstein local ring. $\square$

We are now in a position to finish the proof of Theorem 3.4.1.

Proof of the implication (2) $\Rightarrow$ (3) *in Theorem 3.4.1.* It suffices to show that $\mathcal{R}_{\mathfrak{M}}$ is not an almost Gorenstein local ring if $\mathrm{o}(Q) \geq 3$. We write $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ with $f_{ij} \in \mathfrak{m}^2$ $(i, j = 1, 2)$ and set $c = \det \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$. Then $Q : c = \mathfrak{m}$ and $Q : \mathfrak{m} = Q + (c)$. We have

$$(-f_{22})a + f_{12}b + cx = 0 \quad \text{and} \quad f_{21}a + (-f_{11})b + cy = 0.$$

Hence by Theorem 3.4.2 $\mathcal{R}_{\mathfrak{M}}$ is not an almost Gorenstein local ring, which completes the proof of Theorem 3.4.1. $\square$

Chapter 4

Ulrich ideals and almost Gorenstein rings

4.1 Introduction

This chapter studies Ulrich ideals of Cohen–Macaulay local rings and almost Gorenstein local rings.

Ulrich ideals are newcomers. They were introduced by [30] in 2014. Typical examples of Ulrich ideals are the maximal ideal of a Cohen–Macaulay local ring with minimal multiplicity. The syzygy modules of Ulrich ideals are known to be well-behaved [30]. We refer the reader to [30] for a basic theory of Ulrich ideals and [31] for the results about the ubiquity of Ulrich ideals of two-dimensional rational singularities and the representation-theoretic aspects of Ulrich ideals.

Almost Gorenstein rings are also newcomers. They form a class of Cohen–Macaulay rings, which are not necessarily Gorenstein but still good, hopefully next to the Gorenstein rings. The notion of almost Gorenstein local rings dates back to the article [8] of Barucci and Fröberg in 1997. They introduced almost Gorenstein rings in the case where the local rings are of dimension one and analytically unramified. We refer the reader to [8] for a well-developed theory of almost symmetric numerical semigroups. The notion of almost Gorenstein local rings in the present chapter is, however, based on the definition given by the authors [36] in 2015 for Cohen–Macaulay local rings of arbitrary dimension. See [26] for a basic theory of almost Gorenstein local rings of dimension one which might be analytically ramified.

One of the purposes of this chapter is to clarify the structure of Ulrich ideals of almost Gorenstein local rings. The motivation for the research comes from a recent result of

Kei-ichi Watanabe, which asserts that non-Gorenstein almost Gorenstein numerical semigroup rings possess no Ulrich monomial ideals except the maximal ideal. This result essentially says that there should be some restriction of the distribution of Ulrich ideals of an almost Gorenstein but non-Gorenstein local ring. Our research started from the attempt to understand this phenomenon. Along the way, we recognized that his result holds true for every one-dimensional almost Gorenstein non-Gorenstein local ring, and finally reached new knowledge about the behavior of Ulrich ideals, which is reported in this chapter.

Let us state the results of this chapter, explaining how this chapter is organized. In Section 4.2 we shall prove the following structure theorem of the complex $\mathbf{R}\mathrm{Hom}_R(R/I,R)$ for an Ulrich ideal I.

Theorem 4.1.1. *Let R be a Cohen–Macaulay local ring of dimension $d \geq 0$. Let I be a non-parameter Ulrich ideal of R containing a parameter ideal of R as a reduction. Denote by $\nu(I)$ the minimal number of generators of I, and put $t = \nu(I) - d$. Then there is an isomorphism*

$$\mathbf{R}\mathrm{Hom}_R(R/I,R) \cong \bigoplus_{i\in\mathbb{Z}} (R/I)^{\oplus u_i}[-i]$$

in the derived category of R, where

$$u_i = \begin{cases} 0 & (i < d), \\ t & (i = d), \\ (t^2-1)t^{i-d-1} & (i > d). \end{cases}$$

In particular, one has $\mathrm{Ext}^i_R(R/I,R) \cong (R/I)^{\oplus u_i}$ for each integer i.

This theorem actually yields a lot of consequences and applications. Let us state some of them. The Bass numbers of R are described in terms of those of R/I and the u_i, which recovers a result in [30]. Finiteness of the G-dimension of I is characterized in terms of $\nu(I)$, which implies that if R is *G-regular* in the sense of [?] (e.g., R is a non-Gorenstein ring with minimal multiplicity, or is a non-Gorenstein almost Gorenstein ring), then one must have $\nu(I) \geq d+2$. For a non-Gorenstein almost Gorenstein ring with prime Cohen–Macaulay type, all the Ulrich ideals have the same minimal number of generators. For every one-dimensional non-Gorenstein almost Gorenstein local ring the only non-parameter Ulrich ideal is the maximal ideal. This recovers the result of

Watanabe mentioned above, and thus our original aim of the research stated above is achieved.

Now we naturally get interested in whether or not the minimal numbers of generators of Ulrich ideals of an almost Gorenstein non-Gorenstein local ring are always constant. We will explore this in Section 4.3 to obtain some supporting evidence for the affirmativity. By the way, it turns out to be no longer true if the base local ring is not almost Gorenstein. In Section 4.4 we will give a method of constructing Ulrich ideals which possesses different numbers of generators.

Notation 4.1.2. In what follows, unless otherwise specified, R stands for a d-dimensional Cohen–Macaulay local ring with maximal ideal $\mathfrak{m}$ and residue field k. For a finitely generated R-module M, denote by $\ell_R(M)$, $\nu_R(M)$, $r_R(M)$ and $\mathrm{e}^0_{\mathfrak{m}}(M)$ the length of M, the minimal number of generators of M, the Cohen–Macaulay type of M and the multiplicity of M with respect to $\mathfrak{m}$. Let $v(R)$ denote the embedding dimension of R, i.e., $v(R) = \nu_R(\mathfrak{m})$. For each integer i we denote by $\mu_i(R)$ the i-th Bass number of R, namely, $\mu_i(R) = \dim_k \operatorname{Ext}^i_R(k, R)$. Note that $\mu_d(R) = r(R)$. The subscript indicating the base ring is often omitted.

4.2 The structure of $\mathbf{R}\mathrm{Hom}_R(R/I, R)$ for an Ulrich ideal I

In this section, we establish a structure theorem of $\mathbf{R}\mathrm{Hom}_R(R/I, R)$ for an Ulrich ideal I of a Cohen–Macaulay local ring, and derive from it a lot of consequences and applications. First of all, we fix our notation and assumptions on which all the results in this section are based.

Setup 4.2.1. Throughout this section, let I be a non-parameter $\mathfrak{m}$-primary ideal of R containing a parameter ideal Q of R as a reduction. Suppose that I is an *Ulrich ideal*, that is, $I^2 = QI$ and I/I^2 is R/I-free. Put $t = \nu(I) - d > 0$ and

$$u_i = \begin{cases} 0 & (i < d), \\ t & (i = d), \\ (t^2-1)t^{i-d-1} & (i > d). \end{cases}$$

Remark 4.2.2. (1) The condition that I contains a parameter ideal Q of R as a reduction is automatically satisfied if k is infinite.
(2) The condition $I^2 = QI$ is independent of the choice of minimal reductions Q of I.

The following is the main result of this section.

Theorem 4.2.3. *There is an isomorphism*

$$\mathbf{R}\mathrm{Hom}_R(R/I, R) \cong \bigoplus_{i\in\mathbb{Z}} (R/I)^{\oplus u_i}[-i]$$

in the derived category of R. Hence for each integer i one has an isomorphism

$$\mathrm{Ext}^i_R(R/I, R) \cong (R/I)^{\oplus u_i}$$

of R-modules. In particular, $\mathrm{Ext}^i_R(R/I, R)$ is a free R/I-module.

Proof. Let us first show that $\mathrm{Ext}^i_R(R/I, R) \cong (R/I)^{\oplus u_i}$ for each i. We do it by making three steps.

Step 1. As I is an $\mathfrak{m}$-primary ideal, R/I has finite length as an R-module. Hence we have $\mathrm{Ext}^{<d}_R(R/I, R) = 0$.

Step 2. There is a natural exact sequence $0 \to I/Q \xrightarrow{f} R/Q \xrightarrow{g} R/I \to 0$, which induces an exact sequence

$$\begin{array}{cccccc} & \mathrm{Ext}^d_R(R/I, R) & \longrightarrow & \mathrm{Ext}^d_R(R/Q, R) & \longrightarrow & \mathrm{Ext}^d_R(I/Q, R) \\ \longrightarrow & \mathrm{Ext}^{d+1}_R(R/I, R) & \longrightarrow & \mathrm{Ext}^{d+1}_R(R/Q, R). & & \end{array}$$

Since Q is generated by an R-sequence, we have $\mathrm{Ext}^{d+1}_R(R/Q, R) = 0$. There is a commutative diagram

$$\begin{array}{ccccccc} & & \mathrm{Ext}^d_R(R/I, R) & \longrightarrow & \mathrm{Ext}^d_R(R/Q, R) & \longrightarrow & \mathrm{Ext}^d_R(I/Q, R) \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ 0 & \longrightarrow & \mathrm{Hom}_{R/Q}(R/I, R/Q) & \xrightarrow{g^*} & \mathrm{Hom}_{R/Q}(R/Q, R/Q) & \xrightarrow{f^*} & \mathrm{Hom}_{R/Q}(I/Q, R/Q) \\ & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & (Q : I)/Q & \xrightarrow{h} & R/Q & & \end{array}$$

with exact rows, where the vertical maps are natural isomorphisms and h is an inclusion map. Thus we get an exact sequence

$$0 \to (Q : I)/Q \xrightarrow{h} R/Q \to \mathrm{Hom}_{R/Q}(I/Q, R/Q) \to \mathrm{Ext}^{d+1}_R(R/I, R) \to 0.$$

Note here that $I/Q \cong (R/I)^{\oplus t}$ and $Q : I = I$ hold; see [30, Lemma 2.3 and Corollary 2.6]. Hence $\mathrm{Ext}_R^d(R/I, R) \cong (Q : I)/Q = I/Q \cong (R/I)^{\oplus t}$. We have isomorphisms $\mathrm{Hom}_{R/Q}(I/Q, R/Q) \cong \mathrm{Hom}_{R/Q}(R/I, R/Q)^{\oplus t} \cong (I/Q)^{\oplus t} \cong (R/I)^{\oplus t^2}$, and therefore we obtain an exact sequence

$$0 \to R/I \to (R/I)^{\oplus t^2} \to \mathrm{Ext}_R^{d+1}(R/I, R) \to 0.$$

This exact sequence especially says that $\mathrm{Ext}_R^{d+1}(R/I, R)$ has finite projective dimension as an R/I-module. Since R/I is an Artinian ring, it must be free, and we see that $\mathrm{Ext}_R^{d+1}(R/I, R) \cong (R/I)^{\oplus t^2-1}$.

Step 3. It follows from [30, Corollary 7.4] that $\mathrm{Syz}_R^i(R/I) \cong \mathrm{Syz}_R^d(R/I)^{\oplus t^{i-d}}$ for each $i \geq d$. Hence we have

$$\begin{aligned}\mathrm{Ext}_R^{i+1}(R/I, R) &\cong \mathrm{Ext}_R^1(\mathrm{Syz}_R^i(R/I), R) \cong \mathrm{Ext}_R^1(\mathrm{Syz}_R^d(R/I)^{\oplus t^{i-d}}, R)\\ &\cong \mathrm{Ext}_R^{d+1}(R/I, R)^{\oplus t^{i-d}} \cong (R/I)^{\oplus (t^2-1)t^{i-d}}\end{aligned}$$

for all $i \geq d$.

Combining the observations in Steps 1, 2 and 3 yields that $\mathrm{Ext}_R^i(R/I, R) \cong (R/I)^{\oplus u_i}$ for all $i \in \mathbb{Z}$.

Take an injective resolution E of R. Then note that $C := \mathrm{Hom}_R(R/I, E)$ is a complex of R/I-modules with $\mathrm{H}^i(C) \cong \mathrm{Ext}_R^i(R/I, R) \cong (R/I)^{\oplus u_i}$ for every $i \in \mathbb{Z}$. Hence each homology $\mathrm{H}^i(C)$ is a projective R/I-module. Applying [1, Lemma 3.1] to the abelian category $\mathrm{Mod}R/I$, the category of (all) R/I-modules, we obtain isomorphisms $\mathbf{R}\mathrm{Hom}_R(R/I, R) \cong C \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(C)[-i] \cong \bigoplus_{i\in\mathbb{Z}}(R/I)^{\oplus u_i}[-i]$ in the derived category of R/I. This completes the proof of the theorem. □

The remainder of this section is devoted to producing consequences and applications of the above theorem. First, we investigate vanishing of Ext modules.

Corollary 4.2.4. *Let M be a (possibly infinitely generated) R/I-module. There is an isomorphism*

$$\mathrm{Ext}_R^i(M, R) \cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Ext}_{R/I}^{i-j}(M, R/I)^{\oplus u_j}$$

for each integer i. In particular, if $\mathrm{Ext}_R^{\gg 0}(M, R) = 0$, then $\mathrm{Ext}_{R/I}^{\gg 0}(M, R/I) = 0$.

Proof. There are isomorphisms

$$\mathbf{R}\mathrm{Hom}_R(M,R) \cong \mathbf{R}\mathrm{Hom}_{R/I}(M, \mathbf{R}\mathrm{Hom}_R(R/I,R)) \cong \bigoplus_{j\in\mathbb{Z}} \mathbf{R}\mathrm{Hom}_{R/I}(M,R/I)^{\oplus u_j}[-j],$$

where the first isomorphism holds by adjointness (see [13, (A.4.21)]) and the second one follows from Theorem 4.2.3. Taking the ith homologies, we get an isomorphism $\mathrm{Ext}^i_R(M,R) \cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Ext}^{i-j}_{R/I}(M,R/I)^{\oplus u_j}$ for all integers i. Since $u_d = t > 0$, the module $\mathrm{Ext}^{i-d}_{R/I}(M,R/I)$ is a direct summand of $\mathrm{Ext}^i_R(M,R)$. Therefore, when $\mathrm{Ext}^{\gg 0}_R(M,R) = 0$, one has $\mathrm{Ext}^{\gg 0}_{R/I}(M,R/I) = 0$. □

Now we can calculate the Bass numbers of R in terms of those of R/I.

Theorem 4.2.5. *There are equalities*

$$\mu_i(R) = \sum_{j\in\mathbb{Z}} u_j \mu_{i-j}(R/I) = \begin{cases} 0 & (i<d), \\ \sum_{j=d}^{i} u_j\mu_{i-j}(R/I) & (i \geq d). \end{cases}$$

In particular, one has

$$t \cdot r(R/I) = r(I/Q) = r(R).$$

Proof. Applying Corollary 4.2.4 to the R/I-module k gives rise to an isomorphism $\mathrm{Ext}^i_R(k,R) \cong \bigoplus_{j\in\mathbb{Z}} \mathrm{Ext}^{i-j}_{R/I}(k,R/I)^{\oplus u_j}$ for each integer i. Compaing the k-dimension of both sides, we get $\mu_i(R) = \sum_{j\in\mathbb{Z}} u_j\mu_{i-j}(R/I)$. Hence we have $r(R) = \mu_d(R) = u_d\mu_0(R/I) = t\cdot r(R/I) = r(I/Q)$, where the last equality comes from the isomorphism $I/Q \cong (R/I)^{\oplus t}$ (see [30, Lemma 2.3]). Thus all the assertions follow. □

The above theorem recovers a result of Goto, Ozeki, Takahashi, Watanabe and Yoshida.

Corollary 4.2.6. *[30, Corollary 2.6(b)] The following are equivalent.*

(1) *R is Gorenstein.*

(2) *R/I is Gorenstein and $\nu(I) = d+1$.*

Proof. Theorem 4.2.5 implies $t\cdot r(R/I) = r(R)$. Hence $r(R) = 1$ if and only if $t = r(R/I) = 1$. This shows the assertion. □

To state our next results, let us recall some notions.

A *totally reflexive* R-module is by definition a finitely generated reflexive R-module G such that $\operatorname{Ext}_R^{>0}(G, R) = 0 = \operatorname{Ext}_R^{>0}(\operatorname{Hom}_R(G, R), R)$. Note that every finitely generated free R-module is totally reflexive. The *Gorenstein dimension* (*G-dimension* for short) of a finitely generated R-module M, denoted by $\operatorname{Gdim}_R M$, is defined as the infimum of integers $n \geq 0$ such that there exists an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$$

of R-modules with each G_i totally reflexive.

A Noetherian local ring R is called *G-regular* if every totally reflexive R-module is free. This is equivalent to saying that the equality $\operatorname{Gdim}_R M = \operatorname{pd}_R M$ holds for all finitely generated R-modules M.

Remark 4.2.7. The following local rings are G-regular.

- Regular local rings.
- Non-Gorenstein Cohen–Macaulay local rings with minimal multiplicity.
- Non-Gorenstein almost Gorenstein local rings.

For the proofs, we refer to [71, Proposition 1.8], [5, Examples 3.5] (see also [80, Corollary 2.5]) and [36, Corollary 4.5], respectively.

Suppose that R admits a canonical module K_R. We say that R is *almost Gorenstein* if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that C is an *Ulrich R-module*, i.e., C is a Cohen–Macaulay R-module (of dimension $d-1$) with $\mathrm{e}_{\mathfrak{m}}^0(C) = \nu_R(C)$.

Using our Theorem 4.2.3, we establish a characterization of finiteness of the G-dimension of R/I in terms of the minimal number of generator of I.

Theorem 4.2.8. *One has*

$$\nu(I) = d+1 \iff \operatorname{Gdim}_R R/I < \infty.$$

In particular, if R is G-regular, then $\nu(I) \geq d+2$.

Proof. As to the first assertion, it suffices to show that $t = 1$ if and only if R/I has finite G-dimension as an R-module.

The 'if' part: As R/I has depth 0, it has G-dimension d by [13, (1.4.8)], and hence $\mathrm{Ext}_R^{>d}(R/I, R) = 0$ by [13, (1.2.7)]. It follows from Theorem 4.2.3 that $u_i = 0$ for all $i > d$. In particular, we have $t^2 - 1 = u_{d+1} = 0$, which implies $t = 1$.

The 'only if' part: By Theorem 4.2.3 we have $\mathbf{R}\mathrm{Hom}_R(R/I, R) \cong R/I[-d]$. It is observed from this that the homothety morphism $R/I \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(R/I, R), R)$ is an isomorphism. It follows from [13, (2.2.3)] that the R-module R/I has finite G-dimension.

Thus the first assertion of the theorem follows. As for the second assertion, suppose that $t = 1$. Then R/I has finite G-dimension, and so does I by [13, (1.2.9)]. Since R is G-regular, I has finite projective dimension. As I is an Ulrich ideal, I/I^2 is a free R/I-module. Hence we see from [11, Theorem 2.2.8] that I is generated by an R-sequence, which contradicts the assumption that I is a non-parameter $\mathfrak{m}$-primary ideal. Therefore we have $t \geq 2$, which means $\nu(I) \geq d + 2$. □

As a consequence of the above theorem, we have a characterization of Gorenstein local rings.

Corollary 4.2.9. *The following are equivalent.*

(1) *R is Gorenstein.*

(2) *There is an Ulrich ideal I of R with finite G-dimension such that R/I is Gorenstein.*

Proof. (1) ⇒ (2): Any parameter ideal I of R is such an ideal as in the condition (2).

(2) ⇒ (1): It is trivial if I is a parameter ideal, so suppose that I is not so. The Gorensteinness of R/I implies $\mu_0(R/I) = 1$ and $\mu_{>0}(R/I) = 0$, and hence $\mu_i(R) = u_i$ for all $i \geq d$ by Theorem 4.2.5. Since R/I has finite G-dimension, we have $t = 1$ by Theorem 4.2.8, whence $u_d = 1$ and $u_{>d} = 0$. Thus we get $\mu_d(R) = 1$ and $\mu_{>d}(R) = 0$, which shows that R is Gorenstein. □

Remark 4.2.10. Corollary 4.2.9 is a special case of [70, Theorem 2.3], which implies that a (not necessarily Cohen–Macaulay) local ring R is Gorenstein if and only if it possesses a (not necessarily Ulrich) ideal I of finite G-dimension such that R/I is Gorenstein.

Using our theorems, we observe that the minimal numbers of generators of Ulrich ideals are constant for certain almost Gorenstein rings.

Corollary 4.2.11. *Let R be a non-Gorenstein almost Gorenstein local ring such that $r(R)$ is a prime number. Then R/I is a Gorenstein ring and $\nu(I) = r(R) + d$.*

Proof. It follows from Theorem 4.2.5 that $t \cdot r(R/I) = r(R)$. Since $t > 1$ by Theorem 4.2.8 and Remark 4.2.7, we must have that $r(R/I) = 1$ and $r(R) = t = \nu(I) - d$. □

Corollary 4.2.12. *Let R be a two-dimensional rational singularity. Then $\nu(I) = r(R) + 2$.*

Proof. Because $r(R/I) = 1$ by [31, Corollary 6.5], the equality follows from Theorem 4.2.5. □

The following corollary is another consequence of Theorem 4.2.3. Note that such an exact sequence as in the corollary exists for every almost Gorenstein ring.

Corollary 4.2.13. *Suppose that R admits a canonical module K_R, and that there is an exact sequence $0 \to R \to \mathrm{K}_R \to C \to 0$ of R-modules. If $\nu(I) \geq d + 2$, then $\operatorname{Ann}_R C \subseteq I$.*

Proof. We set $\mathfrak{a} = \operatorname{Ann}_R C$ and $M = \operatorname{Syz}_R^d(R/I)$. Then M is a maximal Cohen–Macaulay R-module. Hence $\operatorname{Ext}_R^{>0}(M, \mathrm{K}_R) = 0$, and in particular there is a surjection $\operatorname{Hom}_R(M, C) \twoheadrightarrow \operatorname{Ext}_R^1(M, R)$. Since $\operatorname{Ext}_R^1(M, R) \cong \operatorname{Ext}_R^{d+1}(R/I, R)$ and $t > 1$, the ideal $\mathfrak{a}$ annihilates R/I by Theorem 4.2.3, whence I contains $\mathfrak{a}$. □

Now we state the last theorem in this section, whose first assertion is proved by Kei-ichi Watanabe in the case where R is a numerical semigroup ring over a field and all ideals considered are monomial.

Theorem 4.2.14. *Let R be a non-Gorenstein local ring of dimension d. Assume that R is almost Gorenstein, that is, there exists an exact sequence $0 \to R \to \mathrm{K}_R \to C \to 0$ such that C is Ulrich.*

(1) *If $d = 1$, then $I = \mathfrak{m}$.*

(2) *Suppose that k is infinite. If $\mathfrak{m}C = IC$, then $I = \mathfrak{m}$.*

Proof. (1) As $C \neq 0 = \mathfrak{m}C$, we have $\nu(I) \geq d+2$ by Theorem 4.2.8 and Remark 4.2.7. Hence $I = \mathfrak{m}$ by Corollary 4.2.13.

(2) We may assume by (1) that $d > 1$ and that our assertion holds true for $d-1$. We set $\mathfrak{a} = \mathrm{Ann}_R C$ and $S = R/\mathfrak{a}$. Then $\mathfrak{m}S$ is integral over IS, because $\mathfrak{m}C = IC$. Therefore, without loss of generality, we may assume that $a = a_1$ (a part of a minimal basis of a reduction $Q = (a_1, a_2, \dots, a_d)$ of I) is a superficial element of C with respect to $\mathfrak{m}$. Let $\overline{R} = R/(a)$, $\overline{I} = I/(a)$, and $\overline{C} = C/aC$. Then $\overline{R}$ is a non-Gorenstein almost Gorenstein ring, $\overline{C}$ is an Ulrich $\overline{R}$-module, and we have an exact sequence $0 \to \overline{R} \to \mathrm{K}_{\overline{R}} \to \overline{C} \to 0$ of $\overline{R}$-modules ([36, Proof of Theorem 3.7 (2)]), because $\mathrm{K}_{\overline{R}} \cong \mathrm{K}_R/a\mathrm{K}_R$ ([43, Korollar 6.3]). Consequently, since $\mathfrak{m}\overline{C} = I\overline{C}$, we get $I\overline{R} = \mathfrak{m}\overline{R}$ by the hypothesis of induction, so that $I = \mathfrak{m}$. $\square$

4.3 The expected core of Ulrich ideals

In this section let $(R, \mathfrak{m}, k)$ be a d-dimensional Cohen-Macaulay local ring with canonical module K_R. We denote by $\mathcal{X}_R$ the set of non-parameter Ulrich ideals of R. Let

$$\mathfrak{a} = \sum_{f \in K_R \text{ such that } 0:_R f = 0} [Rf :_R \mathrm{K}_R],$$

which is the *expected core* of Ulrich ideals in the case where R is an almost Gorenstein but non-Gorenstein ring. In fact we have the following.

Theorem 4.3.1. *Suppose that R is a non-Gorenstein almost Gorenstein local domain. Then the following assertions hold true.*

(1) *If $I \in \mathcal{X}_R$, then $\mathfrak{a} \subseteq I$.*

(2) *Suppose that $R_\mathfrak{p}$ is a Gorenstein ring for every $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. Then $\sqrt{\mathfrak{a}} = \mathfrak{m}$.*

(3) *Suppose that $\dim R = 2$ and $r(R) = 2$. If $R_\mathfrak{p}$ is a Gorenstein ring for every $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$, then $\mathcal{X}_R$ is a finite set and every $I \in \mathcal{X}_R$ is minimally generated by four elements.*

Proof. (1) For each $f \in \mathrm{K}_R$ such that $0 :_R f = 0$ we have an exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

with $\varphi(1) = f$ and applying Corollary 4.2.13 to the sequence, we get $\mathfrak{a} \subseteq I$ by Theorem 4.2.8.

(2) Let $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. Then $[\mathrm{K}_R]_\mathfrak{p} = \mathrm{K}_{R_\mathfrak{p}} \cong R_\mathfrak{p}$, since $R_\mathfrak{p}$ is a Gorenstein ring. Choose an element $f \in \mathrm{K}_R$ so that $[\mathrm{K}_R]_\mathfrak{p} = R_\mathfrak{p}\frac{f}{1}$. Then $0 :_R f = 0$ and $Rf :_R \mathrm{K}_R \not\subseteq \mathfrak{p}$. Hence $\mathfrak{a} \not\subseteq \mathfrak{p}$ and therefore $\sqrt{\mathfrak{a}} = \mathfrak{m}$.

(3) We have an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $C = R/\mathfrak{p}$ is a DVR ([36, Corollary 3.10]). Then $\mathfrak{p} \subseteq \mathfrak{a}$ by the definition of $\mathfrak{a}$. Since $\mathfrak{a} \neq \mathfrak{p}$ by assertion (2), we have $\mathfrak{a} = \mathfrak{p} + x^n R$ for some $n > 0$ where $x \in \mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{p} + (x)$. Let $I \in \mathcal{X}_R$. Then because $\mathfrak{a} \subseteq I$ by assertion (1), we get $I = \mathfrak{p} + x^\ell R$ with $1 \leq \ell \leq n$. Hence the set $\mathcal{X}_R$ is finite. By Corollary 4.2.11 every $I \in \mathcal{X}_R$ is minimally generated by four elements. $\square$

Corollary 4.3.2. *Let R be a two-dimensional normal local ring. Assume that R is a non-Gorenstein almost Gorenstein ring with $\mathrm{r}(R) = 2$. Then $\mathcal{X}_R$ is a finite set and every $I \in \mathcal{X}_R$ is minimally generated by four elements.*

Remark 4.3.3. We know no examples of non-Gorenstein almost Greenstein local rings in which Ulrich ideals do not possess a common number of generators.

We explore a few examples. Let $S = k[X, Y, Z, W]$ be the polynomial ring over a field k. Let $n \geq 1$ be an integer and consider the matrix $\mathbb{M} = \left(\begin{smallmatrix} X^n & Y & Z \\ Y & Z & W \end{smallmatrix}\right)$. We set $T = S/\mathrm{I}_2(\mathbb{M})$ where $\mathrm{I}_2(\mathbb{M})$ denotes the ideal of S generated by two by two minors of $\mathbb{M}$. Let x, y, z, w denote the images of X, Y, Z, W in T respectively. We set $R = T_M$, where $M = (x, y, z, w)T$.

Theorem 4.3.4. *We have the following.*

(1) *R is a non-Gorenstein almost Gorenstein local integral domain with $\mathrm{r}(R) = 2$.*

(2) *$\mathcal{X}_R = \{(x^\ell, y, z, w)R \mid 1 \leq \ell \leq n\}$.*

(3) *R is a normal ring if and only if $n = 1$.*

Proof. We regard S as a $\mathbb{Z}$-graded ring so that $\deg X = 1, \deg Y = n+1, \deg Z = n+2$, and $\deg W = n + 3$. Then $T \cong k[s, s^n t, s^n t^2, s^n t^3]$ where s, t are indeterminates over k.

Hence T is an integral domain, and T is a normal ring if and only if $n = 1$. The graded canonical module K_T of T has the presentation of the form

$$S(-(n+3)) \oplus S(-(n+4)) \oplus S(-(n+5)) \xrightarrow{\left(\begin{smallmatrix} X^n & Y & Z \\ Y & Z & W \end{smallmatrix}\right)} S(-3) \oplus S(-2) \xrightarrow{\varepsilon} K_R \to 0. \quad (4.1)$$

Since

$$\mathrm{K}_T/T{\cdot}\varepsilon(\mathbf{e}_1) \cong [S/(Y, Z, W)](-2) \quad \text{and} \quad \mathrm{K}_T/T{\cdot}\varepsilon(\mathbf{e}_2) \cong [S/(X^n, Y, Z)](-3)$$

where $\mathbf{e}_1, \mathbf{e}_2$ is the standard basis of $S(-3) \oplus S(-2)$, R is an almost Gorenstein local ring with $\mathrm{r}(R) = 2$. By Theorem 4.3.1 (1) every $I \in \mathcal{X}_R$ contains $(x^n, y, z, w)R$, so that $I = (x^\ell, y, z, w)R$ with $1 \le \ell \le n$. It is straightforward to check that $(x^\ell, y, z, w)R$ is actually an Ulrich ideal of R for every $1 \le \ell \le n$. □

Let k be a field and let $T = k[X^n, X^{n-1}Y, \dots, XY^{n-1}, Y^n]$ be the Veronesean subring of the polynomial ring $S = k[X, Y]$ of degree $n \ge 3$. Let $R = T_M$ and $\mathfrak{m} = MT_M$, where M denotes the graded maximal ideal of T. Then R is a non-Gorenstein almost Gorenstein normal local ring ([36, Example 10.8]) and we have the following. Let us note a brief proof in our context.

Example 4.3.5 (cf. [31, Example 7.3])**.** $\mathcal{X}_R = \{\mathfrak{m}\}$.

Proof. Since $M^2 = (X^n, Y^n)M$, we have $\mathfrak{m} \in \mathcal{X}_R$. Because

$$\mathrm{K}_R = (X^{n-1}Y, X^{n-2}Y^2, \dots, X^2Y^{n-2}, XY^{n-1})R,$$

it is direct to check that $M \subseteq \sum_{i=1}^{n-1} [R{\cdot}X^iY^{n-i} :_R \mathrm{K}_R]$. Hence $\mathcal{X}_R = \{\mathfrak{m}\}$ by Theorem 4.3.1 (1). □

Let $S = k[[X, Y, Z]]$ be the formal power series ring over an infinite field k. We choose an element $f \in (X, Y, Z)^2 \setminus (X, Y, Z)^3$ and set $R = S/(f)$. Then R is a two-dimensional Cohen–Macaulay local ring of multiplicity 2. Let $\mathfrak{m}$ denote the maximal ideal of R and consider the Rees algebra $\mathcal{R} = \mathcal{R}(\mathfrak{m}^\ell)$ of $\mathfrak{m}^\ell$ with $\ell \ge 1$. Hence

$$\mathcal{R} = R[\mathfrak{m}^\ell{\cdot}t] \subseteq R[t]$$

where t is an indeterminate over R. Let $\mathfrak{M} = \mathfrak{m} + \mathcal{R}_+$ and set $A = \mathcal{R}_{\mathfrak{M}}$, $\mathfrak{n} = \mathfrak{M}\mathcal{R}_{\mathfrak{M}}$. Then A is not a Gorenstein ring but almost Gorenstein ([34, Example 2.4]). We furthermore have the following.

Theorem 4.3.6. $\mathcal{X}_A = \{\mathfrak{n}\}$.

Proof. We have $\mathfrak{m}^2 = (a,b)\mathfrak{m}$ with $a, b \in \mathfrak{m}$. Let $Q = (a, b - a^\ell t, b^\ell t)$. Then $Q \subseteq \mathfrak{M}$ and $\mathfrak{M}^2 = Q\mathfrak{M}$, so that $\mathfrak{n} \in \mathcal{X}_A$. Conversely, let $J \in \mathcal{X}_A$ and set $K = J \cap \mathcal{R}$. We put $I = \mathfrak{m}^\ell$ and choose elements $x_1, x_2, \ldots, x_q \in I$ ($q := \nu(I) = 2\ell + 1$) so that the ideal (x_i, x_j) of R is a reduction of I for each pair (i,j) with $1 \leq i < j \leq q$. (Hence x_i, x_j form a super-regular sequence with respect to I, because $\mathrm{gr}_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a Cohen–Macaulay ring.) Then

$$\sum_{i=1}^{q} [x_i\mathcal{R} :_{\mathcal{R}} I\mathcal{R}] \subseteq K$$

by Corollary 4.2.13, since $\mathrm{K}_{\mathcal{R}}(1) \cong I\mathcal{R}$ ([34, Proposition 2.1]). We have for each $1 \leq i \leq q$ that

$$x_i\mathcal{R} :_{\mathcal{R}} I\mathcal{R} = \sum_{n \geq 0} \left(x_i I^{n-1}\right) t^n,$$

because each pair x_i, x_j $(i \neq j)$ forms a super-regular sequence with respect to I. Consequently,

$$(x_i, x_i t)\mathcal{R} \subseteq K \text{ for all } 1 \leq i \leq q.$$

Thus $I + \mathcal{R}_+ \subseteq K$ and hence $K = \mathfrak{b} + \mathcal{R}_+$ for some $\mathfrak{m}$-primary ideal $\mathfrak{b}$ of R. Notice that $[K/K^2]_1 = I/\mathfrak{b}I$ is $R/\mathfrak{b}$-free, since K/K^2 is $\mathcal{R}/K$- free and $\mathcal{R}/K = [\mathcal{R}/K]_0 = R/\mathfrak{b}$. We then have $\mathfrak{b} = \mathfrak{m}$. In fact, let us write $\mathfrak{m} = (a, b, c)$ so that each two of a, b, c generate a reduction of $\mathfrak{m}$. Set $\mathfrak{q} = (a, b)$. Then since $\mathfrak{q}$ is a minimal reduction of $\mathfrak{m}$, the elements $\{a^{\ell-i}b^i\}_{0 \leq i \leq \ell}$ form a part of a minimal basis, say $\{a^{\ell-i}b^i\}_{0 \leq i \leq \ell}$ and $\{c_i\}_{1 \leq i \leq \ell}$, of $I = \mathfrak{m}^\ell$. Let $\{\mathbf{e}_i\}_{0 \leq i \leq 2\ell}$ be the standard basis of $R^{\oplus(2\ell+1)}$ and let

$$\varphi : R^{\oplus(2\ell+1)} = \bigoplus_{i=0}^{2\ell} R\mathbf{e}_i \to \mathfrak{m}^\ell$$

be the R-linear map defined by $\varphi(\mathbf{e}_i) = a^{\ell-i}b^i$ for $0 \leq i \leq \ell$ and $\varphi(\mathbf{e}_{i+\ell}) = c_i$ for $1 \leq i \leq \ell$. Then setting $Z = \operatorname{Ker}\varphi$, we get $\xi = \begin{bmatrix} b \\ -a \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in Z$ and $\xi \notin \mathfrak{m}Z$ because $\xi \notin \mathfrak{m}^2 {\cdot} R^{\oplus(2\ell+1)}$. Hence $b, -a \in \mathfrak{b}$, because $I/\mathfrak{b}I$ is $R/\mathfrak{b}$-free. We similarly have $c, -a \in \mathfrak{b}$. Hence $\mathfrak{b} = \mathfrak{m}$, so that $K = \mathfrak{m} + \mathcal{R}_+ = \mathfrak{M}$. Thus $J = \mathfrak{n}$. $\square$

4.4 A method of constructing Ulrich ideals with different number of generators

This section purposes to show that in general the numbers of generators of Ulrich ideals are not necessarily constant. To begin with, we note the following.

Lemma 4.4.1. *Let $\varphi : (A, \mathfrak{n}) \to (R, \mathfrak{m})$ be a flat local homomorphism of Cohen–Macaulay local rings of the same dimension. Let $\mathfrak{q}$ be a parameter ideal of A and assume that $\mathfrak{n}^2 = \mathfrak{q}\mathfrak{n}$. Then $J = \mathfrak{n}R$ is an Ulrich ideal of R.*

Proof. We set $Q = \mathfrak{q}R$. Then

$$J/Q \cong R \otimes_A (\mathfrak{n}/\mathfrak{q}) \cong R \otimes_A (A/\mathfrak{n})^{\oplus t}$$

where $t = v(A) - \dim A \geq 0$. Hence J is an Ulrich ideal of R, as $J^2 = QJ$. □

When $\dim R = 0$ and R contains a field, every Ulrich ideal of R is obtained as in Lemma 4.4.1.

Proposition 4.4.2. *Let $(R, \mathfrak{m})$ be an Artinian local ring which contains a coefficient filed k. Let $I = (x_1, x_2, \ldots, x_n)$ $(n = \nu(I))$ be an Ulrich ideal of R. We set $A = k[x_1, x_2, \ldots, x_n] \subseteq R$ and $\mathfrak{n} = (x_1, x_2, \ldots, x_n)A$. Then $\mathfrak{n}$ is the maximal ideal of A, $I = \mathfrak{n}R$, and R is a finitely generated free A-module.*

Proof. Let $r = \dim_k R/I$. Hence $r = \nu_A(R)$, as $I = \mathfrak{n}R$. Let $\varphi : A^{\oplus r} \to R$ be an epimorphism of A-modules. Then $(A/\mathfrak{n}) \otimes_A \varphi : (A/\mathfrak{n})^{\oplus r} \to R/I$ is an isomorphism, so that the induced epimorphism $\phi : \mathfrak{n}^{\oplus r} \to I$ must be an isomorphism, because $\dim_k \mathfrak{n}^{\oplus r} = nr$ (remember that $\nu_A(\mathfrak{n}) = n$) and $\dim_k I = \dim_k (R/I)^{\oplus n} = nr$. Hence $\varphi : A^{\oplus r} \to R$ is an isomorphism. □

When $\dim R > 0$, the situation is more complicated, as we see in the following.

Example 4.4.3. Let $V = k[[t]]$ be the formal power series ring over a field k and set $R = k[[t^4, t^5]]$ in V. Then $(t^4, t^{10}), (t^8, t^{10})$ are Ulrich ideals of R. We set $A = k[[t^4, t^{10}]]$ and $B = k[[t^8, t^{10}, t^{12}, t^{14}]]$. Let $\mathfrak{m}_A$ and $\mathfrak{m}_B$ be the maximal ideals of A and B, respectively. Then A and B are of minimal multiplicity with $(t^4, t^{10}) = \mathfrak{m}_A R$ and $(t^8, t^{10}) = \mathfrak{m}_B R$. Notice that $R \cong A^{\oplus 2}$ as an A-module, while R is not a free B-module. We actually have $\operatorname{rank}_B R = 2$ and $\nu_B(R) = 4$.

Let $(A, \mathfrak{n})$ be a Noetherian local ring and $S = A[X_1, X_2, \ldots, X_\ell]$ $(\ell > 0)$ the polynomial ring. We choose elements $a_1, a_2, \ldots, a_\ell \in A$ and set

$$\mathfrak{c} = (X_i^2 - a_i \mid 1 \leq i \leq \ell) + (X_i X_j \mid 1 \leq i, j \leq \ell \text{ such that } i \neq j).$$

We put $R = S/\mathfrak{c}$. Then R is a finitely generated free A-module of $\mathrm{rank}_A R = \ell + 1$. We get $R = A{\cdot}1 + \sum_{i=1}^{\ell} A{\cdot}x_i$, where x_i denotes the image of X_i in R. With this notation we readily get the following. We note a brief proof.

Lemma 4.4.4. *Suppose that $a_1, a_2, \ldots, a_\ell \in \mathfrak{n}$. Then R is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{n}R + (x_i \mid 1 \leq i \leq \ell)$.*

Proof. Let $M \in \operatorname{Max} R$ and $1 \leq i \leq \ell$. Then $a_i \in M$ since $a_i \in \mathfrak{n} = M \cap A$, while $x_i \in M$ since $x_i^2 = a_i$. Thus $\mathfrak{n}R + (x_i \mid 1 \leq i \leq \ell) \subseteq M$. Hence we get the result, because $\mathfrak{n}R + (x_i \mid 1 \leq i \leq \ell) \in \operatorname{Max} R$. $\square$

The following Theorem 4.4.5 and Lemma 4.4.1 give a simple method of constructing of Ulrich ideals with different numbers of generators. In fact, suppose that A has maximal embedding dimension and let $\mathfrak{q}$ be a parameter ideal of A such that $\mathfrak{n}^2 = \mathfrak{q}\mathfrak{n}$. Then the ideals I in Theorem 4.4.5 and J in Lemma 4.4.1 are both Ulrich ideals of R but the numbers of generators are different, if one takes the integer $\ell \geq 1$ so that $\ell \neq \mathrm{e}_{\mathfrak{n}}^0(A) - 1$.

Theorem 4.4.5. *Let $\mathfrak{q}$ be a parameter ideal of A and assume that*

(1) *A is a Cohen–Macaulay ring of dimension d and*

(2) *$a_i \in \mathfrak{q}^2$ for all $1 \leq i \leq \ell$.*

Let $I = \mathfrak{q}R + (x_i \mid 1 \leq i \leq \ell)$. Then I is an Ulrich ideal of R with $\nu(I) = d + \ell$.

Proof. We set $Q = \mathfrak{q}R$, $\mathfrak{a} = (a_i \mid 1 \leq i \leq \ell)$, and $\mathfrak{b} = (x_i \mid 1 \leq i \leq \ell)$. Then Q is a parameter ideal of R and $I = Q + \mathfrak{b}$. Therefore $I^2 = QI$ since $\mathfrak{b}^2 = \mathfrak{a} \subseteq \mathfrak{q}^2$. We set $m = \ell_A(A/\mathfrak{q})$. Hence $\ell_A(R/I) \leq m$, as R/I is a homomorphic image of $A/\mathfrak{q}$. We consider the epimorphism

$$(R/I)^{\oplus \ell} \xrightarrow{\varphi} I/Q \to 0$$

of R-modules defined by $\varphi(\mathbf{e}_i) = \overline{x_i}$ for each $1 \leq i \leq \ell$, where $\{\mathbf{e}_i\}_{1 \leq i \leq \ell}$ denotes the standard basis of $(R/I)^{\oplus \ell}$ and $\overline{x_i}$ denotes the image of x_i in I/Q. Then φ is an isomorphism, since

$$\ell_A(I/Q) = \ell_A(R/Q) - \ell_A(R/I) \geq (\ell+1)m - m = \ell m \geq \ell_A((R/I)^{\oplus \ell}).$$

Thus I is an Ulrich ideal of R with $\nu(I) = d + \ell$. $\square$

Example 4.4.6. Let $0 < a_1 < a_2 < \cdots < a_n$ $(n \geq 3)$ be integers such that $\gcd(a_1, a_2, \ldots, a_n) = 1$. Let $L = \langle a_1, a_2, \ldots, a_n \rangle$ be the numerical semigroup generated by $a_1, \mathfrak{a}_2, \ldots, a_n$. Then $c(L) = a_n - a_1 + 1$, where $c(L)$ denotes the conductor of L. We set $A = k[[t^{a_i} \mid 1 \leq i \leq n]]$ (k a field) and assume that $v(A) = \mathrm{e}^0_{\mathfrak{n}}(A) = n$, where $\mathfrak{n}$ denotes the maximal ideal of A (hence $a_1 = n$). We choose an odd integer $b \in L$ so that $b \geq a_n + a_1 + 1$ and consider the semigroup ring $R = k[[\{t^{2a_i}\}_{1 \leq i \leq n}, t^b]]$ of the numerical semigroup $H = 2L + \langle b \rangle$. Let $\varphi : A \to R$ be the homomorphism of k-algebras such that $\varphi(t^{a_i}) = t^{2a_i}$ for each $1 \leq i \leq n$. We set $f = t^b$. Then $f^2 = \varphi(f) \in \varphi(A)$ but $f \notin \varphi(A)$, since $b \in L$ is odd. Therefore R is a finitely generated free A-module with $\mathrm{rank}_A R = 2$ and $R = A{\cdot}1 + A{\cdot}f$. Since $f \in t^{2a_1}A$ by the choice of the integer b, we get by Theorem 4.4.5 and Lemma 4.4.1 that $I = (t^{2a_1}, t^b)$ and $J = (t^{2a_i} \mid 1 \leq i \leq n)$ are both Ulrich ideals of R with $\nu(I) = 2$ and $\nu(J) = n > 2$.

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