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Graduate School of Advanced Mathematical Sciences

Resume of Doctoral Dissertation

A study on visualization methods of Kleinian groups and their spread into artistic expression

Major: Frontier Media Science

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1 Research goal

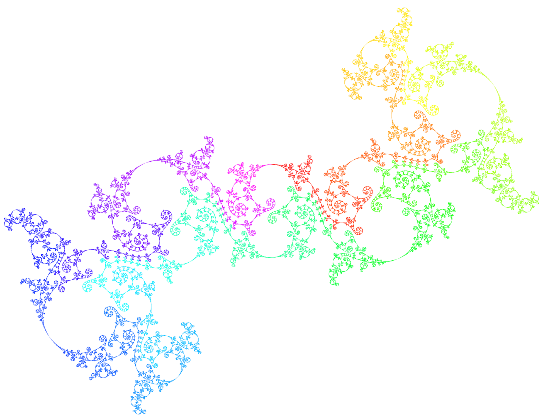

Figure 1

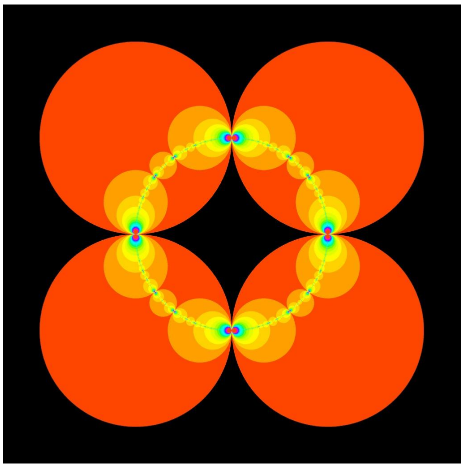

Figure 2

Fractal is a geometrical concept advocated by Mandelbrot. Fractal has a feature called self-similarity. It is a feature that all of the shapes are similar to a part of itself. The fractal structure is often complicated and beautiful. The researchers were interested in the visualized fractals such as the Mandelbrot set and studied them. Moreover, after personal computers became popular, many people enjoyed fractals as art.

The Kleinian group is a geometrical concept related to hyperbolic geometry. The visualized Kleinian group often has a beautiful fractal structure; For example, it has spiral structures originating from Möbius transformations as shown in Figure 1. So, the artists have paid attention to visualized images of the Kleinian group. There is a perspective that the research of Kleinian groups is advanced by computer experiments including visualization. So, programmers are also interested in visualizing Kleinian groups.

Generally, computing and rendering fractals often take much time. Efficiently visualizing Kleinian groups is a challenging problem. Computational efficiency leads to ease of experimentation. Actually, four-dimensional Kleinian groups composed of two by two quaternion matrices have many unsolved problems compared to low-dimensional Kleinian groups because of the difficulties of experiments.

There are many kinds of Kleinian groups. So, in order to develop a new efficient algorithm to visualize Kleinian groups, we focused on circle inversion fractals as shown in Figure 2, and we invented an algorithm called Iterated Inversion System (IIS.) To render circle inversion fractals, we have to compute the centers and radii of circles and draw the disks in order. On the other hand, in IIS, we compute the nesting depth of the disks pixel by pixel and color the pixels. The algorithm can be performed in parallel, and we can render the fractals in real-time. Also, IIS can be extended to the three-dimensional sphere inversion fractals. IIS is an important part of our work. We work on Kleinian groups from the perspective of computer graphics and arts.

There are many solved problems about Kleinian groups in two dimensions. On the other hand, four-dimensional Kleinian groups have many unsolved problems. One of the reasons why is that it is difficult to experiment with four-dimensional Kleinian groups efficiently. IIS is a tool to solve such a situation. The author invented many software and artworks about Kleinian groups with IIS.

There are geometrical concepts that are not visualized. We deal with such a concept with

IIS. Sphairahedron and its fractals are one of them. Visualized sphairahedron and fractals show various forms. They are not only beautiful but also mathematically valuable. We found the seed of mathematical problems. The visualization shows another perspective of the objects. We create new kinds of fractal arts from visualized Kleinian groups.

The software developed by the author has common properties. We need a software framework to make this kind of software easily. The author developed the framework called Flower. It is a flow-based programming environment to construct inversion fractals.

Our main goal is to propose a new kind of mathematical visualization and push forward the study of Kleinian groups and new possibilities of mathematical arts. Also, mathematical results inspire mathematical arts. Both mathematics and arts affect each other. We develop new algorithms and software.

2 Summary of the chapters

In our paper, first of all, we introduce the background of the relationship between Kleinian groups and their visualization and arts. Also, we describe our research goal.

In section 2, we summarize each section in the paper.

In section 3, we introduce the concept, fractals. There are many kinds of fractals and their rendering methods. We mainly use a shader-based rendering and sphere tracing technique. We also show escape-time fractals and famous distance-estimated three-dimensional fractals. Some fractals have Kleinian group-like structures. We describe why such structures appear.

Section 4 denotes the basic mathematical matters of Möbius transformations and the Kleinian groups. We describe inversions, Möbius transformations and their classification, Kleinian groups, and the limit set.

In section 5, we introduce well-known visualizing methods for Kleinian groups. It uses the Cayley graph composed of elements of the Kleinian group. We traverse the graph with breadth first search and depth first search. Then, we enumerate the elements of the Kleinian groups and apply them to the seed of the orbit or fixed points. There are also random search approaches. We also discuss the faults of these methods.

Then, in section 6, we show our efficient algorithm called Iterated Inversion System (IIS.) The algorithm is developed to render fractals based on circle inversion. We can perform computation in parallel and render circle inversion fractals in real-time. We also describe a three-dimensional extension. IIS is used in almost all of our works.

In section 7, we show the application of IIS. Möbius transformation is composed of even numbers of circle or sphere inversions. So, we can express Möbius transformations as geometrical components such as circles, lines, or spheres. We can handle Möbius transformations intuitively and easily. Also, we can optimize applying Möbius transformations using modulo or logarithm operators.

Section 8 shows a sphairahedron. It is a polyhedron with spherical faces. We can render fractals based on sphairahedra using IIS in real-time. We have drawn many fractals that have never been seen before. We also consider materializing sphairahedra and their fractals. We challenged three-dimensional printing. We succeeded in printing not only monochrome objects but also full-colored objects.

Section 9 introduces a software called Flower. It was developed by the author, and it enables us to make circle inversion fractals by flow-based programming; that is, we compose the structure of geometrical construction by connecting nodes with edges. We aim to develop a framework for mathematical visualization.

Section 10 shows the Fractal Flame algorithm. It is a kind of random search approach to generate fractals, but it has many ingenuities to make fractals more beautiful. We apply points on the planes to various transformations repeatedly. We integrate Möbius transformations into transformations for Fractal Flame.

Finally, section 11 is dedicated to a summary of this paper.