A New Type of Impossible Object That Becomes Partly Invisible in a Mirror

Kokichi Sugihara*
Meiji University, 4-21-1 Nakano, Nakano-ku, Tokyo 164-8525, Japan

Abstract

This paper presents a new class of anomalous objects that partly disappear when their reflection is viewed in a mirror. They are constructed from a combination of the mathematical properties of mirror symmetry and the psychological properties of an optical illusion. A general method for designing these objects is proposed, and examples are presented.

1 Introduction

Optical illusions are phenomena in which what we perceive by our eyes are different from physical reality. We can experience them easily by just seeing special objects, but still they are mysterious. Hence they can be potential resources for various industries such as education industry, museum industry, and event and entertainment industry. Indeed optical illusions are used as materials to consider what is seeing and more generally to attract people by showing mystery with easy display.

Historically optical illusions were discovered and studies for 2D pictures first but gradually illusions related to 3D objects were found and have been used in various ways for both art and entertainment. Among these, there is a class of anomalous pictures depicting "impossible objects". Typical examples include Penroses' triangle and endless stairs [4]. These were used by the Dutch graphic artist M. C. Escher in works such as "Ascending and Descending" (1960) and "Belvedere" (1958) [2].

There have also been attempts to realize impossible objects as 3D structures. One familiar approach is to create a discontinuity that appears to be connected when the object is seen from a specific viewpoint [3], and another trick is to use curved surfaces that appear to be planar [1]. With appropriate techniques, it is also possible to realize impossible objects in which apparent connections are actual connections, and apparent planar surfaces are actually planar [6, 7], and this same trick can also be used for creating "impossible motion" illusions [8].

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Another class of optical illusion related to 3D objects is that of depth reversal, in which the perceived depth is the reverse of the actual depth, and hence, for example, a convex shape appears to be concave. This class includes the crater illusion [5], the hollow face illusion [3], reverse perspective [11] and height reversal [10]. The reverse perspective in the art of Patrick Hughes is notable for the fact that binocular stereo vision is useless in resolving our perception; even if we see the object with both of our eyes and from a short distance, the surface depth is reversed, and the surface appears to deform when we move our heads.

In this paper we present a new class of 3D objects whose behavior appears to be impossible, and hence they might be useful for art and entertainment. An ordinary plane mirror generates a mirror symmetric image of an object. This is a basic concept in optical physics, and we cannot defy it. However, we can create 3D objects that appear as if this physical law has been broken: we can create an object such that part of it disappears when it is reflected in a mirror, that is, part of the object is invisible in its mirror image. We call this class of objects "partly invisible objects." The trick to producing this effect is a combination of optical physics and optical illusion. We begin by discussing the various behaviors that are observed in this class of anomalous objects (Section 2), and then we present the "height-reversal property," a basic property of horizontal pictures (Section 3). Using this property, we present a general method for designing partly invisible objects (Section 4).

2 Behaviors of Partly Invisible Objects

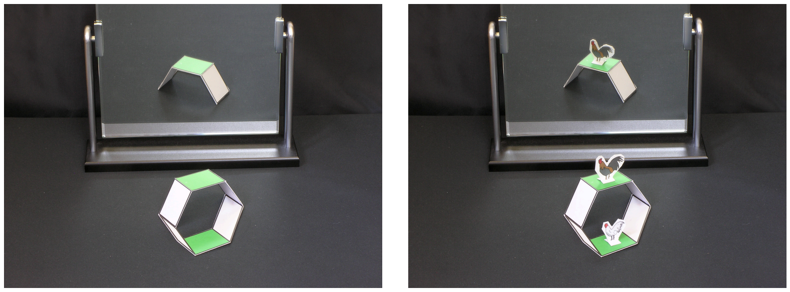
Fig. 1(a) shows a very simple example of a partly invisible object with a plane mirror positioned vertically behind it. In the direct view, the object appears to be a regular hexagonal cylinder turned sideways to the viewer, but in its mirror image, the lower half disappears as if it is buried under the desk surface. The mirror image is quite different from what we expect for an ordinary hexagonal cylinder and an ordinary plane mirror. This is a typical example of the behavior of a partly invisible object.

Similar mysterious behaviors can also be created by adding supplementary objects. As shown in Fig. 1(b), if we place an image of a rooster on the top plate of the cylinder and an image of a hen on the bottom plate, the hen disappears in the mirror.

A second example is shown in Fig. 2(a), where an object is placed and a plane mirror is positioned behind it vertically. In the direct view, the object appears to be the same hexagonal cylinder as in Fig. 1, but in its mirror image, the upper half, instead of the lower half, disappears.

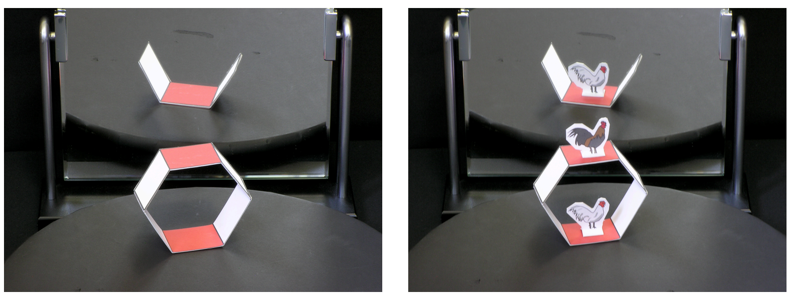
Next, as shown in Fig. 2(b), if we place an image of a rooster on the top plate and an image of a hen on the bottom plate, in the mirror, the rooster disappears together with the upper half of the cylinder.

A third example is shown in Fig. 3. A hexagonal cylinder with images of roosters on the top and bottom plates is placed in front of a vertical mirror, in the same manner as in Figs. 1 and 2. However, in this example, although the



(a) (b)

Figure 1. Regular hexagonal cylinder whose lower half becomes invisible in a mirror: (a) basic behavior of the object; (b) behavior of additional small objects.



(a) (b)

Figure 2. Regular hexagonal cylinder whose upper half becomes invisible in a mirror: (a) basic behavior of the object; (b) behavior of additional small objects.

hexagonal cylinder is reflected in the mirror normally, in the sense that no part becomes invisible, the image of the rooster on the bottom plate disappears in the mirror. In other words, the cylindrical object does not change, but one of the additional small objects becomes invisible.

Fig. 4 shows a fourth example. A regular 12-gonal cylinder is placed sideways, and a plane mirror is positioned vertically behind it. Note that in the direct view, an image of a hen is on the bottom plate, but in the mirror, the hen disappears, and an image of a rooster appears on the top plate of the cylinder. In other words, the cylindrical object does not change, but the additional small object becomes invisible and a new small object becomes visible in the mirror.

These four examples show typical behaviors of partly invisible objects. In some cases, part of an object becomes invisible in a mirror, and in other cases, the main object does not change, but additional small objects become invisible,

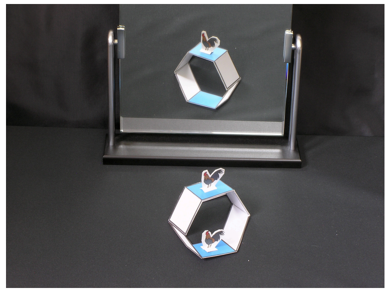


Figure 3. Regular hexagonal cylinder that does not change, but an additional object becomes invisible in a mirror.

Figure 4. Impossible behaviors of a hen and a rooster on a regular 12-gonal cylinder.

or new small objects become visible.

The behavior of partly invisible objects is not consistent with our ordinary understanding of mirror symmetry. What is really occurring is an optical illusion. Indeed, what we understand when we view the object is different from physical reality; more specifically, the main object that appears to be a regular cylinder is actually not a regular cylinder, but a certain unusual shape. Consequently, its mirror image is different from what we expect to see. We will show the actual shapes of the objects and the method used to design them in the following sections.

3 Height-Reversal Property of Horizontal Pictures

As shown in Fig. 5, let (x, y, z) be a 3D Cartesian coordinate system such that the xy plane is horizontal and the positive direction of the z axis orients upward. Let $\boldsymbol{v}_1 = (0, \cos\theta, -\sin\theta)$ and $\boldsymbol{v}_2 = (0, -\cos\theta, -\sin\theta)$ be vectors representing two view directions, where $0 < \theta < \pi/2$. Note that these view directions are parallel to the yz plane, opposite in the y direction, and are slanted downward by the same angle θ measured from the horizontal. Suppose that we place a 3D object on the xy plane, and then observe it from the two view directions.

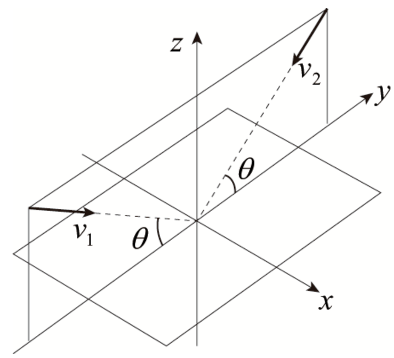


Figure 5. Mutually opposite view directions.

Alternatively, we place a vertical mirror parallel to the xz plane behind the object, and view the scene in the direction $\boldsymbol{v}_1$. Then, the direct view is the view in the direction $\boldsymbol{v}_1$, and the mirror image is the view in the direction $\boldsymbol{v}_2$. Thus, the two views can be realized simultaneously, as we have seen in Figs. 1 to 4.

Next, as shown in Fig. 6, let S be the surface of an object placed on the xy plane, and let S' be its plane-symmetric counterpart with respect to the xy plane. From this, we obtain the following property.

Height-reversal property [10]. Suppose that the entire surface S is visible from the direction $\boldsymbol{v}_1$. Then, the projection, say D, of the surface S in the direction $\boldsymbol{v}_1$ onto the xy plane coincides with the projection of the plane-symmetric counterpart S' in the direction $\boldsymbol{v}_2$ onto the xy plane.

Therefore, if we place the projected picture D on the xy plane, we perceive the surface S when we view it in the direction $\boldsymbol{v}_1$, and we perceive the surface S' when we view it in the direction $\boldsymbol{v}_2$. This property is used to design partly invisible objects as shown in the following section.

4 Design of Partly Invisible Objects

We are ready to design partly invisible objects. As shown in Fig. 7, consider a regular hexagonal cylinder fixed in space sideways in such a way that the xy plane passes through the middle of the cylinder, and it does this in such a way that the cylinder is divided into upper and lower halves that are mutually symmetric with respect to the xy plane. Let us denote the upper half by S and

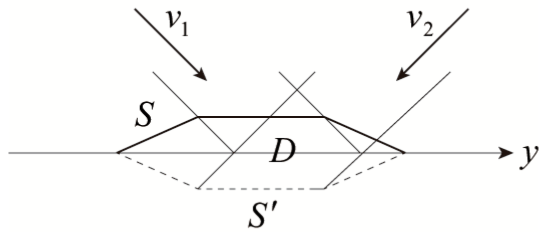


Figure 6. Height-reversal property.

the lower half by S'. Moreover let D_1 be the image of the upper half S and D_2 be the image of the lower half S', which are obtained by projecting S and S', respectively, in the direction $\boldsymbol{v}_1$ onto the xy plane.

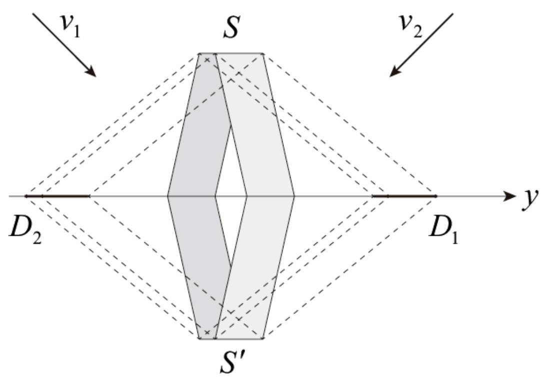


Figure 7. Design of a partly invisible object.

As shown in Fig. 7, the original hexagonal cylinder consists of the upper half S and the lower half S'. Suppose that we replace the lower half S' with the horizontal picture D_2, and obtain the object, say O, consisting of S and D_2. Then, if we view O in the direction $\boldsymbol{v}_1$, its appearance coincides with the hexagonal cylinder, because S is the upper half and D_2 has the same appearance as does the lower half. If we view O in the direction $\boldsymbol{v}_2$, on the other hand, its appearance is just the upper half of the cylinder, because D_2 coincides with S, and consequently, it is hidden behind the real upper half S. This is shown in Fig. 1. A general view of the object in Fig. 1(b) is shown in Fig. 8. From this figure, we can see that the image of the rooster and that of the hen are both positioned vertically, and that the hen is hidden behind the rooster when they are seen in the direction $\boldsymbol{v}_2$. This is the trick of the partly invisible object shown in Fig. 1.

Fig. 9 shows a diagram of the unfolded surfaces from which the object can

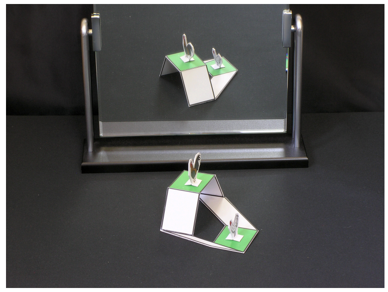


Figure 8. General view of the solid shown in Fig. 1(b).

be constructed by paper crafting. Part A corresponds to the upper half of the hexagonal cylinder. We fold it along two lines at 120 degrees, so that it forms half of a hexagonal cylinder. Part B corresponds to the lower half, but it is a picture that is to be placed on the horizontal plane. Hence, B should not be folded; it is used as it is. These two parts are glued together so that the same symbols (“a” or “b”) are placed together. We suggest that the reader construct such an object, since this will make it easier to understand the shape of the object and the relation between the hidden part and the part that obscures it.

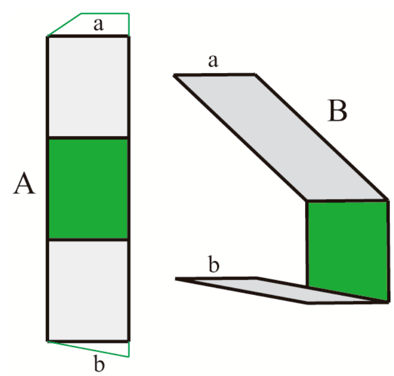


Figure 9. Unfolded surfaces from which the object in Fig. 1(a) can be constructed.

Next, let us change the original hexagonal cylinder by replacing the upper half S with the horizontal picture D_1. Let us denote the resulting object by O'. That is, the object O' consists of the lower half S' and the horizontal picture D_1. If we view O' in the direction $\boldsymbol{v}_1$, it appears to be a hexagonal cylinder, because the picture D_1 coincides with the upper half S. If we view O' in the direction $\boldsymbol{v}_2$, on the other hand, it appears to be the lower half of the cylinder, because the horizontal picture D_1 has the same appearance as the lower half S', and the actual lower half S' is hidden behind the picture D_1. A general view of the situation displayed in Fig. 2(b) is shown in Fig. 10.

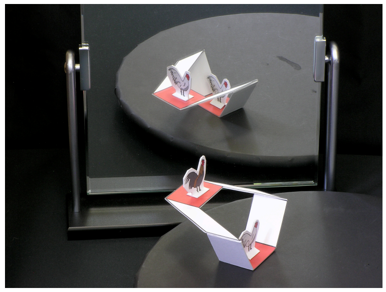


Figure 10. General view of the solid shown in Fig. 2(b).

Readers may ask why the image of the hen remains standing in the mirror image and where the image of the rooster has gone. The answer is that the small object on the top plate of the cylinder in Fig. 2(b) has an image of a rooster on the front side and an image of a hen on the backside. Hence, the image hen we see in the mirror image in Fig. 2(b) is the backside of the image rooster; the original image hen is hidden behind it.

The third object shown in Fig. 3 consists of the upper half S and both horizontal pictures D_1 and D_2, as shown in Fig. 11. Therefore, the object appears to be a hexagonal cylinder when viewed from directions $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$. The image roosters are positioned on the top plate and on one of the horizontal pictures. Hence, both roosters are visible when they are viewed from one direction, whereas one is hidden by the other when they are viewed from the other direction.

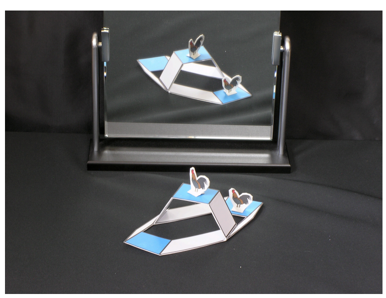


Figure 11. General view of the solid shown in Fig. 3.

The forth object shown in Fig. 4 consists of the two horizontal pictures D_1 and D_2, as shown in Fig. 12, where the original hexagonal cylinder is replaced with a regular 12-gon cylinder. Thus, both the upper half and the lower half are replaced with the horizontal pictures. Hence, because of the height-reversal property, the upper half in the direct view corresponds with the lower half in the mirror, and the lower half in the direct view corresponds with the upper half in the mirror. In this example, the image of the rooster is pasted onto the reverse side of the image of the hen. Thus, the hen at the bottom in the direct view corresponds to the rooster at the top in the mirror image.

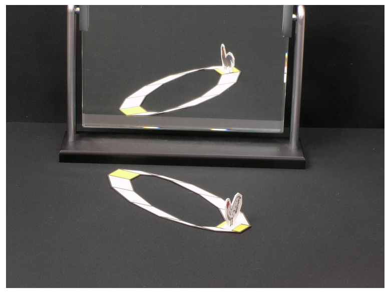
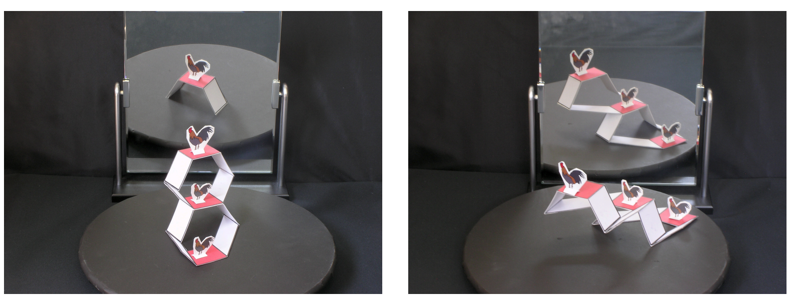


Figure 12. General view of the solid shown in Fig. 4.

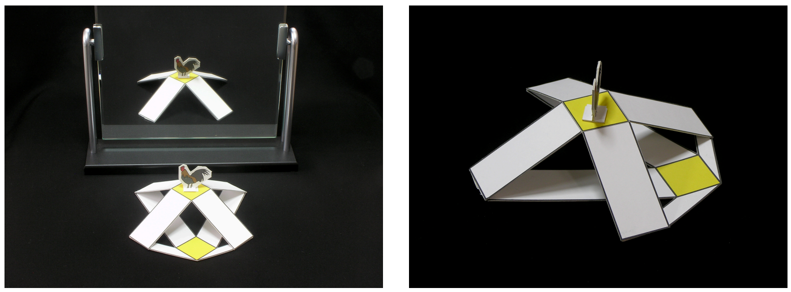
More-complicated structures can be constructed by combining two or more objects, one on top of the other. An example is shown in Fig. 13(a); two hexagonal cylinders, one on top of the other, appear to be only the upper half of a single cylinder when viewed in the mirror. Three-quarters of the structure disappears. The actual structure of this object consists of two copies of the object in Fig. 8, as shown in Fig. 13(b). Similarly we can repeat the basic structure as many times as we want to create complicated examples of partly invisible cylinders.



(a) (b)

Figure 13. More-complicated partly invisible object: (a) the object and its mirror image seen in a specific view direction; (b) the same object seen in a general view direction.

Another direction of generalization of partly invisible cylinders is shown in Fig. 14(a), where two apparent cylinders cross each other. Both of the cylinders are of the type shown in Fig. 1; the upper half is the exact shape of the upper half of the hexagonal cylinders, and the lower half is a horizontal picture. Therefore, both of the lower parts disappear in the mirror. The actual shape of the object is shown in Fig. 14(b).



(a) (b)

Figure 14. Partly invisible object composed of two crossing cylinders: (a) the object and its mirror image seen in a specific view direction; (b) the same object seen in a general view direction.

5 Concluding Remarks

We have presented the principles and a design method for partly invisible objects. This class of objects is based on the height-reversal property of pictures on the horizontal plane [10]. We begin with a 3D object that is symmetric with respect to a horizontal plane, and replace the upper half or the lower half with a picture obtained by projecting the object onto the horizontal plane. Then, the upper and lower halves coincide when they are seen from the second viewpoint, and consequently, half of the object disappears, because it is hidden behind the other.

In addition to the examples presented in this paper, partly invisible cylinders might be generalized in various ways; for example, the hexagon can be replaced with other polygons, and any number of cylinders might be combined or made to intersect.

The invisible objects proposed in this paper are sensitive to the angle from which they are viewed. Part of an object becomes invisible because it is hidden by another part, and although these parts coincide when they are seen from a specific viewpoint, even a small deviation of the viewing angle reveals the hidden part and thus reveals the trick. For this reason, it is important to precisely determine the viewing angle; otherwise, the visual effect is lost.

It would be preferable if the visual effect could be maintained within a certain range of viewing angles, since this would make it easier to create a display. For the present objects, it is necessary to create a structure to guide the viewing angle, such as only allowing it to be viewed through a small hole. If the illusion could be maintained over a larger viewing angle, however, this would not be necessary. This could be achieved, for example, if the hidden part were smaller than the part obscuring it. An area of further research will be to find a principle for creating invisible objects that can be viewed from wider angles.

Acknowledgements

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