系列的 Cohen-Macaulay グラフの研究とその応用

明治大学大学院理工学研究科

２０１６年度

博士学位請求論文

Study of sequentially Cohen—Macaulay graphs and its applications

（系列的 Cohen—Macaulay グラフの研究とその応用）

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Study of sequentially Cohen–Macaulay graphs and its applications

A Dissertation

Presented to

The School of Science and Technology

Meiji University

Department of Mathematics

Professor Yukio Nakamura, Advisor

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

by

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February, 2017

Acknowledgments

First of all, I would like to thank the people concerned with this research life for many wonderful helps.

I would like to give my particular thanks Professor Yukio Nakamura of Meiji University, who provided mathematical valuable advices and further provided various supports during this research life. I also would like to thank Professor Shiro Goto for giving me opportunities for my mathematics and for making important advices to the preparation of this research. I also want to send my gratitude to my colleagues.

Many thanks also go to the Graduate School of Science and Technology and Department of Mathematics of School of Science and Technology at Meiji University.

Finally, I would like to present my thanks my parents for watching and supporting this research life.

Contents

Introduction

This thesis deals with the Stanley–Reisner ring over a field. In particular, we investigate the Stanley–Reisner ring defined by the edge ideals. First of all, we state necessary notations. Let $S = K[X_1, \ldots, X_n]$ be the polynomial ring of n variables over a field K. Let G be a simple graph with the set of vertices $V(G) = \{1, \ldots, n\}$. We denote by $E(G)$ the set of edges of G, where we describe each edge as a two-element subset $\{i, j\}$ of $V(G)$. The square-free monomial ideal $I(G) = (X_iX_j \mid \{i, j\} \in E(G))$ of S is called the edge ideal of G. We say that G is Cohen–Macaulay (or sequentially Cohen–Macaulay) if $S/I(G)$ is a Cohen–Macaulay ring (or a sequentially Cohen–Macaulay ring, respectively).

The study of edge ideals was started by Villarreal in [30]. Estrada and Villarreal gave the characterization of Cohen–Macaulay bipartite graphs in terms of simplicial complexes ([7]). Herzog and Hibi gave the characterization for bipartite graphs to be Cohen–Macaulay in terms of the edges ([12]). After that, Van Tuyl and Villarreal gave the characterization of sequentially Cohen–Macaulay bipartite graphs in terms of simplicial complexes ([28]) and studied the behavior of the edges of sequentially Cohen–Macaulay bipartite graphs that have a perfect matching ([28], cf.[25]). In the case of complete multipartite graphs, Kiani and Seyyedi studied the Cohen–Macaulayness and the sequential Cohen–Macaulayness of G ([21], cf.[26]).

The main purpose of this thesis is to detect a characterization for graphs to be sequentially Cohen–Macaulay in terms of the behavior of the edges. In this thesis, we introduce the almost complete multipartite graph whose class contains the bipartite graph and the complete multipartite graph, and we describe a sufficient condition for the graphs to be sequentially Cohen–Macaulay in terms of the edges. Moreover, we give a characterization for the graphs to be sequentially Cohen–Macaulay in terms of the simplicial complex.

The Castelnuovo–Mumford regularity is one of the most important invariants in the commutative ring theory. We denote by $\mathrm{reg}(S/I(G))$ the Castelnuovo–Mumford regularity of $S/I(G)$ and we call $\mathrm{reg}(S/I(G))$ the regularity of G. Finding the estimate of $\mathrm{reg}(S/I(G))$ in terms of G is an interesting problem. In particular, we are interested in a characterization for $\mathrm{reg}(S/I(G))$ in terms of the induced matching number of G (denoted by $\mathrm{im}(G)$). Katzman showed in [17] that the inequality $\mathrm{reg}(G) \geq \mathrm{im}(G)$ holds in general. Several classes of graphs satisfying the equality $\mathrm{reg}(G) = \mathrm{im}(G)$ have been found by researchers. One of the purposes of this thesis is to give new classes satisfying the equality $\mathrm{reg}(G) = \mathrm{im}(G)$. For example, we will show that the equality $\mathrm{reg}(G) = \mathrm{im}(G)$ holds if G belongs to the sequentially Cohen–Macaulay almost complete multipartite graph.

In the rest, we discuss the organization of this thesis.

Chapter 1 is a preliminary for the later argument. We first state notations and terminologies of the graph and the simplicial complex. Next, we recall the definition of sequential Cohen–Macaulayness. Then, we give properties related to not only the sequential Cohen–Macaulayness, purity, shellability, vertex decomposability and so on. We will list the results about such properties. The definitions of some invariants (the Castelnuovo–Mumford regularity and the induced matching number) are stated.

Chapter 2 is based on [15]. We study sequentially Cohen–Macaulay graphs which contain cycles. We extend the concept of cycles in order to give a characterization for bipartite graphs to be sequentially Cohen–Macaulay. We also study the shellability of graphs intersecting with cyclic graph at one vertex.

Chapter 3 is based on [16]. We introduce the concept of almost complete multipartite graphs. We study the purity, vertex decomposability, Cohen–Macaulayness, and sequential Cohen–Macaulayness of almost complete multipartite graphs. The main result of Chapter 3 is a characterization of sequentially Cohen–Macaulay almost complete multipartite graphs. As an application, we compute the regularity of the graphs.

In Chapter 4, we introduce the semi-unmixed graph. We explore a relation between of the purity of almost complete multipartite graphs and semi-unmixed graphs. Then, we give a characterization of sequentially Cohen–Macaulay semi-unmixed graphs. Moreover, from such an argument, we obtain a sufficient condition for bipartite graphs to be sequentially Cohen–Macaulay in terms of the edges. As applications, an alternative proof of the characterization of Herzog–Hibi ([12]) is given. Besides, we obtain a sufficient condition for almost complete multipartite graphs to be sequentially Cohen–Macaulay in terms of the edges.

Chapter 1

Preliminaries

In this chapter, we state notations and basic properties which are needed afterward in our arguments. In Section 1, we present the notations and terminologies of graphs and simplicial complexes. After, we state the operations of graphs corresponding to the operations (link, star and deletion) of simplicial complexes. In Section 2, we recall the basic conditions (vertex decomposable, shellable, and sequentially Cohen–Macaulay) of the simplicial complex. Several results on these conditions are listed. In Section 3, we recall two invariants. One is the Castelnuovo–Mumford regularity for commutative algebras. The other is the induced matching number for graphs. We state the relations between the two invariants.

1.1 Notations of graphs and simplicial complexes

We first state notations of the graph. For more details of graph concepts, we refer the reader to [4]. Let G be a simple graph. A simple graph means that it has no loops and no multiple edges. In this thesis, G is simply called a graph. We denote by $V(G)$ the set of vertices of G and by $E(G)$ the set of edges of G. Let V be a subset of $V(G)$. The subgraph induced of G by V, denoted by G_V, means the graph whose $V(G_V)$ coincides with V and $E(G_V)$ consists of $\{x, y\} \in E(G)$, where $x, y \in V$. We denote by $G \setminus V$ the subgraph of G induced by $V(G) \setminus V$. Let $H_1, \ldots, H_r$ be induced subgraphs of G. If $V(G) = V(H_1) \sqcup \cdots \sqcup V(H_r)$ and

$E(G) = E(H_1) \sqcup \cdots \sqcup E(H_r)$, we write $G = H_1 \sqcup \cdots \sqcup H_r$.

The set of neighbors of a vertex v in G is denoted by $N_G(v)$. We set $N_G[v] = N_G(v) \cup \{v\}$. Generally, for a subset $V \subseteq V(G)$, the intersection of $V(G) \setminus V$ and the union of neighbors of vertices in V is called the neighbors of V, which is denoted by $N_G(V)$. Similarly, $N_G[V]$ is defined as $N_G(V) \cup V$. For the induced subgraph H of G, we put $N_G(H) = N_G(V(H))$ and $N_G[H] = N_G[V(H)]$. The degree of v in G is the number of elements of $N_G(v)$, which is denoted by $\deg_G(v)$. A vertex v in G with $\deg_G(v) = 0$ is called an isolated vertex. The set of isolated vertices is denoted by $\mathrm{iso}(G)$.

A subset V of $V(G)$ is called independent if $E \not\subseteq V$ for any $E \in E(G)$. Let $c \geq 1$ be an integer. G is called a c-partite graph, if there exist independent sets $V_1, \ldots, V_c$ of G such that $V(G)$ coincides with the disjoint union $V_1 \sqcup \cdots \sqcup V_c$. When G is a c-partite graph with the partition $V_1, \ldots, V_c$, we denote it by $(G\,;\,V_1, \ldots, V_c)$. In particular, $(G\,;V_1, V_2)$ means a bipartite graph with the partitions V_1 and V_2. A bipartite graph $(H\,;\,X, Y)$ is called complete if $\{x, y\} \in E(H)$ for all $x \in X$ and all $y \in Y$. Let $(G\,;\,V_1, \ldots, V_c)$ be a c-partite graph; then, the induced subgraph $G_{V_i \sqcup V_j}$ is a bipartite graph for any i, j with $i < j$. A c-partite graph $(G\,;\,V_1, \ldots, V_c)$ is called a complete c-partite graph if every $G_{V_i \sqcup V_j}$ is a complete bipartite graph for all i, j with $i < j$. When we do not specify the number c, $G = (G\,;\,V_1, \ldots, V_c)$ is simply called a complete multipartite graph.

We denote by $\Delta(G)$ the set of independent sets of G and by $\mathcal{F}(G)$ the set of maximal independent sets among $\Delta(G)$. We regard $\emptyset$ as an independent set. The number of maximum size of independent sets of G is denoted by $\dim(G)$. It easily follows that $\Delta(G)$ forms a simplicial complex on $V(G)$. In the rest of this section, we recall some definitions and fundamental properties of simplicial complexes, which we will need later. A simplicial complex Δ on $[n] = \{1, \ldots, n\}$ is a collection of the subsets of $[n]$ such that $F \in \Delta$ whenever $F \subseteq F'$ for some $F' \in \Delta$ and such that $\{i\} \in \Delta$ for all $i \in [n]$. The element of Δ is called a face, and the dimension, $\dim F$, of a face F is the number $|F| - 1$. Note that $\emptyset$ is a face of dimension -1 for any nonempty simplicial complex. Faces of dimensions 0 and 1 are called a vertex and an edge, respectively. A maximal face of Δ is called a facet of Δ. Let $\mathcal{F}(\Delta)$ stand for the set of facets of Δ. The dimension of Δ is the maximum number of $\dim H$ for all $H \in \mathcal{F}(\Delta)$, which is denoted by $\dim \Delta$. Note that $\dim \Delta = -1$ if $\Delta = \{\emptyset\}$. For each graph G, it follows that $\mathcal{F}(\Delta(G)) = \mathcal{F}(G)$. We say that Δ is pure if $\dim F = \dim \Delta$ for all $F \in \mathcal{F}(\Delta)$. We denote the minimum number of $\dim H$ for all $H \in \mathcal{F}(\Delta)$ by $\text{min-dim}\,\Delta$. Therefore, one can say that Δ is pure if and only if $\text{min-dim}\,\Delta = \dim \Delta$. Let $\mathcal{N}(\Delta)$ be the set of minimal elements among the subsets $F \subseteq [n]$ with $F \notin \Delta$. We note that $\mathcal{N}(\Delta(G)) = E(G)$ for any graph G.

Let $\mathcal{P}$ be a collection of subsets of $[n]$. Then, there exists the smallest simplicial complex, say $\langle \mathcal{P} \rangle$, which contains $\mathcal{P}$. $\langle \mathcal{P} \rangle$ is called the simplicial complex generated by $\mathcal{P}$, whose vertex set consists of $\{i\,;\, i \in X \in \mathcal{P}\}$. Under this notation, obviously $\Delta = \langle \mathcal{F}(\Delta) \rangle$. Δ is called a simplex if $|\mathcal{F}(\Delta)| = 1$. Let Δ be a simplicial complex and Δ' be a subset of Δ. Δ' is called a subcomplex of Δ if $\Delta' = \langle \Delta' \rangle$. For $F \in \Delta$, we recall the definitions of the star of F, the deletion of F and the link of F, which are subcomplexes of Δ:

$$\begin{aligned} \text{star}_\Delta F &= \{H \in \Delta\,;\, F \cup H \in \Delta\}, \\ \text{del}_\Delta F &= \{H \in \Delta\,;\, F \cap H = \emptyset\}, \\ \text{link}_\Delta F &= \text{star}_\Delta F \cap \text{del}_\Delta F. \end{aligned}$$

For $0 \leq i \leq \dim\Delta$, let $\Delta(i)$ denote the simplicial complex generated by the faces of Δ of dimension i. Then, $\Delta(i)$ is the pure subcomplex of Δ and called the i-th pure skeleton of Δ. We say that Δ is a flag complex if $|V| = 2$ for any $V \in \mathcal{N}(\Delta)$. Let $\Delta(G)$ be the simplicial complex associated with a graph G. Then $\Delta(G)$ is a flag complex. For $F \in \Delta(G)$, the star, deletion, and link of F in $\Delta(G)$ are flag complexes associated with certain induced subgraphs of G. Namely, we have the following relations:

$$\begin{aligned}
&\operatorname{star}_{\Delta(G)} F = \Delta(G \setminus N_G(F)),\\
&\operatorname{del}_{\Delta(G)} F = \Delta(G \setminus F),\\
&\operatorname{link}_{\Delta(G)} F = \Delta(G \setminus N_G[F]).
\end{aligned}$$

1.2 Sequential Cohen–Macaulayness and properties of simplicial complexes

Firstly, we define the sequential Cohen–Macaulayness. Let K be a field and $S = K[X_1, \ldots, X_n]$ be the polynomial ring with the variables $X_1, \ldots, X_n$ over K. Throughout this thesis, we always consider properties of a Stanley–Reisner ring over a fixed field K. The concept of sequential Cohen–Macaulayness is defined by Stanley as follows:

Definition 1.1. ([27]) *Let M be a finite generated $\mathbb{Z}$-graded S-module. We say that M is sequentially Cohen–Macaulay if there exists a finite filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$$

of M by graded submodules M_i satisfying the following two conditions:

(1) *Each quotient M_i/M_{i-1} is Cohen–Macaulay.*

(2) $\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1})$.

Let Σ be a simplicial complex on $[n]$. For $F \in [n]$, we set $X_F = \prod_{i \in F} X_i$. I_Σ is an ideal generated by all monomials X_F, where $F \notin \Sigma$ and is called the Stanley–Reisner ideal of Σ. The residue class ring S/I_Σ is also called the Stanley–Reisner ring of Σ. It is known that $\dim S/I_\Sigma = \dim\Sigma + 1$. We say that Σ is sequentially Cohen–Macaulay if S/I_Σ is a sequentially Cohen–Macaulay ring. We also say that Σ is called Cohen–Macaulay if S/I_Σ is a Cohen–Macaulay ring. In [6], Duval gave another characterization for Σ to be sequentially Cohen–Macaulay as follows:

Theorem 1.2. ([6]) *Σ is sequentially Cohen–Macaulay if and only if the i-th pure skeleton $\Sigma(i)$ is Cohen–Macaulay for all $0 \le i \le \dim \Sigma$.*

By Theorem 1.2, we obtain the following two results. For more details we refer the reader to [13].

Proposition 1.3. *Σ is Cohen–Macaulay if and only if Σ is pure and sequentially Cohen–Macaulay.*

Proposition 1.4. (cf.[28, Theorem 3.3.]) *If Σ is sequentially Cohen–Macaulay, then $\mathrm{link}_\Sigma F$ is sequentially Cohen–Macaulay for all $F \in \Sigma$.*

Secondly, we recall several properties of simplicial complexes that are related to sequentially Cohen–Macaulay algebras. Let Σ be a simplicial complex. The shellability of simplicial complex, which is not necessarily pure, is defined in [1] as follows: A simplicial complex Σ is shellable if Σ is a simplex, or there exists a numbering of $\mathcal{F}(\Sigma) = \{F_1, \ldots, F_t\}$ such that $\langle F_1, \ldots, F_{i-1}\rangle \cap \langle F_i \rangle$ is a pure simplicial complex of dimension $\dim F_i - 1$ for all $2 \le i \le t$. A numbering of $\mathcal{F}(\Sigma)$ satisfying the above conditions is called a shelling of Σ. When Σ is shellable, it is known that the i-th pure skeleton $\Sigma(i)$ is also shellable for all $0 \le i \le \dim\Sigma$ (cf.[1, 2.9. Theorem.]). Combining the above facts and Theorem 1.2, we have the following well-known result.

Proposition 1.5. *If Σ is shellable, then Σ is sequentially Cohen–Macaulay.*

In this thesis, as definition of shellability, we apply the following statement. For more details we refer the reader to [13].

Proposition 1.6. (cf. [13]) *Let Σ be a simplicial complex. Then, the following conditions are equivalent..*

(1) *Σ is shellable.*

(2) *There exists a numbering* $\mathcal{F}(\Sigma) = \{F_1, F_2, \ldots, F_r\}$ *satisfying either the following condition:*

(a) $r = 1$.

(b) $r > 1$ *and for any two facets* F_i *and* F_j *with* $j < i$*, there is an integer* $k < i$ *and a vertex* $l \in F_i \setminus F_j$ *such that* $F_i \setminus F_k = \{l\}$.

In this case, a numbering of $\mathcal{F}(\Sigma)$ *satisfying* (2)(b) *is a shelling of* Σ.

Let v be a vertex of Σ. A vertex v is called a shedding vertex if $\mathcal{F}(\mathrm{del}_\Sigma\{v\}) \subseteq \mathcal{F}(\Sigma)$, which is equivalent to saying that $\mathcal{F}(\mathrm{del}_\Sigma\{v\}) \cap \mathcal{F}(\mathrm{link}_\Sigma\{v\}) = \emptyset$. We say that Σ is vertex decomposable if Σ is a simplex, or there exists a shedding vertex v of Σ such that both $\mathrm{del}_\Sigma\{v\}$ and $\mathrm{link}_\Sigma\{v\}$ are vertex decomposable (cf. [2]). We recall the following famous result.

Proposition 1.7. (cf.[2, 11.3. Theorem.]) *If* Σ *is vertex decomposable, then* Σ *is shellable.*

Next, we introduce several properties in terms of graphs and recall the important results concerning to our argument. Let G be a graph. The square-free monomial ideal $I(G) = (X_iX_j\,;\,\{i,j\} \in E(G))$ of S is called the edge ideal of G. Since $I(G) = (X_iX_j\,;\,\{i,j\} \in \mathcal{N}(\Delta(G))) = I_{\Delta(G)}$, $S/I(G)$ coincides with the Stanley–Reisner ring $S/I_{\Delta(G)}$. Therefore, we say that G is sequentially Cohen–Macaulay, vertex decomposable, shellable and Cohen–Macaulay if the associated simplicial complex $\Delta(G)$ is as well. Besides, the shelling of $\Delta(G)$ is called a shelling of G. Furthermore, if v is a shedding vertex of $\Delta(G)$, v is called shedding vertex of G. From our notations and the above of Section 2, G is vertex decomposable if and only if $E(G) = \emptyset$ or there exists a shedding vertex v such that $G \setminus \{v\}$ and $G \setminus N_G[v]$ are vertex decomposable. The study of the vertex decomposability of graphs was started by Dochtermann–Engström ([5]) and Woodroofe ([32, Remark 8]).

We list the important results applied in our argument.

Lemma 1.8. (cf.[28, Lemma 3.9.]) *Let* $(H\,;\,X, Y)$ *be a bipartite graph. Suppose that* $|X| \leq |Y|$ *and* $\mathrm{iso}(H) = \emptyset$*. If* H *is sequentially Cohen–Macaulay, then there exists* $y \in Y$ *such that* $\deg_H(y) = 1$*.*

In Chapter 2, we will give an alternative proof of Lemma 1.8.

The converse of Proposition 1.5 and Proposition 1.7 is always not true. In the case of bipartite graphs, the following statement is true by virtue of Lemma 1.8.

Theorem 1.9. ([29, Theorem 2.10.], cf.[28, Theorem 3.10.]) *Let* H *be a bipartite graph. The following conditions are equivalent.*

(1) H *is vertex decomposable.*

(2) H *is shellable.*

(3) H *is sequentially Cohen–Macaulay.*

When $\Delta(G)$ is pure, $I(G)$ is an unmixed ideal (i.e., the height of every prime ideal belonging to $I(G)$ is constant). We simply say that G is unmixed if $I(G)$ is as well.

Finally, in this section, we list some other results of [28] about the shellability of graphs.

Proposition 1.10. (cf. [28, Lemma 2.4]) *Let* G *be a gprah and* $H_1, \ldots, H_r$ *be an induced subgraph of* G *with* $G = H_1 \sqcup \cdots \sqcup H_r$*. Then, the following conditions are equivalent..*

(1) G *is shellable.*

(2) *The components* $H_1, \ldots, H_r$ *is shellable.*

Proposition 1.11. (cf. [28, Theorem 2.6]) *Let* G *be a graph. If* G *is shellable, then* $G \setminus N_G[V]$ *is shellable for all* $V \in \Delta(G)$*.*

Theorem 1.12. ([28, Theorem 2.9.]) *Let G be a graph. Suppose that G has vertices x and y with $N_G(x) = \{y\}$. Put $G_1 = G \setminus N_G[x]$ and $G_2 = G \setminus N_G[y]$. Then, the following conditions are equivalent..*

(1) *G is shellable.*

(2) *Both G_1 and G_2 are shellable*

In this case, if the shellings are given as $\mathcal{F}(G_1) = \{F_1, \dots, F_r\}$ and $\mathcal{F}(G_2) = \{H_1, \dots, H_s\}$, then

$$\mathcal{F}(G) = \{F_1 \cup \{x\}, \dots, F_r \cup \{x\}, H_1 \cup \{y\}, \dots, H_s \cup \{y\}\}$$

is a shelling of G. Moreover, for any two facets $H_i \cup \{y\}$ and $F_j \cup \{x\}$, there is a facet $F_k \cup \{x\}$ such that $H_i \cup \{y\} \setminus F_k \cup \{x\} = \{y\}$.

1.3 Regularities and induced matching numbers

Let S be a polynomial ring over a field K. Let M be a finite generated $\mathbb{Z}$-graded S-module. The Castelnuovo–Mumford regularity of M, or simply the regularity of M, is defined as

$$\operatorname{reg}(M) = \max\{j\,;\ \operatorname{Tor}_S^i(K, M)_{i+j} \neq 0 \text{ for some } i\},$$

which is one of the most important invariants in commutative algebra and algebraic geometry (e.g., see [3]). We are interested in determining $\operatorname{reg}(S/I(G))$ in terms of the graph G. For a simplicial complex Σ, we set $\operatorname{reg}(\Sigma) = \operatorname{reg}(S/I_\Sigma)$ and call it the regularity of Σ. For a graph G, we set $\operatorname{reg}(G) = \operatorname{reg}(\Delta(G))$ and call it the regularity of G. By virtue of Hochster's formula, it follows that $\operatorname{reg}(\Sigma') \leq \operatorname{reg}(\Sigma)$ for any subcomplex Σ' of Σ. Therefore, if H is an induced subgraph of G, then $\Delta(H)$ is a subcomplex of $\Delta(G)$; thus, the inequality $\operatorname{reg}(H) \leq \operatorname{reg}(G)$ holds. For a graph G, a subset M of $E(G)$ is called an induced matching if $|M| = 1$ or $N_G[e_1] \cap e_2 = \emptyset$ for any different edges e_1, $e_2 \in M$. The induced matching number of G is defined as

$$\mathrm{im}(G) = \max\{|M|\,;\, M \text{ is an induced matching of } G\},$$

which is closely related to the regularity of G.

Katzman showed in [17] that the inequality $\mathrm{reg}(G) \geq \mathrm{im}(G)$ holds in general. It is known that the equality holds if the graph belongs to one of the following classes: the forest ([33]), chordal graphs ([11]), unmixed bipartite graphs ([18]), Cohen–Macaulay bipartite graphs ([8]), very well covered graphs ([23]), sequentially Cohen–Macaulay bipartite graphs ([29]), C_5-free vertex decomposable graphs ([19]) and Cameron–Walker graphs ([14]). In particular, we state the following claim that is important in our research below:

Proposition 1.13. *Let G be a graph. The inequality* $\mathrm{im}(G) \leq \mathrm{reg}(G)$ *always holds* ([17]). *The equality holds if one of the following conditions are satisfied.*

(1) ([11, Corollary 6.9]) *G is a chordal graph,*

(2) ([18]) *G is a bipartite graph and unmixed,*

(3) ([19]) *G is vertex decomposable and has no cycle of length 5.*

Note that a sequentially Cohen–Macaulay bipartite graph satisfies condition (3) of Proposition 1.13. In particular, so does a forest (cf.[19]).

Chapter 2

Characterization by cyclic graphs

Let $S = K[X_1, \ldots, X_n]$ be the polynomial ring over fixed field K and G be a graph with n vertices. $I(G)$ is the edge ideal of G. We say that G is sequentially Cohen–Macaulay if $S/I(G)$ is a sequentially Cohen–Macaulay ring.

G is called a forest if G has no cycles. Since the theory of graphs, forests are bipartite graphs. If G is a forest, then G is sequentially Cohen–Macaulay([33]). So, we are interested in the sequential Cohen–Macaulayness of bipartite graphs which have cycles. In this chapter, we first give an alternative proof of Lemma 1.8. Next, we investigate the relation between cycles and vertices of degree 1 on bipartite graphs (Theorem 2.3). The conclusion gives an improvement of Lemma 1.8. Besides, we will introduce a concept, which is analogy of cycles, and give a characterization for bipartite graphs to be sequentially Cohen–Macaulay. Furthermore, for (not necessary bipartite) graphs, we give a sufficient condition to be sequentially Cohen–Macaulay. As an application, we show that G is sequentially Cohen–Macaulay if there is a vertex $v \in N_G(C)$ with $\deg_G(v) = 1$ for all $C \in C(G)$ (Corollary 2.6).

We say that G is shellable if $\Delta(G)$ is shellable. Let $\{x, y\}$ be an edge of G with $N_G(x) = \{y\}$. We put $G_1 = G \setminus N_G[x]$ and $G_2 = G \setminus N_G[y]$. In [28], Tuyl and Villarreal show that G is shellable if and only if G_1 and G_2 are shellable (Theorem 1.12). We consider the graphs replaced $\{x, y\}$ by cyclic graph C_5 and show an analogous statement of Theorem 1.12 (Theorem 2.9). As an application, we investigate the shellability of graphs which are two cyclic graphs intersecting at only one vertex.

2.1 Sequential Cohen–Macaulayness of bipartite graphs with cycles

To give an alternative proof of Lemma 1.8, we recall the following condition of simplicial complexes.

Definition 2.1. ([12]) *Let Δ be a pure simplicial complex of the dimension $d-1$. We say that Δ is connected codimension one if for each two facets F and G of Δ, there is a sequence of facets $F = F_0, F_1, \ldots, F_{q-1}, F_q = G$ such that $|F_i \cap F_{i+1}| = d-1$.*

The following statement is a well-known fact. For the proof, we refer the reader to [13]

Proposition 2.2. *Let Δ be a simplicial complex. If Δ is Cohen–Macaulay, then Δ is connected in codimension one.*

In here, we denote by $\Delta(G)^{[i]}$ the $(i-1)$-th pure skeleton for all $0 \le i \le \dim G$. (*Proof of Lemma 1.8*)

By Theorem 1.2, $\Delta(G)^{[i]}$ is Cohen–Macaulay for all $0 \le i \le \dim G$. Applying Proposition 2.2, $\Delta(G)^{[i]}$ is connected in codimension one for all $0 \le i \le \dim G$. In particular, so is $\Delta(G)^{[|X|]}$. We take $Y' \subseteq Y$ with $|Y'| = |X|$. Then, we get a sequence of facets $X = F_0, F_1, \ldots, F_r = Y'$ such that $|F_i \cap F_j| = |X| - 1$. In particular, $|X \cap F_1| = |X| - 1$. We put $\{y\} = F_1 \setminus X \subseteq Y$. Then, we obtain $\deg_G(y) = 1$. □

We give an improvement of Lemma 1.8 by cycles of G. The set of cycles of G is denoted by $C(G)$. For details of graph concepts (e.g., path, cycle, bridge and so on), we refer the reader to [4].

Theorem 2.3. *Let G be a bipartite graph with $C(G) \neq \emptyset$. If G is sequentially Cohen–Macaulay, then there is a vertex $x \in N_G(C)$ with $\deg_G(x) = 1$ for some $C \in C(G)$.*

Proof. We may assume that G is connected. Applying Lemma 1.8, we take a vertex v with $\deg_G(v) = 1$. We put $N_G(v) = \{w\}$, $t = |N_G(w) \setminus \{v\}|$, and $N_G(w) \setminus \{v\} = \{v_1, \ldots, v_t\}$. We may assume that $w \notin V(C)$ for all $C \in C(G)$. First, we consider the case $t \geq 2$. Put $G_1 = G \setminus N_G[v]$ and $G_2 = G \setminus N_G[w]$. By Proposition 1.4, G_1 and G_2 are sequentially Cohen–Macaulay. Let H_i be the connected component of G_1 with $v_i \in V(H_i)$ for all $1 \leq i \leq c$. We put $H_i' = H_i \setminus \{v_i\}$. Then, H_i' is an induced subgraph of G_2. Since $w \notin V(C)$ for all $C \in C(G)$, it follows that $\{w, v_1\}, \ldots, \{w, v_t\}$ are bridges of G. Therefore, we obtain

$$G_1 = H_1 \sqcup \cdots \sqcup H_t, \quad G_2 = H_1' \sqcup \cdots \sqcup H_t'$$

by [4]. We put $G_{(i)} = G_{V(H_i)\cup\{v,w\}}$ and consider the induced subgraphs $G_{(i),1} = G_{(i)} \setminus \{v, w\}$, $G_{(i),2} = G_{(i)} \setminus \{v, w, v_i\}$ for all $1 \leq i \leq t$. Since $V(G_{(i),1}) = V(H_i)$ and $V(G_{(i),2}) = V(H_i')$, then it follows that $G_{(i),1} = H_i$ and $G_{(i),2} = H_i'$. From Proposition 1.10, $G_{(i),1}$ and $G_{(i),2}$ are sequentially Cohen–Macaulay. Therefore, $G_{(i)}$ is sequentially Cohen–Macaulay by Theorem 1.12 and Theorem 1.9. We have $C(G_1) = C(G)$ because $\{v, w\} \cap C = \emptyset$ for all $C \in C(G)$. Therefore, for some $1 \leq k \leq t$, it follows that $C(H_k) \neq \emptyset$; hence, $C(G_{(k)}) \neq \emptyset$. Suppose that there is a vertex $v \in N_{G_{(k)}}(C)$ with $\deg_{G_{(k)}}(x) = 1$ for some $C \in C(G_{(k)})$. We claim that $\deg_G(x) = 1$. Indeed, if $\deg_G(x) \geq 2$, we take $u \in V(G) \setminus V(G_{(k)})$ such that $\{x, u\} \in E(G)$. Since $\{x, u\} \subseteq V(G_1)$, it follows that $\{x, u\} \in E(G_1) = \sqcup_{i=1}^t E(H_i)$. Therefore, we have $\{x, u\} \in E(H_k)$. Besides, we also have $u \in V(H_k) \subseteq V(G_{(k)})$. This is a contradiction. Thus, we need only consider the case $t = 1$.

Fix $t = 1$ and put $N_G(w)\setminus\{v\} = \{x\}$. Suppose that this statement is false. Then, we can find a counter example G for this statement. We may assume that $|V(G)|$ is minimum among the counter examples. By a similar argument (see the case $t \geq 2$),

G_1 is sequentially Cohen–Macaulay with $C(G_1) \neq \emptyset$. By $|V(G_1)| < |V(G)|$, we can take a cycle D of G_1 and a vertex $y \in N_{G_1}(D)$ such that $\deg_{G_1}(y) = 1$. We note $\deg_G(y) \geq 2$ by our assumption. Therefore, $y \in N_G(w)$; hence, $y = x$. The vertex x is not contained in any cycle of G because $C(G) = C(G_1)$ and $\deg_{G_1}(x) = 1$. Since $G_2 = G \setminus \{v, w, x\}$, it follows that $C(G_2) = C(G) \neq \emptyset$. By $|V(G_2)| < |V(G)|$, we can take a cycle E of G_2 and a vertex $z \in N_{G_2}(E)$ such that $\deg_{G_2}(z) = 1$ and $\deg_G(z) \geq 2$. Since $z \notin \{v, w\}$, it follows that $z \in N_{G_1}(x)$. Then, combining $\deg_{G_1}(x) = 1$ and $x \in N_{G_1}(D)$, we get $z \in V(D)$. Now $C(G_1) = C(G) = C(G_2)$, it follows that $\deg_{G_2}(z) \geq 2$, a contradiction. Thus, this statement must be true. □

Let $v, w \in V(G)$ and $X, Y \subseteq V(G)$. The minimum length of paths from v to w is denoted by $\mathrm{dist}_G(v, w)$. We denote by $\mathrm{dist}_G(X, Y)$ the minimum number of $\mathrm{dist}_G(x, y)$ for all $x \in X$ and $y \in Y$. For a positive integer m, we put

$$D_G^m(X) = \{x \in V(G)\,;\, \mathrm{dist}_G(x, X) = m\}.$$

Moreover, for the induced subgraph H of G, we put $D_G^m(H) = D_G^m(V(H))$.

We denote by $T(G)$ the set of the induced subgraphs K of G such that all the degrees of vertices of K are at least two; i.e.,

$$T(G) = \{K : \text{induced subgraph of } G\,;\, \deg_K(v) \geq 2 \text{ for all } v \in V(K)\}.$$

If C is an induced subgraph of G by vertices of a cycle, then it follows that $C \in T(G)$. We give the characterization of sequentially Cohen–Macaulay bipartite graphs in terms of $T(G)$.

Theorem 2.4. *Let G be a bipartite graph. Then, the following conditions are equivalent.*

(1) *G is sequentially Cohen–Macaulay.*

(2) *For all $K \in T(G)$ and $L \subseteq D_G^2(K)$ where $L \in \Delta(G)$, the neighborhood $N_{G \setminus N_G[L]}(K)$ is no empty set.*

Proof. (1) $\Rightarrow$ (2) : Let $K \in T(G)$, $L \subseteq D_G^2(K)$ and L be an independent set of G. We put $G_{(L)} = G \setminus N_G[L]$. K is the induced subgraph of $G_{(L)}$ because $V(K) \subseteq V(G) \setminus N_G[L]$. Consider the connected component H of $G_{(L)}$ such that $V(K) \subseteq V(H)$. Since G is sequentially Cohen–Macaulay, H is also sequentially Cohen–Macaulay. By Lemma 1.8, H has a vertex v with $\deg_H(v) = 1$. We claim that $v \notin V(K)$. Indeed, if $v \in V(K)$, then it follows that

$$2 \leq \deg_K(v) \leq \deg_{G_{(L)}}(v) = \deg_H(v) = 1,$$

a contradiction. Since H is the connected component of $G_{(L)}$, there is a path containing v and some vertices of K. Thus, we obtain $N_{G_{(L)}}(K) \neq \emptyset$.

(2) $\Rightarrow$ (1) : We may assume that G has no isolated vertices. Suppose that $G \in T(G)$. It is always true that $D_G^2(G) = \emptyset$. Taking $L = \emptyset$, we obtain $N_{G_{(L)}}(G) = N_G(G) = \emptyset$. This contradicts to our assumption. Therefore, $G \notin T(G)$; hence, G has a vertex v of degree 1.

Suppose that G is not sequentially Cohen–Macaulay. We put $N_G(v) = \{w\}$, $G_1 = G \setminus N_G[v]$, and $G_2 = G \setminus N_G[w]$. If G_1 and G_2 have no vertices of degree 1, we have nothing to do. If G_1 has a vertex v_1 of degree 1, we put $N_{G_1}(v_1) = \{w_1\}$, $(G_1)_1 = G_1 \setminus N_{G_1}[v_1]$, and $(G_1)_2 = G_1 \setminus N_{G_1}[w_1]$. If G_2 also has a vertex v_2 of degree 1, similarly, we put $N_{G_2}(v_2) = \{w_2\}$, $(G_2)_1 = G_2 \setminus N_{G_2}[v_2]$, and $(G_2)_2 = G_2 \setminus N_{G_2}[w_2]$. We also consider the same operations for $(G_1)_1$, $(G_1)_2$, $(G_2)_1$, and $(G_2)_2$. By $|V(G)| < \infty$, these operations finish in finite steps. We denote by $\mathfrak{X}$ the set of the induced subgraphs without vertices of degree 1 which is given by the above operations.

Let $H \in \mathfrak{X}$. If $E(H) = \emptyset$, H is shellable; hence H is sequentially Cohen–Macaulay. Repeated application of Theorem 1.12, it follows that G_1 and G_2 are sequentially Cohen–Macaulay; hence, we conclude that G is sequentially Cohen–Macaulay. But, this is a contradiction. Therefore, $E(H) \neq \emptyset$. We put $K = H \setminus \mathrm{iso}(H)$. Since K have no vertices of degree ≤ 1, we get $K \in T(G)$. From the definition of the induced subgraph H, we can take $L' \in \Delta(G)$ and can write $H = G \setminus N_G(L')$. Because K is the induced subgraph of H, it follows that $N_G(K) \cap L' = \emptyset$. Put $L = L' \cap D_G^2(K)$. L is an independent set of G.

For $K \in T(G)$ and $L \subseteq D_G^2(K)$, we have $N_{G \setminus N_G[L]}(K) \neq \emptyset$ by our assumption. Take $v \in N_{G \setminus N_G[L]}(K)$. By definition of K, there is a vertex $u \in L'$ such that $v \in N_G(u)$. Since $N_G(K) \cap L' = \emptyset$, it follows that $u \notin N_G(K)$. Therefore, we get $u \in D_G^2(K)$; hence, $u \in L$. Since $v \in N_G(u)$, it follows that $v \notin V(G_{(L)})$. This contradicts the choice of v. Thus, G is sequentially Cohen–Macaulay. □

According the part ((2) ⇒ (1)) of the proof of Theorem 2.4, we have proved without the conditions of bipartite graphs. Therefore, the following statement follows.

Theorem 2.5. *Let G be a graph. If $N_G(K) \setminus N_G[L] \neq \emptyset$ for all $K \in T(G)$ and $L \subseteq D_G^2(K)$ where $L \in \Delta(G)$, then G is shellable and sequentially Cohen–Macaulay.*

The converse of Theorem 2.3 is always not true. As an application of Theorem 2.5 (or Theorem 2.4), we give the following corollary.

Corollary 2.6. *Let G be a graph. If there is a vertex $v \in N_G(C)$ with $\deg_G(v) = 1$ for all $C \in C(G)$, then G is shellable and sequentially Cohen–Macaulay.*

Proof. Let $K \in T(G)$, $L \subseteq D_G^2(K)$ and L be an independent set of G. We can take a cycle C that is a subgraph of K. By our assumption, there is a vertex $v \in N_G(C)$ such that $\deg_G(v) = 1$. Therefore, $v \in N_G(K)$; hence, $v \notin N_G[L]$. Thus, we get $v \in N_G(K) \setminus N_G[L]$. By Theorem 2.5, G is shellable and sequentially Cohen–Macaulay. □

2.2 Shellability of graphs intersecting with a cyclic graph at one vertex

Let C_i be the cyclic graph with i vertices. We first recall the result of sequentially Cohen–Macaulay cyclic graphs.

Theorem 2.7. ([9, Proposition 4.1]) *Let C_i be the cyclic graph with vertices i. Then, the following conditions are equivalent.*

(1) *G is sequentially Cohen–Macaulay.*

(2) *$i = 3$ or $i = 5$.*

Remark 2.8. We can check that all of the vertices of C_3 and C_5 are shedding vertices. Therefore, C_3 and C_5 are vertex decomposable. By Theorem 1.9, C_3 and C_5 are also shellable and sequentially Cohen–Macaulay.

We next give the analogy of Theorem 1.12.

Theorem 2.9. *Let G be a graph. Suppose that G contains a cyclic graph C_5 with $\deg_G(w) = 2$ for all $w \in V(C_5)$ except for one vertex. Then, the following conditions are equivalent.*

(1) *G is shellable.*

(2) *Both $G \setminus C_5$ and $G \setminus N_G[C_5]$ are shellable.*

Proof. Suppose that C_5 is a connected component of G. Then, it follows that $G \setminus C_5 = G \setminus N_G[C_5]$. By Remark 2.8, C_5 is shellable. Therefore, in this case, the proof follows from Proposition 1.10. We may assume that C_5 is not a connected component of G. Then, there is an unique vertex $v \in V(C_5)$ with $\deg_G(v) \geq 3$. We put $v = 1$, $V(C_5) = \{1, 2, 3, 4, 5\}$, and $E(C_5) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}\}$. Then, it follows that

$$\mathcal{F}(C_5) = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}\}.$$

Moreover, we have $G \setminus N_G[A] \in \{G \setminus C_5, G \setminus N_G[C_5]\}$ for all $A \in \mathcal{F}(C_5)$.

(1) $\Rightarrow$ (2) : Suppose that G is shellable. By Proposition 1.11, $G \setminus N_G[A]$ is shellable for all $A \in \mathcal{F}(C_5)$. Therefore, $G \setminus C_5$ and $G \setminus N_G[C_5]$ are shellable.

(2) $\Rightarrow$ (1) : Put $G_1 = G \setminus C_5$ and $G_2 = G \setminus N_G[C_5]$. We can check the following inclusion relation:

$$\bigcup_{A \in \mathcal{F}(C_5)} \{F' \cup A\,;\, F' \in \mathcal{F}(G \setminus N_G[A])\} \subseteq \mathcal{F}(G).$$

We show the converse inclusion. Let $F \in \mathcal{F}(G)$. If $1 \in F$, then $2, 5 \notin F$. Since F is a maximal independent set, it follows that $3 \in F$ or $4 \in F$. If $1 \notin F$ and $2 \in F$, then $3 \notin F$; hence, $4 \in F$ or $5 \in F$. If $1, 2 \notin F$, then $3, 5 \in F$. Combining these facts, we have $A \subseteq F$ for all $A \in \mathcal{F}(C_5)$. Therefore, we see at once that $F \setminus A \in \mathcal{F}(G \setminus N_G[A])$. Thus, the converse holds.

Let $\mathcal{F}(G_1) = \{F_1, \ldots, F_r\}$ and $\mathcal{F}(G_2) = \{H_1, \ldots, H_s\}$ be shellings. Then, $\mathcal{F}(G)$ coincides with the union of

$$\mathcal{F}_1 = \{F_1 \cup \{2,4\}, \ldots, F_r \cup \{2,4\}\}, \mathcal{F}_2 = \{F_1 \cup \{2,5\}, \ldots, F_r \cup \{2,5\}\},$$
$$\mathcal{F}_3 = \{F_1 \cup \{3,5\}, \ldots, F_r \cup \{3,5\}\}, \mathcal{F}_4 = \{H_1 \cup \{1,3\}, \ldots, H_s \cup \{1,3\}\},$$
$$\mathcal{F}_5 = \{H_1 \cup \{1,4\}, \ldots, H_s \cup \{1,4\}\}.$$

We claim that

$$\begin{aligned}\mathcal{F}(G) = &\{F_1 \cup \{2,4\}, \ldots, F_r \cup \{2,4\}, F_1 \cup \{2,5\}, \ldots, F_r \cup \{2,5\},\\ &F_1 \cup \{3,5\}, \ldots, F_r \cup \{3,5\}, H_1 \cup \{1,3\}, \ldots, H_s \cup \{1,3\},\\ &H_1 \cup \{1,4\}, \ldots, H_s \cup \{1,4\}\}\end{aligned}$$

forms a shelling of G in this order. Since G_1 and G_2 is shellable, the numbering of $\mathcal{F}_i$ is a shelling of $\langle \mathcal{F}_i \rangle$ for all $1 \leq i \leq 5$. For the simplicial complex $\Delta(G)$, we check to satisfy the condition (2)(b) of Proposition 1.6. Let $F, F' \in \mathcal{F}(G)$. We may assume that $F \in \mathcal{F}_i$ and $F' \in \mathcal{F}_j$ where $i \neq j$.

Firstly, we consider the case $F, F' \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. If $F \in \mathcal{F}_3$ and $F' \in \mathcal{F}_1$, then there are $1 \leq \alpha$ and $\alpha' \leq r$ such that $F = F_\alpha \cup \{3, 5\}$ and $F' = F_{\alpha'} \cup \{2, 4\}$. It follows that $3 \in F \setminus F'$. Put $F'' = F_\alpha \cup \{2, 5\} \in \mathcal{F}_2$. Then, we have $F \setminus F'' = \{3\}$. For the other choice of F and F', we can also check the condition (2)(b) of Proposition 1.6 in the similar way. Therefore, the numbering of $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ is a shelling. Similarly, we can also check that the numbering of $\mathcal{F}_4 \cup \mathcal{F}_5$ is a shelling.

Finally, we consider the case $F \in \mathcal{F}_4 \cup \mathcal{F}_5$ and $F' \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Let $F \in \mathcal{F}_5$ and $F' \in \mathcal{F}_3$. Then, we can write $F = H_\beta \cup \{1, 4\}$ and $F' = F_\alpha \cup \{3, 5\}$; hence, $1 \in F \setminus F'$. Since G_1 and G_2 are shellable, $G \setminus \{3, 4, 5\}$ is shellable by Theorem 1.12. Moreover, the shelling is

$$\mathcal{F}(G \setminus \{3, 4, 5\}) = \{F_1 \cup \{2\}, \ldots, F_r \cup \{2\}, H_1 \cup \{1\} \ldots, H_s \cup \{1\}\}.$$

Therefore, for $H_\beta \cup \{1\}$ and $F_\alpha \cup \{2\}$, there is $\alpha'(1 \leq \alpha' \leq r)$ such that $H_\beta \cup \{1\} \setminus F_{\alpha'} \cup \{2\} = \{1\}$; hence, it follows that $H_\beta \cup \{1, 4\} \setminus F_{\alpha'} \cup \{2, 4\} = \{1\}$. For the other choice of F and F', we can also check the condition to be shelling in the similar way. Thus, G is shellable. $\square$

According to the proof of Theorem 2.9, we get the following corollary.

Corollary 2.10. *Under the assumption of Theorem 2.9, let* $V(C_5) = \{1, 2, 3, 4, 5\}$, $E(C_5) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}\}$ *and* $\deg_G(1) \geq 3$. *Then, the following conditions are equivalent..*

(1) *G is shellable.*

(2) *Both $G \setminus C_5$ and $G \setminus N_G[C_5]$ are shellable.*

(3) *$G \setminus \{3, 4, 5\}$ is shellable.*

Remark 2.11. Replacing C_5 by C_3 or complete graphs, we obtain the same statement as Theorem 2.9 and Corollary 2.10. The proof is given by the similar arguments.

Let $i, j \geq 3$. We consider a graph G that contains C_i and C_j. If $V(G) = V(C_i) \cup V(C_j)$, $|V(C_i) \cap V(C_j)| = 1$, and $E(G) = E(C_i) \cup E(C_j)$, we write $G = G_i *_1 G_j$.

Corollary 2.12. *Let C_i be the cyclic graph with vertices i. Then, the following conditions are equivalent.*

(1) *$G_i *_1 G_j$ is sequentially Cohen–Macaulay.*

(2) *$i \in \{3, 5\}$ or $j \in \{3, 5\}$.*

Proof. Let $G = G_i *_1 G_j$ and $V(C_i) \cap V(C_j) = \{x\}$.

(1) $\Rightarrow$ (2) : Suppose that i, $j \notin \{3, 5\}$. We put $D^2_G(x) \cap V(C_i) = \{y, z\}$ (if $i = 4$, then $y = z$). The induced subgraph $G \setminus N_G[\{y, z\}]$ contains C_j as the connected component. By Theorem 2.7, C_j is not sequentially Cohen–Macaulay. Therefore, neither is $G \setminus N_G[\{y, z\}]$; hence, G is not sequentially Cohen–Macaulay.

(1) $\Rightarrow$ (2) : Let $i = 5$. We apply Corollary 2.10 to G. Under the assumption of Corollary 2.10, let $G' = G \setminus \{3, 4, 5\}$. Then, it follows that $C(G') = \{C_j\}$ and $2 \in N_G(C_j)$. Since $\deg_{G'}(2) = 1$, G' is shellable by Corollary 2.6. By Corollary 2.10, G is shellable; hence, G is sequentially Cohen–Macaulay. Let $i = 3$. By Remark 2.11, similarly, we can also check that G is sequentially Cohen–Macaulay. □

Chapter 3

Almost complete multipartite graphs

Let $S = K[X_1, \ldots, K_n]$ be the polynomial ring over a fixed field K and G be a graph. We define the almost complete multipartite graph which contain the bipartite graph and the complete multipartite graph.

Definition 3.1. *A graph G is an almost complete multipartite graph if there exist independent sets $V_0, V_1, \ldots, V_c$ of G with $c \geq 1$ satisfying the following conditions:*

(1) $V(G) = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_c$*,*

(2) $G \setminus V_0 = G_{V_1 \sqcup \cdots \sqcup V_c}$ *is a complete c-partite graph.*

When the $(c+1)$-partite graph $(G\,; V_0, V_1, \ldots, V_c)$ satisfies conditions (1) and (2), we simply say that $(G\,; V_0, V_1, \ldots, V_c)$ is an almost complete multipartite graph.

Remark 3.2. Let $(G\,; V_0, V_1, \ldots, V_c)$ be an almost complete multipartite graph. If $V_0 = \emptyset$, then G is a complete multipartite graph. Moreover, bipartite graphs are always almost complete multipartite graphs. Let $c = 2$. If V_1 and V_2 are not the empty set, the complementary graph $\overline{G}$ of G is decomposable (see [13] for the definition).

In this chapter, we investigate the sequential Cohen–Macaulayness of almost complete multipartite graphs.

Let $(G\,;V_0,V_1,\ldots,V_c)$ be an almost complete multipartite graph. We put $G_i = G_{V_0 \sqcup V_i}$ for all $1 \leq i \leq c$. Then, G_i is a bipartite graph for all $1 \leq i \leq c$. Moreover, one can check that $\Delta(G) = \Delta(G_1) \cup \cdots \cup \Delta(G_c)$. So, we consider the union of simplicial complexes in Section 1. Let Σ, Δ, and Γ be simplicial complexes with $\Sigma = \Delta \cup \Gamma$. We give the characterization for $\Sigma = \Delta \cup \Gamma$ to be sequentially Cohen–Macaulay (Theorem 3.15). As a corollary, we obtain the characterization for Σ to be Cohen–Macaulay (Corollary 3.17).

Herzog and Hibi gave a quite interesting characterization for bipartite graphs to be Cohen–Macaulay as follows:

Theorem 3.3. ([13, Corollary 9.1.14], cf.[12]) *Let $(H;\, X,Y)$ be a bipartite graph. Suppose that* $\mathrm{iso}(H) = \emptyset$*. Then, the following conditions are equivalent.*

(1) *H is Cohen–Macaulay.*

(2) *There exists a numbering of the vertices $X = \{x_1,\ldots,x_m\}$, $Y = \{y_1.\ldots,y_m\}$ such that the following three conditions hold:*

   (a) *$\{x_i, y_i\} \in E(H)$ for all $1 \leq i \leq m$,*

   (b) *if $\{x_i, y_j\} \in E(H)$, then $i \leq j$,*

   (c) *if $\{x_i, y_j\}$, $\{x_j, y_k\} \in E(H)$, then $\{x_i, y_k\} \in E(H)$.*

The proof of Theorem 3.3 is given in Chapter 4. In Section 2, we first define the conditions (CM1), (CM2) and (CM3) that are analogous to the conditions (a), (b), and (c) respectively (Definition 3.18). Then, we study in the behavior of edges of sequentially Cohen–Macaulay bipartite graphs. The sequentially Cohen–Macaulay bipartite graph is contained in the class of graphs satisfying the conditions (CM2) (Proposition 3.19). From the study of sequentially Cohen–Macaulay bipartite graphs and the results of Section 1, we obtain the main result of this chapter (Theorem 3.22). As an application of the main result, it is follows that sequentially Cohen–

Macaulay almost complete multipartite graphs are vertex decomposable (Theorem 3.27).

In Section 3, we compute the regularity of the union of simplicial complexes. We also compute the regularity of almost complete multipartite graphs. As an application of the main result, we consider a new class of graphs which satisfies $\mathrm{im}(G) = \mathrm{reg}(G)$. We show that $\mathrm{im}(G) = \mathrm{reg}(G)$ if almost complete multipartite graph G is sequentially Cohen–Macaulay or unmixed (Theorem 3.30).

3.1 The union of simplicial complexes

Let $(G\,; V_0, V_1, \ldots, V_c)$ be an almost complete multipartite graph. We set $G_i = G_{V_0 \sqcup V_i}$ for all $i \neq 0$. Then, one can check that $\Delta(G) = \bigcup_{i=1}^{c} \Delta(G_i)$ and $\Delta(G_i) \cap \Delta(G_j) = \langle V_0 \rangle$ for all $i < j$. Thus, in order to investigate properties of $\Delta(G)$, it is essential to consider the union of simplicial complexes whose intersection is a simplex. Moreover, the argument can be reduced to the case where $c = 2$. In this section, we take the union of two simplicial complexes and consider the sequential Cohen–Macaulayness of these complexes.

Let Δ, Γ, and Σ be simplicial complexes such that $\Sigma = \Delta \cup \Gamma$ and $V(\Sigma) = [n]$. We set $\Pi = \Delta \cap \Gamma$. Then, Π is a subcomplex of Δ, Γ, and Σ. Moreover, it holds that

$$\mathcal{F}(\Delta) \cap \mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Sigma) \subseteq \mathcal{F}(\Delta) \cup \mathcal{F}(\Gamma).$$

Precisely speaking, we have the following proposition. We omit the proof since it is done by the routine process.

Proposition 3.4. $\mathcal{F}(\Sigma) = (\mathcal{F}(\Delta) \setminus \mathcal{F}(\Pi)) \sqcup (\mathcal{F}(\Gamma) \setminus \mathcal{F}(\Pi)) \sqcup (\mathcal{F}(\Delta) \cap \mathcal{F}(\Gamma))$.

For $F \in \Sigma$, it is obvious that $\mathrm{link}_\Sigma F = \mathrm{link}_\Delta F \cup \mathrm{link}_\Gamma F$. We also note that $\mathrm{link}_\Sigma F = \mathrm{link}_\Delta F$ if $F \notin \Gamma$ and that $\mathrm{link}_\Pi F = \mathrm{link}_\Delta F \cap \mathrm{link}_\Gamma F$ if $F \in \Pi$. Moreover, it holds that $\Sigma(i) = \Delta(i) \cup \Gamma(i)$ and $\Pi(i) \subseteq \Delta(i) \cap \Gamma(i)$ for all $0 \leq i \leq \dim \Sigma$.

We need a sufficient condition for the equality $\Pi(i) = \Delta(i) \cap \Gamma(i)$ to hold for all $0 \leq i \leq \dim \Sigma$. We set $\Gamma_1 = \langle \mathcal{F}(\Gamma) \setminus \mathcal{F}(\Pi) \rangle$, where Γ_1 is a subcomplex of Γ.

Lemma 3.5. *Suppose that* $\dim \Gamma_1 \leq \text{min-dim}\,\Pi$. *Then the following conditions hold:*

(1) $\dim \Gamma = \dim \Pi$,

(2) $\mathcal{F}(\Pi) \subseteq \mathcal{F}(\Gamma)$,

(3) $\Pi(i) = \Delta(i) \cap \Gamma(i)$ *for all* $0 \leq i \leq \dim \Sigma$.

Proof. (1) : Since Π is a subcomplex of Γ, it holds that $\dim\Gamma \geq \dim\Pi$. Let $F \in \mathcal{F}(\Gamma)$ with $\dim F = \dim\Gamma$. We may assume that $F \notin \mathcal{F}(\Pi)$. Then, $F \in \mathcal{F}(\Gamma_1)$; hence, $\dim F \leq \dim\Pi$. Thus, $\dim\Gamma = \dim\Pi$.

(2) : Let $F \in \mathcal{F}(\Pi)$. If $F \notin \mathcal{F}(\Gamma)$, there exists $F' \in \mathcal{F}(\Gamma)$ such that $F \subsetneq F'$. Now, $F' \notin \mathcal{F}(\Pi)$; hence, $F' \in \mathcal{F}(\Gamma_1)$. Therefore,

$$\text{min-dim}\Pi \leq \dim F < \dim F' \leq \dim\Gamma_1.$$

This is a contradiction.

(3) : Let $0 \leq i \leq \dim\Sigma$. It is enough to show that $\Pi(i) \supseteq \Delta(i) \cap \Gamma(i)$.

We first suppose that $\dim\Pi < i$; then, $\Pi(i) = \emptyset$. By (1), $\dim\Gamma = \dim\Pi < i$; hence, $\Gamma(i) = \emptyset$. Thus, $\Pi(i) = \Delta(i) \cap \Gamma(i)$.

Next, suppose that $\text{min-dim}\Pi < i \leq \dim\Pi$. It is enough to show that $\Gamma(i) \subseteq \Pi(i)$. Let $H \in \Gamma(i)$. Then, there exists $K \in \mathcal{F}(\Gamma)$ such that $i \leq \dim K$ and $H \subseteq K$. Since $\text{min-dim}\Pi < i$, we have $\dim\Gamma_1 < \dim K$. As $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Pi) \cup \mathcal{F}(\Gamma_1)$, it follows that $K \in \mathcal{F}(\Pi)$; thus, $H \in \Pi(i)$.

Finally, we assume that $i \leq \text{min-dim}\Pi$. Let $H \in \Delta(i) \cap \Gamma(i)$. Since $H \in \Pi$, there exists $K \in \mathcal{F}(\Pi)$ such that $H \subseteq K$ and $i \leq \dim K$. Therefore, $H \in \Pi(i)$. □

Let $\Sigma = \Delta \cup \Gamma$. When $\Delta \cap \Gamma$ is a simplex, one can characterize the purity of Σ as follows.

Proposition 3.6. *Let $H \in \Sigma$ and $\Delta \cap \Gamma = \langle H \rangle$. Suppose that $H \in \mathcal{F}(\Delta) \cap \mathcal{F}(\Gamma)$ or $H \notin \mathcal{F}(\Delta) \cup \mathcal{F}(\Gamma)$. Then, the following conditions are equivalent.*

(1) *Σ is pure.*

(2) *Both Δ and Γ satisfy the following two conditions:*

  (i) *Δ and Γ are pure,*

  (ii) $\dim \operatorname{link}_\Delta H = \dim \operatorname{link}_\Gamma H$.

Proof. By Proposition 3.4, either the condition $H \in \mathcal{F}(\Delta) \cap \mathcal{F}(\Gamma)$ or $H \notin \mathcal{F}(\Delta) \cup \mathcal{F}(\Gamma)$ implies $\mathcal{F}(\Sigma) = \mathcal{F}(\Delta) \cup \mathcal{F}(\Gamma)$.

(1) $\Rightarrow$ (2) : Since Σ is pure, Δ and Γ are pure and of the same dimension. Therefore it holds that $\dim \operatorname{link}_\Delta H = \dim \operatorname{link}_\Gamma H$.

(2) $\Rightarrow$ (1) : From assumptions (i) and (ii), it follows that $\dim \Delta = \dim \Gamma$. Thus, Σ is pure. □

Let Σ be a simplicial complex on $[n]$ and Δ and Γ be subcomplexes of Σ with $\Sigma = \Delta \cup \Gamma$. We set $\Pi = \Delta \cap \Gamma$. We start with the following proposition.

Proposition 3.7. *Suppose that Π is Cohen–Macaulay and $\dim \Pi = \dim \Delta = \dim \Gamma$. Then, Σ is Cohen–Macaulay if and only if Δ and Γ are Cohen–Macaulay.*

Proof. Consider the reduced Mayer–Vietoris exact sequence of $\operatorname{link}_\Delta F$ and $\operatorname{link}_\Gamma F$ for all $F \in \Sigma$:

$$\cdots \to \widetilde{H}_i(\operatorname{link}_\Pi F; K) \to \widetilde{H}_i(\operatorname{link}_\Delta F; K) \oplus \widetilde{H}_i(\operatorname{link}_\Gamma F; K) \to \widetilde{H}_i(\operatorname{link}_\Sigma F; K) \to \cdots .$$

By virtue of Reisner's criterion of Cohen–Macaulayness on simplicial complexes, the proof follows because $\dim \Pi = \dim \Delta = \dim \Gamma = \dim \Sigma$. □

As a corollary of Proposition 3.7, the sequentially Cohen–Macaulay case immediately follows.

Corollary 3.8. *Suppose that* Π *is sequentially Cohen–Macaulay and* $\Pi(i) = \Delta(i) \cap \Gamma(i)$ *for all* $0 \leq i \leq \dim \Sigma$*. Then,* Σ *is sequentially Cohen–Macaulay if and only if* Δ *and* Γ *are sequentially Cohen–Macaulay.*

Proof. Note that $\Sigma(i) = \Delta(i) \cup \Gamma(i)$ and $\Pi(i) = \Delta(i) \cap \Gamma(i)$ for all $0 \leq i \leq \dim \Sigma$. By virtue of Duval's criterion, we can apply Proposition 3.7 to $\Sigma(i)$, $\Delta(i)$, and $\Gamma(i)$ for all $0 \leq i \leq \dim \Sigma$. □

The following proposition might be well-known. However, it is important in our argument; therefore, we give a brief proof.

Proposition 3.9. *Let* Δ *and* Γ *be pure simplicial complexes of dimension* $d\,(\geq 1)$*. Suppose that* $\dim \Pi \leq d - 2$*. Then,* Σ *is not Cohen–Macaulay.*

Proof. Take $F \in \mathcal{F}(\Pi)$ with $\dim F = \dim \Pi$. Since Δ and Γ are pure and $\dim \Pi \leq d - 2$, $\mathrm{link}_\Delta F \neq \{\emptyset\}$ and $\mathrm{link}_\Gamma F \neq \{\emptyset\}$. On the other hand, since $\mathrm{link}_\Pi F = \{\emptyset\}$, it holds that $\mathrm{link}_\Delta F \cap \mathrm{link}_\Gamma F = \{\emptyset\}$; hence, $\mathrm{link}_\Sigma F = \mathrm{link}_\Delta F \cup \mathrm{link}_\Gamma F$ is not connected. Moreover, we have $\dim(\mathrm{link}_\Delta F) = \dim(\mathrm{link}_\Gamma F) = d - (\dim \Pi + 1) \geq 1$. In particular, $\dim(\mathrm{link}_\Sigma F) \geq 1$. This means that $\mathrm{link}_\Sigma F$ is not Cohen–Macaulay. Thus, Σ is not Cohen–Macaulay. □

The following lemma is a sequentially Cohen–Macaulay version of the "Rearrangement lemma"([1]), which plays a key role in proving the main statement in this section.

Lemma 3.10. *Let* Σ *be sequentially Cohen–Macaulay. Suppose that* $\dim \Pi \geq 0$ *and* $\mathcal{F}(\Pi) \subseteq \mathcal{F}(\Gamma)$*. If one of the following conditions is satisfied:*

(1) $\dim \Delta \geq \dim \Gamma$,

(2) $\dim \Delta > \dim \Pi$,

then $\dim \Pi = \dim \Gamma$.

Proof. We suppose that $\dim\Gamma > \dim\Pi$. Let $\Gamma_1 = \langle\, F \in \mathcal{F}(\Gamma) \,|\, F \notin \mathcal{F}(\Pi)\,\rangle$. Then, we get $\dim\Gamma_1 > \dim\Pi$. Take $F \in \mathcal{F}(\Delta)$ and $H \in \mathcal{F}(\Gamma_1)$. Since $F \cap H \in \Pi$, there exists $H' \in \mathcal{F}(\Pi)$ such that $F \cap H \subseteq H'$. If $F \cap H = H'$, then $H' \subseteq H$. Since $\mathcal{F}(\Pi) \subseteq \mathcal{F}(\Gamma)$, both H and H' belong to $\mathcal{F}(\Gamma)$; hence, $H = H' \in \mathcal{F}(\Pi)$, which is a contradiction. Thus, $F \cap H \subsetneq H'$. Therefore, we obtain the following inequalities:

$$\dim(F \cap H) < \dim H' \leq \dim\Pi < \dim\Gamma_1.$$

Now, we set $d = \min\{\dim\Delta, \dim\Gamma_1\}$. By assumption (1) or (2), $\dim\Pi < d$. Since $\Sigma = \Delta \cup \Gamma_1$, it holds that $\Sigma(d) = \Delta(d) \cup \Gamma_1(d)$. Let $\Pi' = \Delta(d) \cap \Gamma_1(d)$. Then, we have $\dim\Pi' \leq d - 2$. In fact, if $X \in \Pi'$, then there exist $F \in \mathcal{F}(\Delta)$ and $H \in \mathcal{F}(\Gamma_1)$ with $X \subseteq F \cap H$. From the above inequalities, it follows that

$$\dim X \leq \dim F \cap H < \dim\Pi < d.$$

By Proposition 3.9, $\Sigma(d)$ is not Cohen–Macaulay. This contradicts the fact that Σ is sequentially Cohen–Macaulay. Thus, $\dim\Pi = \dim\Gamma$. □

Remark 3.11. Obviously, condition (1) in Lemma 3.5 is equivalent to saying that $\dim\Gamma_1 \leq$ min-$\dim\Pi$ if Π is pure. Suppose that $\dim\Delta \geq \dim\Gamma$ or $\dim\Delta > \dim\Pi$. If Σ is sequentially Cohen–Macaulay, then (2) implies (1) in Lemma 3.5 by Lemma 3.10. Therefore, if Σ is sequentially Cohen–Macaulay and Π is pure, then (2) implies (3) in Lemma 3.5.

Theorem 3.12. *Let* $\dim\Delta \geq \dim\Gamma$. *Suppose that* Π *is Cohen–Macaulay such that* $\mathcal{F}(\Pi) \subseteq \mathcal{F}(\Gamma)$. *Then, the following conditions are equivalent.*

(1) Σ *is sequentially Cohen–Macaulay.*

(2) *Both* Δ *and* Γ *are sequentially Cohen–Macaulay and* $\dim\Gamma = \dim\Pi$.

Proof. Suppose that Σ is sequentially Cohen–Macaulay. If $\Pi = \{\emptyset\}$, then $\Sigma = \Delta$ and $\Gamma = \{\emptyset\}$. There is nothing to prove. We assume that $\dim\Pi \geq 0$. Because Π

is pure, condition (3) in Lemma 3.5 holds true by Remark 3.11. Then, the second statement of the theorem follows from the first one by Corollary 3.8. The converse also follows from a similar argument. $\square$

In the rest of this section, we assume that Π is generated by only one face of Σ. Let $\Pi = \langle H \rangle$, where $H \in \Sigma$. We immediately get the following corollary from Theorem 3.12.

Corollary 3.13. *Suppose that* $\dim \Delta \geq \dim \Gamma$*,* $\Delta \cap \Gamma = \langle H \rangle$*, and* $H \in \mathcal{F}(\Gamma)$*. Then, the following conditions are equivalent.*

(1) Σ *is sequentially Cohen–Macaulay.*

(2) *Both* Δ *and* Γ *are sequentially Cohen–Macaulay and* $\dim \Gamma = \dim H$*.*

Next, we consider the case where $H = \emptyset$. The following lemma may be known (cf. [20]). We prove it for the convenience of the readers.

Lemma 3.14. *Suppose that* $\dim \Delta \geq \dim \Gamma$*,* $\Delta \cap \Gamma = \{\emptyset\}$*. Then, the following conditions are equivalent.*

(1) Σ *is sequentially Cohen–Macaulay.*

(2) Δ *is sequentially Cohen–Macaulay and* $\dim \Gamma \leq 0$*.*

Proof. We may assume that $\dim \Sigma \geq 1$. Note that $\dim\Delta = \dim\Sigma$.

(1) $\Rightarrow$ (2) : By (1), $\Sigma(i)$ is Cohen–Macaulay for all $0 \leq i \leq \dim\Sigma$. In particular, $\Sigma(1) = \Delta(1) \cup \Gamma(1)$ is connected; hence, $\Gamma(1) = \emptyset$. Therefore $\dim \Gamma \leq 0$, and for each $0 < i \leq \dim\Sigma$, $\Sigma(i) = \Delta(i)$.

(2) $\Rightarrow$ (1) : As $\dim\Gamma \leq 0$, it holds that $\Sigma(i) = \Delta(i)$ for all $0 < i \leq \dim\Sigma$. Now, $\Delta(i)$ is Cohen–Macaulay for all $0 < i \leq \dim\Sigma$ and so is $\Sigma(i)$. $\square$

We come to the proof of the main statement of this section.

Theorem 3.15. *Suppose that* $\dim \Delta \geq \dim \Gamma$ *and* $\Delta \cap \Gamma = \langle H \rangle$*. Then, the following conditions are equivalent.*

(1) Σ *is sequentially Cohen–Macaulay.*

(2) Δ *and* Γ *satisfy the following three conditions:*

  (i) Δ *and* Γ *are sequentially Cohen–Macaulay,*

  (ii) $\dim \operatorname{link}_\Gamma H \leq \min\{0, \dim \operatorname{link}_\Delta H\}$,

  (iii) $\dim \operatorname{star}_\Gamma H = \dim \Gamma$.

Proof. By Corollary 3.13 and Lemma 3.14, we may assume that $\emptyset \neq H$ and $H \notin \mathcal{F}(\Gamma)$.

(1) $\Rightarrow$ (2) : Suppose that $H \in \mathcal{F}(\Delta)$. As $H \notin \mathcal{F}(\Gamma)$, $\dim H < \dim \Gamma \leq \dim \Delta$. Applying Lemma 3.10(2), it follows that $\dim \Delta = \dim H$, which is a contradiction. Thus, $H \notin \mathcal{F}(\Delta)$.

Let $W \in \mathcal{F}(\operatorname{link}_\Delta H)$. We set $\Gamma' = \Gamma \cup \langle H \cup W \rangle$. Then, it follows that $\Sigma = \Delta \cup \Gamma'$, $\Delta \cap \Gamma' = \langle H \cup W \rangle$, and $H \cup W \in \mathcal{F}(\Gamma')$. Moreover, $\dim \Delta \geq \dim \Gamma'$. Applying Corollary 3.13, Δ and Γ' are sequentially Cohen–Macaulay and $\dim \Gamma' = \dim(H \cup W)$.

We take $V \in \mathcal{F}(\operatorname{link}_\Gamma H)$. Then, $V \neq \emptyset$ since $H \notin \mathcal{F}(\Gamma)$. We set $\Delta' = \Delta \cup \langle H \cup V \rangle$. Then, it follows that $\Sigma = \Delta' \cup \Gamma$, $\Delta' \cap \Gamma = \langle H \cup V \rangle$, and $H \cup V \in \mathcal{F}(\Gamma)$. In addition, $\dim \Delta' \geq \dim \Gamma$. Applying Corollary 3.13, Δ' and Γ are sequentially Cohen–Macaulay and $\dim \Gamma = \dim(H \cup V)$, which also implies that $\dim \operatorname{star}_\Gamma H = \dim \Gamma$. Moreover, it follows that

$$\dim \operatorname{link}_\Gamma H + \dim H = \dim \Gamma \leq \dim \Gamma' \leq \dim H + \dim \operatorname{link}_\Delta H;$$

hence, $\dim \operatorname{link}_\Gamma H \leq \dim \operatorname{link}_\Delta H$. Since $\operatorname{link}_\Sigma H$ is sequentially Cohen–Macaulay, it holds that $\dim \operatorname{link}_\Gamma H \leq 0$ by Lemma 3.14.

(2) $\Rightarrow$ (1) : As $H \notin \mathcal{F}(\Gamma)$, it follows from hypothesis (ii) that $\dim \operatorname{link}_\Gamma H = 0$. By hypothesis (iii), there exists $F \in \Gamma$ such that $H \subseteq F$ and $\dim \Gamma = \dim F = \dim H + 1$.

We set $\Delta' = \Delta \cup \langle F\rangle$; then, it follows that $\Sigma = \Delta' \cup \Gamma$, $\Delta' \cap \Gamma = \langle F\rangle$. By Corollary 3.13, it is enough to show that Δ' is sequentially Cohen–Macaulay.

By hypothesis (ii) again, it holds that $0 \leq \dim \mathrm{link}_\Delta H$; hence, $H \notin \mathcal{F}(\Delta)$. We take $H' \in \mathcal{F}(\Delta)$ with $H \subsetneq H'$, and we set $\Gamma_1 = \langle H', F\rangle$. Because $|F \setminus H'| = 1$, it follows that Γ_1 is shellable and therefore sequentially Cohen–Macaulay. On the other hand, it is holds that $\Delta' = \Delta \cup \Gamma_1$, $\Delta \cap \Gamma_1 = \langle H'\rangle$, and $\dim\Gamma_1 = \dim H'$. Since Δ is sequentially Cohen–Macaulay, applying Corollary 3.13 to $\Delta' = \Delta \cup \Gamma_1$, we obtain that Δ' is sequentially Cohen–Macaulay. Thus, Σ is sequentially Cohen–Macaulay. □

Remark 3.16. When replacing the sequential Cohen–Macaulayness with shellability in Theorem 3.15, the theorem does not hold any more. We construct such an example from [10, Fig. 1.] as follows: Let $V(\Sigma) = \{a, b, c, d, e, f, g_1, g_2, g_3\}$,

$$\Sigma = \langle \{a, c, d, e\}, \{a, b, g_1\}, \{a, b, g_2\}, \{a, b, g_3\}, \{a, c, f\}, \{a, f, g_1\}, \{a, g_2, g_3\}, \\ \{b, c, g_1\}, \{b, c, g_2\}, \{b, c, g_3\}, \{c, f, g_3\}, \{c, g_1, g_2\}, \{f, g_1, g_2\}, \{f, g_2, g_3\}\rangle.$$

Then, Σ is the union of $\Delta = \langle\{a, c, d, e\}\rangle$ and $\Gamma = \langle \mathcal{F}(\Sigma) \setminus \mathcal{F}(\Delta)\rangle$. It is known that both Δ and Γ are shellable, but Σ is not shellable by [10]. Let $H = \{a, c\}$. Then, one can check that $\Delta \cap \Gamma = \langle H\rangle$, $\mathrm{link}_H \Gamma = \langle\{f\}\rangle$, $\mathrm{star}_H \Gamma = \langle\{a, c, f\}\rangle$, and $\mathrm{link}_H \Delta = \langle\{d, e\}\rangle$. Hence, $\dim \mathrm{link}_H \Gamma = 0 \leq 1 = \dim \mathrm{link}_H \Delta$ and $\dim \mathrm{star}_H \Gamma = 2 = \dim \Gamma$. Therefore, conditions (i), (ii), and (iii) of (2) in Theorem 3.15 are all satisfied.

As a corollary of Theorem 3.15, we have the following statement, which implies that Σ becomes the "ridge sum" of Δ and Γ in the sense of [22] when Σ is Cohen–Macaulay.

Corollary 3.17. *Let $\Delta \cap \Gamma = \langle H\rangle$. Suppose that $\Delta \supsetneq \langle H\rangle$ and $\Gamma \supsetneq \langle H\rangle$. Then, the following conditions are equivalent.*

(1) Σ *is Cohen–Macaulay.*

(2) *Both* Δ *and* Γ *satisfy the following two conditions:*

   (i) Δ *and* Γ *are Cohen–Macaulay,*

   (ii) $\dim \operatorname{link}_{\Delta} H = \dim \operatorname{link}_{\Gamma} H \leq 0$.

Proof. If $\Pi = \{\emptyset\}$, the statement is evident. We assume that $\Pi \neq \{\emptyset\}$.

(1) $\Rightarrow$ (2) : We first check that $H \in \mathcal{F}(\Delta) \cap \mathcal{F}(\Gamma)$ or $H \notin \mathcal{F}(\Delta) \cup \mathcal{F}(\Gamma)$. Suppose the contrary. If $H \in \mathcal{F}(\Gamma) \setminus \mathcal{F}(\Delta)$, then there exists $H' \in \mathcal{F}(\Delta)$ with $\dim H' > \dim H$. On the other hand, applying Lemma 3.10(2), it holds that $\dim \Gamma = \dim H$; hence, there exists $F \in \mathcal{F}(\Gamma) \setminus \{H\}$ with $\dim F \leq \dim H$ as $\Gamma \supsetneq \langle H \rangle$. Then, both H' and F belong to $\mathcal{F}(\Sigma)$. This contradicts the purity of Σ. Therefore, from Proposition 3.6, it follows that both Δ and Γ are pure and $\dim \operatorname{link}_{\Delta} H = \dim \operatorname{link}_{\Gamma} H$. Thus, by Theorem 3.15, we get the Cohen–Macaulayness of Δ and Γ. It also follows that $\dim \operatorname{link}_{\Delta} H = \dim \operatorname{link}_{\Gamma} H \leq 0$.

(2) $\Rightarrow$ (1) : Condition (2) (ii) implies that $H \in \mathcal{F}(\Delta) \cap \mathcal{F}(\Gamma)$ or $H \notin \mathcal{F}(\Delta) \cup \mathcal{F}(\Gamma)$. Hence, Σ is pure by Proposition 3.6. Because Γ is pure, the equality $\dim \Gamma = \dim \operatorname{star}_{\Gamma} H$ holds. All of the conditions in (2) of Theorem 3.15 are satisfied; thus, Σ is Cohen–Macaulay. □

3.2 Sequential Cohen–Macaulayness

In this section, we prove the main results of this chapter (Theorem 3.22). We first define some properties that originate from the behavior of the edges of a graph, which is an analogous concept introduced by Herzog and Hibi in [12] and [13].

Definition 3.18. *Let* G *be a graph. Let* U *and* V *be subsets of* $V(G)$ *with* $U \cap V = \emptyset$ *and* $|U| \leq |V|$. *We say that an ordered pair* U, V *satisfies* (CM1) *and/or* (CM2) *and/or* (CM3) *(a total of six patterns) if there is a numbering of the elements of* U

and V, say $U = \{x_1, ..., x_m\}$ and $V = \{y_1, ..., y_n\}$ with $m \leq n$, that satisfies (CM1) *and/or* (CM2) *and/or* (CM3) *as follows:*

(CM1) $\{x_i, y_i\} \in E(G)$ *for all* $1 \leq i \leq m$,

(CM2) *if* $\{x_i, y_j\} \in E(G)$, *then* $i \leq j$.

(CM3) *if* $\{x_i, y_j\}, \{x_j, y_k\} \in E(G)$, *then* $\{x_i, y_k\} \in E(G)$

Let $(H\,; X, Y)$ *be a bipartite graph with* $|X \setminus \mathrm{iso}(H)| \leq |Y \setminus \mathrm{iso}(H)|$. *We say that* H *satisfies* (CM1) *and/or* (CM2) *and/or* (CM3) *(a total of six patterns) if the ordered pair* $X \setminus \mathrm{iso}(H)$, $Y \setminus \mathrm{iso}(H)$ *satisfies* (CM1) *and/or* (CM2) *and/or* (CM3).

In this chapter, we consider the conditions (CM1) and (CM2). About the condition (CM3), we will consider in Chapter 4.

For any graph G, we set $\dim G = \dim\Delta(G) + 1$ and call it the dimension of G. Note that $\dim G = \dim S/I(G)$.

Proposition 3.19. *Let* $(H\,; X, Y)$ *be a bipartite graph. Suppose that* $\mathrm{iso}(H) = \emptyset$ *and* $|X| \leq |Y|$. *If* H *is sequentially Cohen–Macaulay, then* H *satisfies* (CM2).

Proof. We prove this statement by induction on $|X|$. We set $m = |X|$ and $n = |Y|$. When $m = 1$, this statement is always true.

Suppose that $m > 1$. Since H is sequentially Cohen–Macaulay and $|X| \leq |Y|$, we can choose $x_1 \in X$ and $y_1 \in Y$ such that $N_H(y_1) = \{x_1\}$ by Lemma 1.8. We set $H_1 = H \setminus N_H[y_1] = H \setminus \{x_1, y_1\}$. Then, H_1 is sequentially Cohen–Macaulay by Proposition 1.4. We note that $\mathrm{iso}(H_1) \subseteq Y$. Moreover, we set $H' = H_1 \setminus \mathrm{iso}(H_1)$, $X' = X \setminus \{x_1\}$, and $Y' = Y \setminus (\{y_1\} \cup \mathrm{iso}(H_1))$. Then, H' is a sequentially Cohen–Macaulay bipartite graph with the partition $V(H') = X' \sqcup Y'$, and $\mathrm{iso}(H') = \emptyset$.
(Case 1.) : $|X'| \leq |Y'|$.

By the induction hypothesis, $(H'\,; X', Y')$ satisfies (CM2), i.e., there is a numbering $X' = \{x_2, \ldots, x_m\}$ and $Y' = \{y_2, \ldots, y_t\}$ that satisfies (CM2). We set

$\mathrm{iso}(H_1) = \{y_{t+1}, \dots, y_n\}$; then, H_1 satisfies (CM2). Moreover, with the numbering $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$, one can see that H satisfies (CM2).

(Case 2.) : $|Y'| < |X'|$.

We set $\mathrm{iso}(H_1) = \{y_2, \dots, y_s\}$. Note that $|Y'| < |X'| = m - 1$. By the induction hypothesis, $(H'\,; Y', X')$ satisfies (CM2). We set the numbering $Y' = \{y'_1, \dots, y'_{n-s}\}$ and $X' = \{x'_1, \dots, x'_{m-1}\}$ that satisfies (CM2). We set $x_i = x'_{m-i+1}$ for all $2 \leq i \leq m$ and $y_j = y'_{n-j+1}$ for all $s + 1 \leq j \leq n$. Then, one can check that H satisfies (CM2) by the numbering $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_s, y_{s+1}, \dots, y_n\}$. □

Lemma 3.20. *Let* $(H\,; X, Y)$ *be a bipartite graph and assume that* $\mathrm{iso}(H) = \emptyset$. *If* H *is sequentially Cohen–Macaulay, then the following conditions are equivalent:*

(1) $\dim H = |Y|$,

(2) H *satisfies both* (CM1) *and* (CM2).

Proof. (1) ⇒ (2) : The statement is proved by induction on $|X|$. When $|X| = 1$, condition (2) is obvious.

Suppose that $|X| > 1$. By condition (1), it holds that $|X| \leq \dim H = |Y|$; hence, we can take $x_1 \in X$, $y_1 \in Y$, H_1, and $(H'\,; X', Y')$ as in the proof of Proposition 3.19. Since $\dim H = |Y|$, it follows that $\dim H_1 = |Y \setminus \{y_1\}|$; hence, $\dim H' = \dim H_1 - |\mathrm{iso}(H_1)| = |Y'|$. In particular, $|X'| \leq |Y'|$.

By the induction hypothesis, $(H'\,; X', Y')$ satisfies (CM1). From the same argument for (Case 1) in the proof of Proposition 3.19, we have the numbering $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ satisfying (CM1). Obviously, this numbering also satisfies (CM2).

(2) ⇒ (1): Let $Z \in \mathcal{F}(\Delta(H))$. It follows from (CM1) that $|Z \cap X| \leq |N_H(Z \cap X)|$. Since Z is an independent set, it holds that $N_H(Z \cap X) \cap (Z \cap Y) = \emptyset$. Therefore, we get the inequality $|Z| \leq |N_H(Z \cap X)| + |Z \cap Y| \leq |Y|$. Thus, $\dim H = |Y|$. □

If $\mathrm{iso}(H)$ is not the empty set, then we get the following corollary.

Corollary 3.21. *Let $(H\,;X,Y)$ be a bipartite graph. If H is sequentially Cohen–Macaulay, the following conditions are equivalent.*

(1) $\dim(H \setminus N_H(Y)) = \dim H$.

(2) *H satisfies both* (CM1) *and* (CM2).

Proof. Condition (1) is equivalent to saying that $\dim H = |Y \cup \mathrm{iso}(H)|$ by $V(H \setminus N_H(Y)) = Y \cup \mathrm{iso}(H)$. This means that $\dim(H \setminus \mathrm{iso}(H)) = |Y \setminus \mathrm{iso}(H)|$. In particular, $|X \setminus \mathrm{iso}(H)| \leq |Y \setminus \mathrm{iso}(H)|$. Hence, (1) is equivalent to saying that $H \setminus \mathrm{iso}(H)$ satisfies (CM1) and (CM2) by Lemma 3.20, which implies (2) by Definition 3.18. □

Let $(G\,;V_0,V_1,\ldots,V_c)$ be an almost complete multipartite graph, and let $L = G \setminus N_G[V_0]$. Then, $(L\,;\,\emptyset\,,V_1 \setminus N_G[V_0]\,,\ldots,V_c \setminus N_G[V_0])$ is a complete multipartite graph. We set $G_i = G_{V_0 \sqcup V_i}$ and $L_i = L_{V_i \setminus N_G[V_0]}$ for all $1 \leq i \leq c$. It is obvious that $L_i = G_i \setminus N_{G_i}[V_0]$, and it is an edgeless graph for each $1 \leq i \leq c$. The main result is the following statement.

Theorem 3.22. *Suppose that $\dim G = \dim G_1$. Then, the following conditions are equivalent.*

(1) *G is sequentially Cohen–Macaulay.*

(2) *The bipartite graphs $G_1, G_2, ..., G_c$ satisfy the following three conditions:*

  (i) *each G_i is sequentially Cohen–Macaulay for $1 \leq i \leq c$,*

  (ii) *if $i \neq 1$, then $|V(L_i)| \leq \min\{1,\ |V(L_1)|\}$,*

  (iii) *if $i \neq 1$, then $(G_i\,;\,V_i\,,V_0)$ satisfies both* (CM1) *and* (CM2).

Proof. We prove the theorem by induction on c.

The case where $c = 1$ is evident. We suppose that $c > 1$ and set $G' = G \setminus V_c$, $L' = G' \setminus N_{G'}[V_0]$. Both $(G'\,;V_0,V_1,\ldots,V_{c-1})$ and $(L'\,;\,\emptyset\,,\,V_1 \setminus N_{G'}[V_0],\ldots,V_{c-1} \setminus N_{G'}[V_0])$

are almost complete multipartite graphs. We set $G'_i = G'_{V_0 \sqcup V_i}$ and $L'_i = L'_{V_i \setminus N_{G'}[V_0]}$ for all $1 \le i \le c-1$. Note that $G'_i = G_i$ and $L'_i = L_i$ for all $1 \le i \le c-1$. From the argument at the beginning of Section 1, we have $\Delta(G) = \Delta(G') \cup \Delta(G_c)$, $\Delta(G') \cap \Delta(G_c) = \langle V_0 \rangle$, and $\dim \Delta(G') = \dim \Delta(G_1) \ge \dim \Delta(G_c)$. Moreover, it holds that $\dim L_i = |V(L_i)| = \dim \operatorname{link}_{\Delta(G_i)} V_0 + 1$ for all $1 \le i \le c$.

By Theorem 3.15, G is sequentially Cohen–Macaulay if and only if G' and G_c satisfy the following three conditions:

(i)$'$ both G' and G_c are sequentially Cohen–Macaulay,

(ii)$'$ $|V(L_c)| \le \min\{1, \dim L'\}$,

(iii)$'$ $\dim(G_c \setminus N_{G_c}(V_0)) = \dim G_c$.

Applying the induction hypothesis to G', we conclude that G' is sequentially Cohen–Macaulay if and only if the following three conditions are satisfied:

(i)$''$ each G_i is sequentially Cohen–Macaulay for $1 \le i \le c-1$,

(ii)$''$ if $i \ne 1$, c, then $|V(L_i)| \le \min\{1, |V(L_1)|\}$,

(iii)$''$ if $i \ne 1$, c, then $(G_i\,;\,V_i\,,V_0)$ satisfies (CM1) and (CM2).

Condition (ii) of the theorem is equivalent to conditions (ii)$'$ and (ii)$''$. Condition (iii)$'$ is equivalent to saying that $(G_c\,;\,V_c\,,V_0)$ satisfies (CM1) and (CM2) by Corollary 3.21. □

We get Corollary 3.23 as the corollary of Theorem 3.22.

Corollary 3.23. *Suppose that V_0 is an independent set of G such that $|V_0|$ is the maximum number among all independent sets. Then, the following conditions are equivalent:*

(1) *G is sequentially Cohen–Macaulay.*

(2) *The bipartite graphs $G_1, G_2, \ldots, G_c$ satisfy the following two conditions:*

  (i) *each G_i is sequentially Cohen–Macaulay for $1 \le i \le c$,*

  (ii) *each $(G_i\,;V_i, V_0)$ satisfies* (CM1) *and* (CM2) *for $1 \le i \le c$.*

Proof. Since V_0 is a maximal independent set, $V(L_i) = \emptyset$ for all $1 \leq i \leq c$, and condition (2)(ii) of Theorem 3.22 is always true. Further, it follows that $\dim G = \dim G_j$ for all $1 \leq j \leq c$. Applying Theorem 3.22, we get this statement. $\square$

Example 3.24. Let $(H\,; X, Y)$ be a bipartite graph. We set $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3\}$, and $E(H) = \{\{x_1, y_i\}\,; 1 \leq i \leq 3\} \cup \{\{x_i, y_3\}\,; 1 \leq i \leq 3\}$. We set $W_0 = \{x_2, x_3, y_1, y_2\}, W_1 = \{x_1\}$, and $W_2 = \{y_3\}$; then, $(H\,; W_0, W_1, W_2)$ is an almost complete multipartite graph. Applying Corollary 3.23, one can check that H is sequentially Cohen–Macaulay as follows, although it is known because H is a forest([33]). Let $H_i = H_{W_0 \sqcup W_i}$ for all $i = 1, 2$. Since x_1 is a shedding vertex of H_1, H_1 is vertex decomposable. For the same reason, H_2 is also vertex decomposable. Therefore, for each $i = 1, 2$, H_i is sequentially Cohen–Macaulay by Theorem 1.9. Since H_i satisfies (CM1) and (CM2) for $i = 1, 2$, H is sequentially Cohen–Macaulay by Corollary 3.23.

Now, let $Z = \{z_1, z_2, z_3\}$ and $(H'\,; X, Z)$ be a bipartite graph with the edges $E(H') = \{\{x_3, z_i\}\,; 1 \leq i \leq 3\} \cup \{\{x_i, z_1\}\,; 1 \leq i \leq 3\}$. Since H and H' are isomorphic as graphs, H' is sequentially Cohen–Macaulay. We set $V_0 = X$, $V_1 = Y$, and $V_2 = Z$. We consider the almost complete multipartite graph $(G\,; V_0, V_1, V_2)$ such that $G_1 = H$ and $G_2 = H'$. Then, one can check that $\dim G = \dim G_1$, but G_2 does not satisfy (2)(iii) of Theorem 3.22. Therefore, G is not sequentially Cohen–Macaulay.

We next consider the Cohen–Macaulayness on almost complete multipartite graphs. For the unmixedness, we have the following statement, which is a translation of Proposition 3.6.

Proposition 3.25. *Suppose that* $V_0 \in \bigcap_{i=1}^{c} \mathcal{F}(\Delta(G_i))$ *or* $V_0 \notin \bigcup_{i=1}^{c} \mathcal{F}(\Delta(G_i))$. *Then, the following conditions are equivalent.*

(1) *G is unmixed.*

(2) *The bipartite graphs* $G_1, G_2, ..., G_c$ *satisfy the following two conditions:*

(i) *each* G_i *is unmixed for* $1 \leq i \leq c$,

(ii) $|V(L_1)| = \cdots = |V(L_c)|$.

Corollary 3.26. *Under the assumption of Proposition 3.25, let* $c \geq 2$, *and suppose that* V_i *is not an empty set for* $i = 1, 2$. *Then, the following conditions are equivalent.*

(1) G *is Cohen–Macaulay.*

(2) *The bipartite graphs* $G_1, G_2, ..., G_c$ *satisfy the following two conditions:*

(i) *each* G_i *is Cohen–Macaulay for* $1 \leq i \leq c$,

(ii) $|V(L_1)| = \cdots = |V(L_c)| \leq 1$.

Proof. The proof follows from Theorem 3.22 and Proposition 3.25 (or Corollary 3.17). □

Recall that a vertex v of G is shedding vertex if $\mathcal{F}(\Delta(G \setminus N_G[v])) \cap \mathcal{F}(\Delta(G \setminus \{v\})) = \emptyset$. One can check that v is a shedding vertex of G if there exists $w \in V(G)$ with $N_G(w) = \{v\}$. We say that G is vertex decomposable if $E(G) = \emptyset$ or there exists a shedding vertex v such that $G \setminus \{v\}$ and $G \setminus N_G[v]$ are vertex decomposable. Now, we give the proof of Theorem 3.27.

Theorem 3.27. *Let* G *be an almost complete multipartite graph. Then, the following conditions are equivalent..*

(1) G *is sequentially Cohen–Macaulay.*

(2) G *is vertex decomposable.*

Proof. Suppose that G is sequentially Cohen–Macaulay. We may assume that $\dim G = \dim G_1$. If $c = 1$, then $G = G_1$, which is a bipartite graph. The conclusion follows from Theorem 1.9.

Suppose that $c \geq 2$. We may assume that V_i is not an empty set for all $1 \leq i \leq c$. We next use induction on $|V(G)|$. Let $i \neq 1$. By Theorem 3.22, $(G_i\,; V_i, V_0)$ is a sequentially Cohen–Macaulay graph that satisfies (CM1) and (CM2).

We claim that G has a shedding vertex in V_i. If $E(G_i) \neq \emptyset$, then there exist $v \in V_i$ and $w \in V_0$ with $N_{G_i}(w) = \{v\}$ by (CM1) and (CM2). Therefore, v is a shedding vertex of G_i. Then, v is also a shedding vertex of G. In fact, if there exists $F \in \mathcal{F}(\Delta(G \setminus N_G[v])) \cap \mathcal{F}(\Delta(G \setminus \{v\}))$, then $F \subseteq V(G \setminus N_G[v]) = V(G_i \setminus N_{G_i}[v]) \subseteq V(G_i \setminus \{v\})$ while $\Delta(G_i \setminus \{v\}) \subseteq \Delta(G \setminus \{v\})$; hence, $F \in \mathcal{F}(\Delta(G_i \setminus N_{G_i}[v])) \cap \mathcal{F}(\Delta(G_i \setminus \{v\}))$, which is a contradiction.

Suppose that $E(G_i) = \emptyset$. We take $v \in V_i$. By condition (2)(ii) of Theorem 3.22, it holds that $V_i = \{v\}$, and there exists $w \in V_1$ such that $w \in \mathrm{iso}(G_1)$. Then, we get $\mathcal{F}(G \setminus N_G[v]) = \{V_0\}$ and $V_0 \cup \{w\} \in \Delta(G \setminus \{v\})$. Hence, v is a shedding vertex of G.

Let $v \in V_i$ be a shedding vertex of G. Then, $(G \setminus \{v\}; V_0, V_1, \ldots, V_i \setminus \{v\}, \ldots, V_c)$ is an almost complete multipartite graph. In order to see that $G \setminus \{v\}$ is sequentially Cohen–Macaulay, we check conditions in (2) of Theorem 3.22. Since $G_i \setminus \{v\}$ coincides with the graph adding w as an isolated vertex to $G \setminus N_G[w]$, $G_i \setminus \{v\}$ is sequentially Cohen–Macaulay by Proposition 1.4. Thus, condition (2)(i) is satisfied. There is nothing to check for (2)(ii). To check (2)(iii), we may assume that $E(G_i) \neq \emptyset$. Then, $G_i \setminus \{v\}$ satisfies (CM1) and (CM2) by the choice of v. Therefore, $G \setminus \{v\}$ is sequentially Cohen–Macaulay. By the induction hypothesis, $G \setminus \{v\}$ is vertex decomposable. On the other hand, it holds that $G \setminus N_G[v] = G_i \setminus N_{G_i}[v]$, which is also vertex decomposable. Thus, G is vertex decomposable.

The converse follows from Theorem 1.9. □

3.3 Regularities

The purpose of this section is to present examples of almost complete multipartite graphs G such that the equality $\mathrm{reg}(G) = \mathrm{im}(G)$ holds, but G does not belong to the classes stated in Proposition 1.13.

We first compute the regularity of simplicial complexes. Let K be a field. Let Σ be a simplicial complex on $[n]$. Let Δ, Γ, and Π be subcomplexes such that $\Sigma = \Delta \cup \Gamma$ and $\Pi = \Delta \cap \Gamma$. We set $S = K[X_i \,|\, 1 \le i \le n]$, $S_1 = K[X_i \,|\, i \in V(\Delta)]$, $S_2 = K[X_i \,|\, i \in V(\Gamma)]$, and $S_3 = K[X_i \,|\, i \in V(\Pi)]$. As is well known, we have the following Mayer–Vietoris exact sequence:

$$0 \longrightarrow S/I_\Sigma \longrightarrow S_1/I_\Delta \oplus S_2/I_\Gamma \longrightarrow S_3/I_\Pi \longrightarrow 0$$

as S-modules. Then, we get the following lemma.

Lemma 3.28. *Suppose that* $\Pi = \langle H \rangle$. *If* $\Delta \supsetneq \langle H \rangle$ *and* $\Gamma \supsetneq \langle H \rangle$, *then*

$$\mathrm{reg}(\Sigma) = \max\{\mathrm{reg}(\Delta),\, \mathrm{reg}(\Gamma),\, 1\}.$$

Next, we give a method to compute the regularity and induced matching number of almost complete multipartite graphs.

Let $(G; V_0, V_1, \ldots, V_c)$ be an almost complete multipartite graph. In the rest of this section, we use the notation preceding Theorem 3.22. We note that (1) of the following lemma was essentially proved in [24].

Lemma 3.29. *Let* $(G; V_0, V_1, \ldots, V_c)$ *be an almost complete multipartite graph, and let* $E(G) \neq \emptyset$. *Then, we have the following condtions:*

(1) $\mathrm{reg}(G) = \max\{\mathrm{reg}(G_1), \ldots, \mathrm{reg}(G_c), 1\}$,

(2) $\mathrm{im}(G) = \max\{\mathrm{im}(G_1), \ldots, \mathrm{im}(G_c), 1\}$,

(3) *if* $\mathrm{reg}(G_i) = \mathrm{im}(G_i)$ *for all* $1 \le i \le c$, *then* $\mathrm{reg}(G) = \mathrm{im}(G)$.

Proof. (1) : Since $E(G) \neq \emptyset$, it holds that $\operatorname{reg}(G) \geq \operatorname{im}(G) \geq 1$. If $c = 1$, there is nothing to prove. We may assume that $c \geq 2$ and $V_i \neq \emptyset$ for all $1 \leq i \leq c$. By induction on c, this statement follows from Lemma 3.28.

(2) : Let M be an induced matching of G, and let $\operatorname{im}(G) = |M| \geq 2$. Assume that $M \cap E(G \setminus V_0) \neq \emptyset$. Take $e \in E(G \setminus V_0) \cap M$ and $f \in M$ such that $e \neq f$. Then, it holds that $\bigcup_{i=1}^{c} V_i \subseteq N_G[e]$ and $f \nsubseteq V_0$. Hence, $N_G[e] \cap f \neq \emptyset$. This contradicts the fact that M is an induced matching of G. Thus, $M \cap E(G \setminus V_0) = \emptyset$. This implies that $M \subseteq E(G_i)$ for some $1 \leq i \leq c$. Then, we obtain that $\operatorname{im}(G) = \max\{\operatorname{im}(G_1), \ldots, \operatorname{im}(G_c), 1\}$.

Assertion (3) follows from (1) and (2). □

Finally, we prove the main result of this section.

Theorem 3.30. *Let G be an almost complete multipartite graph. Then,* $\operatorname{reg}(G) = \operatorname{im}(G)$ *if one of the following conditions is satisfied.*

(1) *G is sequentially Cohen–Macaulay,*

(2) *G is unmixed.*

Proof. We may assume that $E(G) \neq \emptyset$.

(1) : If G is sequentially Cohen–Macaulay, then G_i is sequentially Cohen–Macaulay by Theorem 3.22. By Theorem 1.9, G_i is vertex decomposable. Since G_i is a bipartite graph, it has no odd cycle. Then, it follows from Proposition 1.13(3) that $\operatorname{reg}(G_i) = \operatorname{im}(G_i)$; hence, $\operatorname{reg}(G) = \operatorname{im}(G)$ by Lemma 3.29(3).

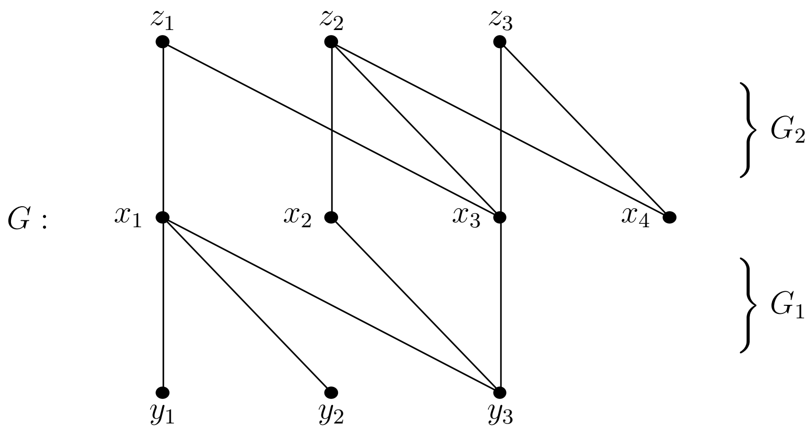
(2) : Suppose that $V_0 \notin \bigcap_{i=1}^{c} \mathcal{F}(\Delta(G_i))$. We may assume that $V_0 \notin \mathcal{F}(\Delta(G_c))$. Let $W = V(L_c)$. Then, $W \neq \emptyset$. Let $W_0 = V_0 \sqcup W$, $W_i = V_i \, (i \neq c)$, and $W_c = V_c \setminus W$. Then, $(G; W_0, W_1, \ldots, W_c)$ is an almost complete multipartite graph. We set $H_i = G_{W_0 \sqcup W_i}$. Since $N_G[w] \supseteq \bigcup_{i=1}^{c} V_i$ for each $w \in W$, we have that $W_0 \in \bigcap_{i=1}^{c} \mathcal{F}(\Delta(H_i))$. Thus, we may assume that $V_0 \in \bigcap_{i=1}^{c} \mathcal{F}(\Delta(G_i))$. Under this condition, G_i is unmixed for all $1 \leq i \leq c$ by Proposition 3.25. Therefore, by

Proposition 1.13(2), $\mathrm{reg}(G_i) = \mathrm{im}(G_i)$ for all $1 \le i \le c$. This implies $\mathrm{reg}(G) = \mathrm{im}(G)$ by Lemma 3.29(3). □

Example 3.31. We consider the almost complete multipartite graph G in Example 3.24. Then, G_1 coincides with the bipartite graph H, and $\mathrm{reg}(G_1) = \mathrm{im}(G_1)$=1. Similarly we have that $G_2 = H'$ and $\mathrm{reg}(G_2) = \mathrm{im}(G_2) = 1$. This means that the assumptions in (3) of Lemma 3.29 are satisfied. Hence, $\mathrm{reg}(G) = \mathrm{im}(G) = 1$.

Example 3.32. Let $V_0 = \{x_1, x_2, x_3, x_4\}$, $V_1 = \{y_1, y_2, y_3\}$, and $V_2 = \{z_1, z_2, z_3\}$. We consider an almost complete multipartite graph $(G\,; V_0, V_1, V_2)$ such that

$$E(G_1) = \{\{y_1, x_1\}, \{y_2, x_1\}, \{y_3, x_1\}, \{y_3, x_2\}, \{y_3, x_3\}\} \text{ and}$$
$$E(G_2) = \{\{z_1, x_1\}, \{z_1, x_3\}, \{z_2, x_2\}, \{z_2, x_3\}, \{z_2, x_4\}, \{z_3, x_3\}, \{z_3, x_4\}\}.$$



Figure

We note that $G \setminus V_0$ is a complete multipartite graph, but we do not draw the edges of $G \setminus V_0$ in Figure.

We claim that G is sequentially Cohen–Macaulay. Now, G_1 is the graph obtained from H of Example 3.24 by adding x_4 as an isolated vertex. Hence, G_1 is sequentially Cohen–Macaulay. On the other hand, z_2 is a shedding vertex of G_2 because $N_{G_2}(x_2) = \{z_2\}$, whereas $G_2 \setminus \{z_2\}$ and $G_2 \setminus N_{G_2}[z_2]$ are trees; in particular, they are vertex decomposable. Hence, G_2 is vertex decomposable. By Theorem 1.9, G_2 is sequentially Cohen–Macaulay. Note that $\dim G = \dim G_1 = 5$ (in fact, $\{x_2, x_3, x_4, y_1, y_2\} \in \mathcal{F}(\Delta(G))$), and $(G_2\,; V_2, V_0)$ satisfies (CM1) and (CM2). Applying Theorem 3.22, G is a sequentially Cohen–Macaulay graph.

By Theorem 3.30, we know that $\operatorname{reg}(G) = \operatorname{im}(G) = 2$. However, G is not unmixed; further, G is not chordal since $\{x_3, z_3, x_4, z_2\}$ is an induced cycle of length 4. Moreover, G has a cycle $\{x_1, z_1, x_3, z_3, y_1\}$ of length 5. Therefore, G does not belong to any class stated in Proposition 1.13.

Chapter 4

Semi-unmixed graphs

Let $(H\,;\,X,Y)$ be a bipartite graph with $|X| \leq |Y|$ and $\mathrm{iso}(H) = \emptyset$. In Section 1, we define the semi-unmixed graph as a class of graphs containing the unmixed graph. Villarreal gave a characterization of the unmixed bipartite graph as follows:

Theorem 4.1. ([31]) *Let $(H;\,X,Y)$ be a bipartite graph. Suppose that* $\mathrm{iso}(H) = \emptyset$. *$H$ is unmixed if and only if there exists a numbering of the vertices $X = \{x_1,\dots,x_m\}$, $Y = \{y_1.\dots,y_m\}$ such that the following two conditions hold:*

(a) $\{x_i, y_i\} \in E(H)$ *for all* $1 \leq i \leq m$,

(c) *if* $\{x_i, y_j\}$, $\{x_j, y_k\} \in E(H)$, *then* $\{x_i, y_k\} \in E(H)$.

Villarreal says that this result is inspired by a characterization of Herzog and Hibi in [31]. The characterization of semi-unmixed bipartite graphs (Theorem 4.4) comes from Theorem 4.1.

Let $(G\,;\,V_0,V_1,\dots,V_c)$ be an almost complete multipartite graph. Put $G_i = G_{V_0 \sqcup V_i}$ for all $1 \leq i \leq c$. In Section 2, we investigate the relation between unmixed almost complete multipartite graphs and semi-unmixed graphs. We will a more refined criterion (Theorem 4.8) than Proposition 3.25. After that, we compute the regularity of semi-unmixed graphs. As an application, we also give a new method to calculate the regularity of unmixed almost complete multipartite graphs.

In Section 3, we give a characterization of sequentially Cohen–Macaulay semi-unmixed graphs in terms of the conditions (CM1), (CM2), and (CM3) (Theorem 4.14). Moreover, from the proof of Theorem 4.14, we will obtain a criterion for bipartite graphs satisfying (CM1) and (CM3) to be sequentially Cohen–Macaulay (Theorem 4.15). As an application, the alternative proof of Theorem 3.3 is given by Theorem 4.15. Finally, as an application of Theorem 3.22 and Theorem 4.15, we present a sufficient condition for almost complete multipartite graph to be sequentially Cohen–Macaulay in terms of edges (Theorem 4.16).

4.1 Behavior of edges

Let $(H;\, X, Y)$ be a bipartite graph. The definition of semi-unmixed graphs is as follows:

Definition 4.2. *Let $(H;\, X, Y)$ be a bipartite graph with $|X \setminus \mathrm{iso}(H)| \leq |Y \setminus \mathrm{iso}(H)|$. Let X_1 be a set with $X_1 \cap V(H) = \emptyset$ and $|X_1| = |Y \setminus \mathrm{iso}(H)| - |X \setminus \mathrm{iso}(H)|$. We say that $(H;\, X, Y)$ is semi-unmixed, if $(H';\, X', Y)$ is unmixed where $X' = X \cup X_1$ and $E(H') = E(H) \cup \{\{x, y\};\, x \in X_1, y \in Y \setminus \mathrm{iso}(H)\}$.*

Remark 4.3. Let $(H;\, X, Y)$ be a semi-unmixed graph. When $|X \setminus \mathrm{iso}(H)| = |Y \setminus \mathrm{iso}(H)|$ holds, i.e., $X_1 = \emptyset$ in Definition 4.2, H is unmixed. Thus, the class of semi-unmixed graphs contains unmixed bipartite graphs.

According to Definition 4.2, $(H;\, X, Y)$ is semi-unmixed if and only if $(H \setminus \mathrm{iso}(H);\, X \setminus \mathrm{iso}(H), Y \setminus \mathrm{iso}(H))$ is semi-unmixed.

We give a condition of edges of semi-unmixed graphs and characterize semi-unmixed graphs in terms of the simplicial complex.

Proposition 4.4. *Let $(H;, X, Y)$ be a bipartite graph with* $\mathrm{iso}(H) = \emptyset$ *and $m = |X| < |Y| = n$. Then, the following conditions are equivalent.*

(1) *$(H;\, X, Y)$ is semi-unmixed.*

(2) *There is a numbering* $X = \{x_1, \ldots, x_m\}$ *and* $Y = \{y_1, \ldots, y_n\}$ *satisfied the following three conditions:*

(CM1) $\{x_i, y_i\} \in E(H)$ *for all* $1 \leq i \leq m$,

(CM3) *if* $\{x_i, y_j\}, \{x_j, y_k\} \in E(H)$, *then* $\{x_i, y_k\} \in E(H)$,

(CP) $H_{N_H(\{y_{m+1},\ldots,y_n\}) \sqcup Y}$ *is a complete bipartite graph.*

(3) If F is a facet of $\Delta(H)$ with $F \neq X$, then $\dim F = \dim \Delta(H)$

Proof. Let X_1 be a set with $X_1 \cap V(H) = \emptyset$. Let $(H';\, X', Y)$ be a bipartite graph with $X' = X \sqcup X_1, X_1 \cap V(H) = \emptyset$, $|X_1| = n - m$ and $E(H') = E(H) \cup \{\{x, y\};\, x \in X_1, y \in Y\}$

(1) $\Rightarrow$ (2) : Suppose that $(H;\, X, Y)$ is semi-unmixed. By Definition 4.2, $(H';\, X', Y)$ is unmixed. By Theorem 4.1, $(H';\, X', Y)$ satisfies (CM1) and (CM3), i.e., there is a numbering $X' = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ which satisfies (CM1) and (CM3). For any $i < j$, the numbering $X' = \{\ldots, x_j, \ldots, x_i, \ldots\}$ and $Y = \{\ldots, y_j, \ldots, y_i, \ldots\}$ also satisfies (CM1) and (CM3), because there is no harm in exchanging i and j. We may assume that $X_1 = \{x_{m+1}, \ldots, x_n\}$. Then, the numbering of vertices of H, $X = X' \setminus X_1 = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$, satisfies (CM1) and (CM3). Therefore, $(H;\, X, Y)$ satisfies (CM1) and (CM3). The remaining is to check the condition (CP) for the numbering. Let $x_i \in H_{N_H(\{y_{m+1},\ldots,y_n\})}$ and $y_k \in Y$. Then, there is y_j such that $m + 1 \leq j \leq n$ and $\{x_i, y_j\} \in E(H) \subseteq E(H')$. By Definition 4.2, it follows that $\{x_j, y_k\}$ is an edge of H'. Therefore, we obtain $\{x_i, y_k\} \in E(H')$ by (CM3); hence $\{x_i, y_k\} \in E(H)$. Thus, $H_{N_H(\{y_{m+1},\ldots,y_n\}) \sqcup Y}$ is complete.

(2) $\Rightarrow$ (3) : Let $W = N_H(\{y_{m+1}, \ldots, y_n\})$ and $Z = \{y_j \in Y\,;\, x_j \in W\} \cup \{y_{m+1}, \ldots, y_n\}$. We claim that $W = N_H(Z)$. Indeed, it is clear that $W \subseteq N_H(Z)$. Conversely, we take $x_i \in N_H(Z)$ where $i \leq m$. Then, there is a $y_j \in Z$ such that $\{x_i, y_j\} \in E(H)$. If $j = m + 1, \ldots, n$, then $x_i \in W$ by definition of W. If

$j = 1, \ldots, m$, then $x_j \in W$ by definition of Z, i.e., there is a integer $k > m$ such that $\{x_j, y_k\} \in E(H)$. By the condition (CM3), we obtain $\{x_i, y_k\} \in E(H)$. Therefore, it follows that $x_i \in W$.

We put $H_0 = H \setminus N_H[Z]$, $X_0 = X \setminus W$, and $Y_0 = Y \setminus Z$. $(H_0; X_0, Y_0)$ is a bipartite graph. Since $W = N_H(Z)$, one can check that $\alpha = |X_0| = |Y_0|$; hence, we write $X_0 = \{x_{i1}, \ldots, x_{i\alpha}\}$ and $Y_0 = \{y_{i1}, \ldots, y_{i\alpha}\}$. Since H_0 is an induced subgraph of H, it follows that H_0 satisfies the conditions (CM1) and (CM3). Therefore, H_0 is unmixed.

Let F be a maximal independent set of H with $F \neq X$; then, $F \cap Y \neq \emptyset$. By the condition (CP), it follows that $F \cap W = \emptyset$. Since $W = N_H(Z)$, we obtain $F \cup Z \in \Delta(H)$. Then, the inclusion relation $F \supseteq Z$ holds because F is maximal independent set. Therefore, $F \setminus Z$ is a maximal independent set of H_0. Since H_0 is unmixed, it follows that the equality $\dim F \setminus Z = \dim Y_0 = \dim Y \setminus Z$; hence, $\dim F = \dim Y$. Thus, we obtain $\dim F = \dim \Delta(H)$.

(3) $\Rightarrow$ (1) : We show that $\Delta(H')$ is pure. Let $F \in \mathcal{F}(\Delta(H'))$. If $F = X'$, then $\dim F = \dim Y$. Suppose that $F \neq X'$; then, $F \cap Y \neq \emptyset$ and $F \neq X$. By definition of $E(H')$, we have $F \cap X_1 = \emptyset$. Since H is an induced subgraph of H', it follows that $F \in \mathcal{F}(\Delta(H))$. By our assumption, we have $\dim F = \dim \Delta(H)$. In particular, when we take $F = Y$, it follows that $\dim Y = \dim \Delta(H)$. Thus, $\Delta(H')$ is pure. $\square$

4.2 Relationship with almost complete multipartite graphs

In this section, we investigate the relation between unmixed almost complete multipartite graphs and semi-unmixed graphs. Firstly, we begin with proving the following two statements about the union of simplicial complexes.

Lemma 4.5. *Let Σ be a simplicial complex and both Δ and Γ be subcomplexes of Σ with $\Sigma = \Delta \cup \Gamma$ and $\Delta \cap \Gamma = \langle F \rangle$. If $X \in \mathcal{F}(\Delta)$ and there is a vertex $v \in X$ such* *that $v \notin F$, then it follows that $X \in \mathcal{F}(\Sigma)$.*

Proof. If there is a vertex $v \in X$ such that $v \notin F$, then we get $X \notin \Gamma$; hence there is no $X' \in \mathcal{F}(\Gamma)$ such that $X \subseteq X'$. It is always true that $\mathcal{F}(\Sigma) \subseteq \mathcal{F}(\Delta) \cup \mathcal{F}(\Gamma)$. Therefore, $X \in \mathcal{F}(\Delta)$ implies $X \in \mathcal{F}(\Sigma)$. $\square$

Proposition 4.6. *Let Σ be a simplicial complex, and $\Delta_1, \dots, \Delta_c$ be subcomplexes of Σ with $\Sigma = \cup_{i=1}^{c} \Delta_i$ and $c \geq 2$. Suppose that $F \in \Sigma$, $\Delta_i \cap \Delta_j = \langle F \rangle$, and $\Delta_i \supsetneq \langle F \rangle$ for all $i < j$. If Σ is pure, then for each i the following conditions are equivalent:*

(1) *A subcomplex Δ_i is pure.*

(2) $\dim \operatorname{link}_\Sigma F = \dim \operatorname{link}_{\Delta_i} F$.

Proof. Note that $\operatorname{link}_\Sigma F, \operatorname{link}_{\Delta_1} F, \dots, \operatorname{link}_{\Delta_c} F$ are always pure.

(1) $\Rightarrow$ (2) : Suppose that $\dim \operatorname{link}_\Sigma F > \dim \operatorname{link}_{\Delta_i} F$. Since $\operatorname{link}_\Sigma F$ is pure and $\operatorname{link}_{\Delta_i} F$ is connected component of $\operatorname{link}_\Sigma F$, it follows that $\operatorname{link}_{\Delta_i} F = \{\emptyset\}$; hence, $F \in \mathcal{F}(\Delta_i)$. By our assumption $\Delta_i \supsetneq \langle F \rangle$, there is a face $F' \in \mathcal{F}(\Delta_i)$ such that $F' \neq F$. From Lemma 4.5, $F' \in \mathcal{F}(\Sigma)$. Since $\dim \operatorname{link}_\Sigma F \geq 0$, i.e., $F \notin \mathcal{F}(\Sigma)$, we have $\dim F' > \dim F$. Thus, Δ_i is not pure.

(2) $\Rightarrow$ (1) : Let $X \in \mathcal{F}(\Delta_i)$. If $X \supseteq F$, then $X \setminus F \in \mathcal{F}(\operatorname{link}_{\Delta_i} F)$. Since $\operatorname{link}_{\Delta_i} F$ is pure, it follows that $\dim X \setminus F = \dim \operatorname{link}_{\Delta_i} F$. By our assumption (2), we have $\dim X \setminus F = \dim \operatorname{link}_\Sigma F$. Therefore, $X \setminus F \in \mathcal{F}(\operatorname{link}_\Sigma F)$; hence, $X \in \mathcal{F}(\Sigma)$. Suppose that $X \not\supseteq F$. By Lemma 4.5, we obtain $X \in \mathcal{F}(\Sigma)$. $\square$

Let $(G\,; V_0, V_1, \dots, V_c)$ be an almost complete multipartite graph. We put $G_i = G_{V_0 \sqcup V_i}$ and $L_i = G_i \setminus N_{G_i}[V_0]$ for all $1 \leq i \leq c$.

Lemma 4.7. *Let $(G; V_0, V_1, \dots, V_c)$ be an almost complete multipartite graph with $c \geq 2$. Suppose that $V_i \neq \emptyset$ for all $1 \leq i \leq c$. If G is unmixed, then each G_i is semi-unmixed for all $1 \leq i \leq c$.*

Proof. We put $G_i = G_{V_0 \sqcup V_i}$, $L = G \setminus N_G[V_0]$, and $L_i = G_i \setminus N_{G_i}[V_0]$. If $\dim L_i = \dim L$, then G_i is unmixed by Lemma 4.5. Suppose that $\dim L > \dim L_i$. Since $\Delta(L)$ is pure and $\Delta(L_i)$ is a component of $\Delta(L)$; then, $\dim L_i = 0$. Besides, $\operatorname{iso}(G_i) \subseteq V_0$.

We show that the bipartite graph $(G_i \setminus \operatorname{iso}(G_i);\, V_0 \setminus \operatorname{iso}(G_i), V_i)$ is semi-unmixed. First, we check that $|V_0 \setminus \operatorname{iso}(G_i)| < |V_i|$. Since G_i is bipartite graph, it follows that $V_i \cup \operatorname{iso}(G_i) \in \mathcal{F}(G_i)$. Besides, $V_i \cup \operatorname{iso}(G_i) \in \mathcal{F}(G)$ by Lemma 4.5. Therefore, $|V_i \cup \operatorname{iso}(G_i)| = \dim G$, hence $|V_i| = \dim G - |\operatorname{iso}(G_i)|$. Since $\dim L > 0$, one has $V_0 \notin \mathcal{F}(G)$. Then, it follows that $\dim G > |V_0|$, therefore we get $|V_i| > |V_0| - |\operatorname{iso}(G_i)|$. Next, we take $X \in \mathcal{F}(G_i \setminus \operatorname{iso}(G_i))$ with $X \neq V_0 \setminus \operatorname{iso}(G_i)$. Then, $X \sqcup \operatorname{iso}(G_i) \in \mathcal{F}(G_i)$ and $X \cap V_i \neq \emptyset$. By Lemma 4.5, we get $X \sqcup \operatorname{iso}(G_i) \in \mathcal{F}(G)$. Since G is unmixed, it follows that $\dim G = |X \sqcup \operatorname{iso}(G_i)|$. Therefore, $\dim G_i = |X \sqcup \operatorname{iso}(G_i)|$; hence, $\dim(G_i \setminus \operatorname{iso}(G_i)) = |X|$. Then, we can apply Proposition 4.4(3) to $(G_i \setminus \operatorname{iso}(G_i);\, V_0 \setminus \operatorname{iso}(G_i), V_i)$. Thus, $(G_i \setminus \operatorname{iso}(G_i);\, V_0 \setminus \operatorname{iso}(G_i), V_i)$ is semi-unmixed. $\square$

We give the characterization of unmixed almost complete multipartite graph in terms of semi-unmixedness.

Theorem 4.8. *Let $(G; V_0, V_1, \ldots, V_c)$ be an almost complete multipartite graph. Suppose that $V_i \neq \emptyset$ and $G_i = G_{V_0 \sqcup V_i}$ for all $1 \leq i \leq c$. Then, the following conditions are equivalent:*

(1) *G is unmixed.*

(2) *There is an unmixed bipartite graph G_j. Moreover, the bipartite graphs $G_1, G_2, \ldots, G_c$ satisfy the following conditions:*

   (a) *$(G_i\,; V_0, V_i)$ is semi-unmixed for all $1 \leq i \leq c$,*

   (b) *if G_i is not unmixed, then $\operatorname{iso}(G_i)$ is contained in V_0,*

   (c) $\dim G_1 = \cdots = \dim G_c$.

Proof. (1) $\Rightarrow$ (2) : Suppose that G is unmixed. By Lemma 4.7, G_i is semi-unmixed for all $1 \leq i \leq c$. Moreover, according to the proof of Lemma 4.7, there is an unmixed bipartite graph G_j. (2)(b) also follows. Fix $1 \leq i \leq c$. Since $V_i \neq \emptyset$, there is a maximal independent set F_i of G_i with $V_i \cap F_i \neq \emptyset$. From Lemma 4.5, F_i is a maximal independent set of G. Therefore, it follows that $\dim F_i = \dim G$. Thus, $\dim G = \dim G_i$.

(2) $\Rightarrow$ (1) : Let $F \in \mathcal{F}(\Delta(G))$. There is the induced subgraph G_i such that $F \in \mathcal{F}(\Delta(G_i))$. If $(F \setminus \mathrm{iso}(G_i)) \cap V_i \neq \emptyset$, $|F| = \dim G_i$ by Proposition 4.4(3). Suppose that $(F \setminus \mathrm{iso}(G_i)) \cap V_i = \emptyset$. Then, it follows that $F \supseteq V_0$. Let G_j be an unmixed graph. If $F = V_0$, then $|F| = |V_0| = \dim\, G_j$. Suppose that $F \supsetneq V_0$. Then, G_i is unmixed by (2)(b). Therefore, it follows that $\dim\, G_i = |F|$. By (2)(c), it follows that $|F|$ is a constant . Thus, G is unmixed. □

Remark 4.9. Proposition 3.25 follows from Theorem 4.8 by Proposition 4.6.

Finally, we calculate the regularity of unmixed almost complete multipartite graphs. We first compute the regularity of semi-unmixed graphs.

Theorem 4.10. *Let $(H\,; X, Y)$ be a bipartite graph. If H is semi-unmixed, then $\mathrm{reg}(H) = \mathrm{im}(H)$.*

Proof. We may assume that $\mathrm{iso}(H) = \emptyset$. Then, it follows that $\mathrm{im}(H) \geq 1$, because $E(G) \neq \emptyset$. If $|X| = |Y|$, then $\mathrm{reg}(H) = \mathrm{im}(H)$ by Proposition 1.13(2). Suppose that $|X| < |Y|$. By definition of semi-unmixed, $(H'\,;\, X', Y)$ is unmixed, i.e., we consider the graph H' such that $V(H') = X' \sqcup Y$, $X' = X \sqcup X_1$ with $X_1 \cap V(H) = \emptyset$, and $E(H') = E(H) \cup \{\{x, y\};\, x \in X_1, y \in Y\}$. Then, it follows that $\mathrm{reg}(H) \leq \mathrm{reg}(H') = \mathrm{im}(H')$. On the other hand, we take an induced matching M of H'. If $M \nsubseteq E(H)$, then there is an edge $\{x_1, y_1\} \in M$ such that $x_1 \in X_1$. Since $N_G(x_1) = Y$, it follows that $M = \{\{x_1, y_1\}\}$; hence, $|M| = 1 \leq \mathrm{im}(H)$. If $M \subseteq E(H)$, then M is an induced matching of H. In any case, it follows that $|M| \leq \mathrm{im}(H)$. Therefore, we get $\mathrm{im}(H') \leq \mathrm{im}(H)$. Thus, $\mathrm{reg}(H) = \mathrm{im}(H)$. □

Let $(G\,; V_0, V_1, \dots, V_c)$ be an almost complete multipartite graph. We put $G_i = G_{V_0 \sqcup V_i}$. As an application for almost complete multipartite graph, we give the following statement.

Corollary 4.11. *Let $(G\,; V_0, V_1, \dots, V_c)$ be an almost complete multipartite graph with $E(G) \neq \emptyset$. If G_i is semi-unmixed for all $1 \leq i \leq c$, then $\mathrm{reg}(G) = \mathrm{im}(G)$.*

Proof. We can check the statement by Theorem 4.10 and Lemma 3.29(3). □

We can get Theorem 3.30(2) as a corollary of the above statement.

4.3 Sequential Cohen–Macaulayness as bipartite graphs

In this section, we characterize of sequentially Cohen–Macaulay semi-unmixed graphs in terms of the condition of edges. The conditions (CM1), (CM2), and (CM3) are introduced in Definition 3.18.

Lemma 4.12. *$(H\,; X, Y)$ be a bipartite graph and $\mathrm{iso}(H) = \emptyset$. Suppose that H has a vertex $y \in Y$ of degree 1. If $(H\,; X, Y)$ satisfies (CM1) and (CM3), then there is a numbering $X = \{x'_1, \dots, x'_m\}$ and $Y = \{y'_1, \dots, y'_n\}$ which satisfies (CM1), (CM3), and $y = y'_1$.*

Proof. Suppose that a numbering $X = \{x_1, \dots, x_m\}$ and $Y = \{y'_1, \dots, y'_n\}$ which satisfies (CM1) and (CM3). We put $y_r = y$ and $\{x_s\} = N_H(y_r)$. If $r > m$, then $\deg_H y_s = 1$ by (CM3). We may assume that $r \leq m$. Let a permutation φ on m. We put $x'_{\varphi(i)} = x_i$ and $y'_{\varphi(i)} = y_i$ for all $1 \leq i \leq m$ and $y'_j = y_j$ for all $j > m$. Then, one can check that the numbering $X = \{x'_1, \dots, x'_m\}$ and $Y = \{y'_1, \dots, y'_n\}$ satisfies (CM1) and (CM3). Taking φ such that $\varphi(r) = 1$, then $y = y_r = y'_1$. This completes the proof. □

Lemma 4.13. *Let $(H\,; X, Y)$ be a bipartite graph and $\mathrm{iso}(H) = \emptyset$. If $(H\,; X, Y)$ satisfies (CM1), (CM2), and (CM3), then G is sequentially Cohen–Macaulay.*

Proof. By Theorem 1.9, it is enough to show that H is vertex decomposable. We prove this statement by induction on $|X|$. When $|X| = 1$, G is always vertex decomposable.

Let $m = |X| > 1$. Suppose that a numbering $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ satisfies (CM1), (CM2), and (CM3). We put $H_1 = G \setminus N_H[y_1]$ and $H_2 = H \setminus N_H[x_1]$. By definition of vertex decomposable, we show that H_1 and H_2 are vertex decomposable. Since H is a bipartite graph, it follows that $\mathrm{iso}(H_1) \subseteq Y$. Moreover, $\mathrm{iso}(H_1) \subseteq \{y_{m+1}, \ldots, y_n\}$ because H satisfies (CM1). Therefore, $H_1 \setminus \mathrm{iso}(H_1)$ is a bipartite graph with a vertex partition $V(H_1 \setminus \mathrm{iso}(H_1)) = \{x_2, \ldots, x_m\} \cup (\{y_2, \ldots, y_m\} \cup \{y_j\, ;\, y_j \notin \mathrm{iso}(H_1)\})$. Hence, one can check that $H_1 \setminus \mathrm{iso}(H_1)$ satisfies (CM1), (CM2), and (CM3) on the numbering induced by that of $V(H)$. By induction, $H_1 \setminus \mathrm{iso}(H_1)$ is vertex decomposable; then, H_1 is vertex decomposable.

It remain to prove that H_2 is vertex decomposable. We have $\mathrm{iso}(H_2) \subseteq X$ because H is a bipartite graph. If $x_i \in \mathrm{iso}(H_2)$, then $y_i \in N_H(x_1)$. Moreover, by (CM3), we can check that the converse holds for $2 \leq i \leq m$, i.e., it follows that $\mathrm{iso}(H_2) = \{x_i\, ;\, y_i \in N_H(x_1), 2 \leq i \leq m\}$. Therefore, we can write $X \setminus \mathrm{iso}(H_2) = \{x_{i_1}, \ldots, x_{i_t}\}$ and $Y \setminus N_H(x_1) = \{y_{i_1}, \ldots, y_{i_t}\} \cup \{y_j\, ;\, y_j \notin N_H(x_1), m+1 \leq j \leq n\}$, hence $H_2 \setminus \mathrm{iso}(H_2)$ is a bipartite graph with $V(H_2 \setminus \mathrm{iso}(H_2)) = \{x_{i_1}, \ldots, x_{i_t}\} \cup (\{y_{i_1}, \ldots, y_{i_t}\} \cup \{y_j\, ;\, y_j \notin N_H(x_1), m+1 \leq j \leq n\})$. Then, one can check that $H_2 \setminus \mathrm{iso}(H_2)$ satisfies (CM1), (CM2), and (CM3). By induction, $H_2 \setminus \mathrm{iso}(H_2)$ is vertex decomposable, hence H_2 is also vertex decomposable. □

The main result in this section is given by the following statement:

Theorem 4.14. *Let $(H\, ; X, Y)$ be a semi-unmixed graph and $\mathrm{iso}(H) = \emptyset$. Then, the following conditions are equivalent:*

(1) *H is sequentially Cohen–Macaulay.*

(2) *$(H\, ;\, X, Y)$ satisfies* (CM1)*,* (CM2)*, and* (CM3)*.*

(3) *H is vertex decomposable.*

Proof. (2) $\Rightarrow$ (3) : By the proof of Lemma 4.13.

(3) $\Rightarrow$ (1) : By Theorem 1.9.

(1) $\Rightarrow$ (2) : We prove this statement by induction on $|X|$. When $|X| = 1$, H always satisfies (CM1), (CM2), and (CM3). Suppose that $m = |X| > 1$. Since H is sequentially Cohen–Macaulay, H has a vertex $y \in Y$ of degree 1. By Proposition 4.4, there is a numbering $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ satisfying (CM1) and (CM3). We may assume that $y = y_1$ by Lemma 4.12. Put $H_1 = H \setminus N_H[y_1]$. Applying Proposition 1.4, H_1 is sequentially Cohen–Macaulay; hence, $H_1 \setminus \mathrm{iso}(H_1)$ is also sequentially Cohen–Macaulay. By the proof of Lemma 4.13, $(H_1 \setminus \mathrm{iso}(H_1)\,;\, \{x_2, \dots, x_m\}, \{y_2, \dots, y_m\} \cup \{y_j\,;\, y_j \notin \mathrm{iso}(H_1)\}$ satisfies (CM1) and (CM3). By induction, $H_1 \setminus \mathrm{iso}(H_1)$ satisfies (CM1), (CM2), and (CM3), i.e., there is a numbering $X \setminus \{x_1\} = \{x'_2, \dots, x'_m\}$ and $Y \setminus (\{y_1\} \cup \mathrm{iso}(H_1)) = \{y'_2, \dots, y'_t\}$ which satisfies (CM1), (CM2), and (CM3), and $t \leq n$. we put $\mathrm{iso}(H_1) = \{y'_{t+1}, \dots, y'_n\}$, $x'_1 = x_1$, and $y'_1 = y_1$. Then, one can check that a numbering $X = \{x'_1, x'_2, \dots, x'_m\}$ and $Y = \{y'_1, y'_2, \dots, y'_t, y'_{t+1}, \dots, y'_n\}$ satisfies (CM1), (CM2), and (CM3). Thus, $(H\,;\, X, Y)$ satisfies (CM1), (CM2), and (CM3). □

On the proof of the part ((1) $\Rightarrow$ (2)) of Theorem 4.14, we have proved without the condition (CP) of definition of semi-unmixed. Therefore, generally, we see the following statement:

Theorem 4.15. *Let $(H\,;X, Y)$ be a bipartite graph satisfying* (CM1) *and* (CM3). *Then, the following conditions are equivalent:*

(1) *H is sequentially Cohen–Macaulay.*

(2) *$(H\,;\, X, Y)$ satisfies* (CM1), (CM2), *and* (CM3).

(3) *H is vertex decomposable.*

We give the proof of Theorem 3.3.

(*Proof of Theorem 3.3*) (1) $\Rightarrow$ (2) : Suppose that H is Cohen–Macaulay with $\mathrm{iso}(H) = \emptyset$. By Theorem 4.14, $(H\,;\,X, Y)$ satisfies (CM1), (CM2), and (CM3), i.e., there exists a numbering $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ such that (CM1), (CM2), and (CM3). Since X and Y is maximal, then $|X| = \dim H = |Y|$. Therefore, the numbering satisfies (a), (b), and (c) of Theorem 3.3 by Definition 3.18.

(2) $\Rightarrow$ (1) : There exists a numbering $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$ such that (a), (b), and (c). By Theorem 4.1, H is unmixed. On the other hand, by Definition 3.18 and Theorem 4.15, H is sequentially Cohen–Macaulay. Thus, H is Cohen–Macaulay. □

Finally, in this section, we give a sufficient condition for almost complete multipartite graph to be sequentially Cohen–Macaulay in terms of the edges.

Theorem 4.16. *Let* $(G\,;\,V_0, V_1, \ldots, V_c)$ *be an almost complete multipartite graph,* $G_i = G_{V_0 \sqcup V_i}$*, and* $L_i = G_i \setminus N_{G_i}[V_0]$ *for all* $1 \leq i \leq c$*. Suppose that* $\dim G = \dim G_1$*, either* $(G_1\,;\,V_0, V_1)$ *or* $(G_1\,;\,V_1, V_0)$ *satisfies* (CM1) *and* (CM3)*, and* $(G_i\,;\,V_i, V_0)$ *satisfies* (CM1) *and* (CM3) *for all* $i \neq 1$*. Then, the following conditions are equivalent.*

(1) *G is sequentially Cohen–Macaulay.*

(2) *The bipartite graphs $G_1, G_2, \ldots, G_c$ satisfy the following three conditions:*

  (i) *Either* $(G_1\,;\,V_0, V_1)$ *or* $(G_1\,;\,V_1, V_0)$ *satisfies* (CM1), (CM2), *and* (CM3),

  (ii) *if* $i \neq 1$, $(G_i\,;\,V_i, V_0)$ *satisfies* (CM1), (CM2), *and* (CM3),

  (iii) *if* $i \neq 1$*, then* $|V(L_i)| \leq \min\{1,\, |V(L_1)|\}$.

Proof. (1) $\Rightarrow$ (2) : Suppose that G is sequentially Cohen–Macaulay. We can apply Theorem 3.22 to G. Therefore, (iii) of this statement is clear. By Theorem 4.15, $(G_i\,;\,V_i, V_0)$ satisfies (CM1), (CM2), and (CM3) for all $i \neq 1$. In the similar argument, $(G_1\,;\,V_0, V_1)$ or $(G_1\,;\,V_1, V_0)$ satisfies (CM1), (CM2), and (CM3).

(2) $\Rightarrow$ (1) : By Theorem 4.15, all of the condition of Theorem 3.22 is satisfied. The proof is complete. $\square$

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